12. 8 Optical Parametric Amplifiers and Oscillators

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12. 8 Optical Parametric Amplifiers and Oscillators

12.8.1 Optical Parametric Generation (OPG)

Degeneracy: \( \omega_i = \omega_s = \omega_p / 2 \)

Energy Conservation: \[ \hbar \omega_p = \hbar \omega_s + \hbar \omega_i. \]

Momentum Conservation: \[ \hbar \vec{k}_p = \hbar \vec{k}_s + \hbar \vec{k}_i. \]
Optical Parametric Oscillator (OPO)

Double resonant: **Signal** and **idler** resonant

Single resonant: **Only Signal** resonant

Advantage: Widely tunable, both signal and idler can be used!

For OPO to operate, less gain is necessary in contrast to an OPA.
12.8.2 Nonlinear Optical Susceptibilities

Total field: Pump, signal and idler:

\[ \vec{E}(\vec{r}, t) = \sum_{\omega > 0} \sum_{i=1}^{3} \frac{1}{2} \left\{ \hat{E}_i(\omega) e^{j(\omega t - \vec{k}_a \vec{r})} + c.c. \right\} \vec{e}_i. \]

Drives polarization in medium:

\[ \vec{P}(\vec{r}, t) = \sum_n \vec{P}^{(n)}(\vec{r}, t) \]

Polarization can be expanded in power series of the electric field:

\[ \vec{P}^{(n)}(\vec{r}, t) = \sum_{\omega > 0} \sum_{i=1}^{3} \frac{1}{2} \left\{ P_i^{(n)}(\omega) e^{j(\omega t - \vec{k}_b \vec{r})} + c.c. \right\} \vec{e}_i. \]

Defines susceptibility tensor:

\[ P_i^{(n)}(\omega_b) = \frac{\varepsilon_0}{2^{m-1}} \sum_P \sum_{j...k} \chi_{ij...k}^{(n)}(\omega_b : \omega_1, \ldots, \omega_n) E_j(\omega_1) \cdots E_k(\omega_n); \]

\[ \omega_b = \sum_{i=1}^{n} \omega_i \text{ and } k'_b = \sum_{i=1}^{n} k_i \]
Special Cases

\[ \hat{P}_i^{(2)}(\omega_3) = \varepsilon_0 \sum_{jk} \chi_{ijk}^{(2)}(\omega_3 : \omega_1, -\omega_2) \hat{E}_j(\omega_1) \hat{E}_k^*(\omega_2), \]

\[ \omega_3 = \omega_1 - \omega_2 \text{ und } k'_3 = k_1 - k_2. \]

(\rightarrow \text{Difference Frequency Generation (DFG)})

\[ \hat{P}_i^{(2)}(\omega_2) = \varepsilon_0 \sum_{jk} \chi_{ijk}^{(2)}(\omega_2 : \omega_3, -\omega_1) \hat{E}_j(\omega_3) \hat{E}_k^*(\omega_1), \]

\[ \omega_2 = \omega_3 - \omega_1 \text{ und } k'_2 = k_3 - k_1. \]

(\rightarrow \text{Parametric Generation (OPG)})

\[ \hat{P}_i^{(3)}(\omega_4) = \frac{6\varepsilon_0}{4} \sum_{jkl} \chi_{ijkl}^{(3)}(\omega_4 : \omega_1, \omega_2, -\omega_3) \hat{E}_j(\omega_1) \hat{E}_k(\omega_2) \hat{E}_l^*(\omega_3) \]

\[ \omega_4 = \omega_1 + \omega_2 - \omega_3 \text{ und } k'_4 = k_1 + k_2 - k_3. \]

(\rightarrow \text{Four Wave Mixing (FWM)})
12.8.3 Continuous Wave OPA

Wave equation (2.7) :

$$
\left( \Delta - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \right) \vec{E} = \mu_0 \frac{\partial^2}{\partial t^2} \vec{P}.
$$

Include linear and second order terms:

$$
\left( \Delta - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \right) \vec{E} = \mu_0 \frac{\partial^2}{\partial t^2} \left( \vec{P}^{(1)}(\vec{r}, t) + \vec{P}^{(2)}(\vec{r}, t) \right)
$$

Changes group and phase velocities of waves
Nonlinear interaction of waves

z-propagation only:

$$
\vec{E}_{p,s,i}(z, t) = \text{Re} \left\{ E_{p,s,i}(z) e^{j(\omega_{p,s,i} t - k_{p,s,i} z)} \vec{e}_{p,s,i} \right\}
$$

Wave amplitudes
\[ \bar{P}_{p,s,i}^{(2)}(z, t) = \text{Re} \left\{ P_{p,s,i}^{(2)}(z) e^{j(\omega_{p,s,i}t-k'_{p,s,i}z)} \hat{e}_{p,s,i} \right\} \]

Separate into three equations for each frequency component:

**Slowly varying amplitude approximation:**

\[ \frac{d^2 E_{p,s,i}(z)}{dz^2} \ll 2k \frac{d E_{p,s,i}(z)}{dz}, \]

\[ \frac{\partial E_{p,s,i}(z)}{\partial z} = -\frac{j c_0^2 \omega_{p,s,i}}{2n(\omega_{p,s,i})} P_{p,s,i}^{(2)}(z) e^{-j(k'_{p,s,i}-k_{p,s,i})z} \]

**Introduce phase mismatch:**

\[ \Delta k = k(\omega_p) - k(\omega_s) - k(\omega_i) \]

**and eff. nonlinearity and coupling coefficients:**

\[ d_{eff} = \frac{1}{2} \chi_{ijk}^{(2)}(\omega_p : \omega_s, \omega_i), \quad \kappa_{p,s,i} = \omega_{p,s,i} d_{eff} / (n_{p,s,c_0}) \]
Coupled wave equations:

\[
\frac{\partial E_p(z)}{\partial z} = -j\kappa_p E_s(z) E_i(z) e^{j\Delta k z}, \\
\frac{\partial E_s(z)}{\partial z} = -j\kappa_s E_p(z) E_i^*(z) e^{-j\Delta k z}, \\
\frac{\partial E_i(z)}{\partial z} = -j\kappa_i E_p(z) E_s^*(z) e^{-j\Delta k z}.
\]

Intensity of waves:

\[
I_{p,s,i} = \frac{n_{p,s,i}}{2Z_F} |E_{p,s,i}|^2
\]

Manley-Rowe Relations:

\[
-\frac{1}{\omega_p} \frac{dI_p}{dz} = \frac{1}{\omega_s} \frac{dI_s}{dz} = \frac{1}{\omega_i} \frac{dI_i}{dz}
\]
12.8.4 Theory of Optical Parametric Amplification

Undepleted pump approximation: \( E_p = \text{const.} \)

\[
\begin{align*}
\frac{\partial E_s(z)}{\partial z} &= -j\kappa_s \ E_p E_i^*(z) \ e^{-j\Delta k z}, \\
\frac{\partial E_i(z)}{\partial z} &= -j\kappa_i \ E_p E_s^*(z) \ e^{-j\Delta k z}.
\end{align*}
\]

with:

\[
E_s(z = 0) = E_s(0) \quad E_i(z = 0) = 0
\]

\[
E_s(z) \sim E_s(0) \ e^{g z - j\Delta k z/2} \text{ and } E_i(z) \sim E_i(0) \ e^{g z - j\Delta k z/2}
\]

\[
\begin{vmatrix}
  g - j\frac{\Delta k}{2} & j\kappa_s \ E_p \\
j\kappa_i \ E_p^* & g + j\frac{\Delta k}{2}
\end{vmatrix} = 0
\]

\[
g = \sqrt{\Gamma^2 - \left(\frac{\Delta k}{2}\right)^2}, \text{ with } \Gamma = \sqrt{\kappa_i \kappa_s |E_p|^2}.
\]

Gain \quad \text{Max. gain, when phase matched}
Maximum Gain

\[ \Gamma^2 = \frac{\omega_s \omega_i}{n_s n_i c_0^2} d_{eff}^2 \text{ } |E_p|^2 = \frac{2Z_{F0} \omega_s \omega_i}{n_p n_s n_i c_0^2} d_{eff}^2 \text{ } |I_p|^2 \]

\[ FOM = \frac{d_{eff}}{\sqrt{\lambda_s \lambda_i n_p n_s n_i}} \]

General solutions:

\[ E_s(z) = \{ E_s(0) \cosh g z + B \sinh g z \} e^{-j \Delta k z / 2} \]

\[ E_i(z) = \{ E_i(0) \cosh g z + D \sinh g z \} e^{-j \Delta k z / 2} \]

Here:

\[ I_s(L) = I_s(0) \left[ 1 + \frac{\Gamma^2}{g^2} \sinh^2 g L \right] \]

\[ I_i(L) = I_s(0) \frac{\omega_i \Gamma^2}{\omega_s g^2} \sinh^2 g L. \]

For large gain: \( \Gamma L >> 1 \)

\[ I_s(L) = \frac{1}{4} I_s(0) e^{2 \Gamma L}, \]

\[ I_i(L) = \frac{1}{4} I_s(0) \frac{\omega_i}{\omega_s} e^{2 \Gamma L} \]

\[ G = \frac{I_s(L)}{I_s(0)} = \frac{1}{4} e^{2 \Gamma L} \]
Figure of merit:

\[ FOM = \frac{d_{eff}}{\sqrt{\lambda_s \lambda_i n_p n_s n_i}} \]
Fig. Exact solution for signal gain, plotted together with hyperbolic secant and exponential function solutions, approximate solutions derived by assuming the pump is undepleted. The exact solution for the pump intensity is also shown.
**Fig. 12.24** Parametric gain for an OPA at the pump wavelength $\lambda_p = 0.8 \, \mu m$ and the signal wavelength $\lambda_s = 1.2 \, \mu m$, using type I phase matching in BBO ($d_{\text{eff}} = 2 \, \text{pm/V}$).
Fig. 12.25 Parametric gain for an OPA at the pump wavelength $\lambda_p = 0.4 \, \mu m$ and the signal wavelength $\lambda_s = 0.6 \, \mu m$, using type I phase matching in BBO ($d_{eff} = 2 \, pm/V$).
12.8.5 Phase Matching

\[ \Delta k = 0 \quad \Rightarrow \quad n_p = \frac{n_s \omega_s + n_i \omega_i}{\omega_p} \]

Uniaxial Crystal: \( n_e < n_o \)

**Type I: noncritical**

**Type I: critical**

Fig. 12.26 Type I noncritical phase matching.

Fig. 12.27 Type I critical phase matching by adjusting the angle \( \theta \) between wave vector of the propagating beam and the optical axis.
12.8.5 Phase Matching

Critical Phase Matching

\[ n_{ep}(\theta) \omega_p = n_{os} \omega_s + n_{oi} \omega_i \]

\[ \frac{1}{n_{ep}(\theta)^2} = \frac{\sin^2 \theta}{n_{ep}^2} + \frac{\cos^2 \theta}{n_{op}^2} \]

\[ \theta = \arcsin \left[ \frac{n_{ep}}{n_{ep}(\theta)} \sqrt{\frac{n_{op}^2 - n_{ep}^2(\theta)}{n_{op}^2 - n_{ep}^2}} \right] \]
Fig. 12.28 Angle tuning curves for a BBO OPA at the pump wavelength $\lambda_p = 0.8 \, \mu m$ for type I phase matching (dotted line), type II ($o_s + e_i \rightarrow e_p$) phase matching (solid line), and type II ($e_s + o_i \rightarrow e_p$) phase matching (dashed line).
Fig. 12.28 Angle tuning curves for a BBO OPA at the pump wavelength $\lambda_p = 0.4 \, \mu m$ for type I phase matching (dotted line), type II ($o_s + e_i \rightarrow e_p$) phase matching (solid line), and type II ($e_s + o_i \rightarrow e_p$) phase matching (dashed line).
12.8.6 Quasi Phase Matching

Fig. 12.30: Variation of $d_{eff}$ in a quasi phase matched material as a function of propagation distance.

$$d_{eff}(z) = \sum_{m=-\infty}^{+\infty} d_m e^{jm\kappa z}$$

$$\frac{\partial E_p(z)}{\partial z} = -j\kappa_p E_s(z) E_i(z) e^{j\Delta k z}$$
12.8.7 Ultrashort Pulse Optical Parametric Amplification

\[ \vec{E}_{p,s,i}(z,t) = \text{Re} \left \{ E_{p,s,i}(z,t) e^{i(\omega_{p,s,i} t - k_{p,s,i} z)} \vec{e}_{p,s,i} \right \} \]

**Pulse envelopes**

\[ \frac{\partial E_p}{\partial z} + \frac{1}{v_p} \frac{\partial E_p}{\partial t} = -j \kappa_p E_s E_i e^{i \Delta k z} , \]
\[ \frac{\partial E_s}{\partial z} + \frac{1}{v_s} \frac{\partial E_s}{\partial t} = -j \kappa_s E_p E^*_i e^{-j \Delta k z} , \]
\[ \frac{\partial E_i}{\partial z} + \frac{1}{v_i} \frac{\partial E_s}{\partial t} = -j \kappa_i E_p E^*_s e^{-j \Delta k z} , \]

\[ v_{p,s,i} = \frac{dk}{d\omega} |_{\omega_{p,s,i}} \] are the corresponding group velocities

\[ t' = t - \frac{z}{v_p} \]
\[ \frac{\partial E_p}{\partial z} = -j \kappa_p E_s E_i e^{i \Delta k z} , \]
\[ \frac{\partial E_s}{\partial z} + \left( \frac{1}{v_s} - \frac{1}{v_p} \right) \frac{\partial E_s}{\partial t} = -j \kappa_s E_p E^*_i e^{-j \Delta k z} , \]
\[ \frac{\partial E_i}{\partial z} + \left( \frac{1}{v_i} - \frac{1}{v_p} \right) \frac{\partial E_s}{\partial t} = -j \kappa_i E_p E^*_s e^{-j \Delta k z} . \]
Temporal walkoff
Group Velocity Mismatch (GVM)

\[ \ell_{jp} = \frac{\tau}{\delta_{jp}}, \quad \text{with} \quad \delta_{jp} = \left( \frac{1}{v_j} - \frac{1}{v_p} \right) \]

Fig. 12.31: Pump-signal ($\delta_{sp}$) and pump-idler ($\delta_{ip}$) group velocity mismatch curves for a BBO OPA at the pump wavelength $\lambda_p=0.8 \ \mu\text{m}$ for type I phase matching (solid line) and type II ($o_s + e_i \rightarrow e_p$) phase matching (dashed line).
Fig. 12.32: Pump-signal ($\delta_{sp}$) and pump-idler ($\delta_{ip}$) group velocity mismatch curves for a BBO OPA at the pump wavelength $\lambda_p=0.4 \, \mu m$ for type I phase matching (solid line) and type II ($o_s + e_i \rightarrow e_p$) phase matching (dashed line).
Figure 12.34: Signal pulse evolution for a BBO type I OPA with $\lambda_p = 0.4 \, \mu\text{m}$, $\lambda_s = 0.7 \, \mu\text{m}$, for different lengths $L$ of the nonlinear crystal. Pump intensity is $20 \, \text{GW/cm}^2$. Time is normalized to the pump pulse duration and the crystal length to the pump-signal pulse splitting length. [5]
Figure 12.35: Signal pulse evolution for a BBO type II OPA with $\lambda_p = 0.8$ μm, $\lambda_s = 1.5$ μm, for different lengths $L$ of the nonlinear crystal. Pump intensity is 20 GW/cm$^2$. Time is normalized to the pump pulse duration and the crystal length to the pump-signal pulse splitting length. [5]
OPA Bandwidth

\[ \omega_s \rightarrow \omega_s + \Delta \omega \quad \omega_i \rightarrow \omega_i - \Delta \omega. \]

\[ \Delta k = -\frac{dk_s}{d\omega} \Delta \omega + \frac{dk_i}{d\omega} \Delta \omega = \left( \frac{1}{v_i} - \frac{1}{v_s} \right) \Delta \omega \]

Bandwidth limitation due to GVM

\[ \Delta f = -\frac{2\sqrt{\ln 2}}{\pi} \sqrt{\frac{\Gamma}{L}} \cdot \frac{1}{\left| \frac{1}{v_i} - \frac{1}{v_s} \right|} \]

For vanishing dispersion:

\[ \Delta f = -\frac{2^4 \sqrt{\ln 2}}{\pi} \sqrt{\frac{\Gamma}{L}} \cdot \frac{1}{\left| \frac{d^2 k_s}{d\omega^2} + \frac{d^2 k_i}{d\omega^2} \right|}. \]
Figure 12.35: Phase matching bandwidth for a BBO OPA at the pump wavelength $\lambda_p=0.8$ $\mu$m for type I phase matching (solid line) and type II ($o_s + e_i \rightarrow e_p$) phase matching (dashed line). Crystal length is 4 mm and pump intensity 50 GW/cm$^2$. 
Figure 12.36: Phase matching bandwidth for a BBO OPA at the pump wavelength $\lambda_p = 0.4 \, \mu m$ for type I phase matching (solid line) and type II ($o_s + e_i \rightarrow e_p$) phase matching (dashed line). Crystal length is 2 mm and pump intensity 100 GW/cm$^2$. 
12.8.8 Optical Parametric Amplifier Designs

Figure 12.37: Scheme of an ultrafast optical parametric amplifier. SEED: seed generation stage; DL1, DL2: delay lines; OPA1, OPA2 parametric amplification stages; COMP: compressor.
Near-IR OPA

Figure 12.38: Scheme of a near-IR OPA DL: delay lines; WL: white light generation stage; DF: dichroic filter. [5]
12.8.9 Noncollinear Optical Parametric Amplifier (NOPA)

**Figure 12.39:** a) Schematic of a noncollinear interaction geometry; b) representation of signal and idler pulses in the case of collinear interaction; and c) same as b) for noncollinear interaction.

**Phase Matching Condition: Vector Condition:**

\[
\Delta k_{par} = k_p \cos \alpha - k_s - k_i \cos \Omega = 0 \\
\Delta k_{perp} = k_p \sin \alpha - k_i \sin \Omega = 0
\]
Variation on phase matching condition by $\Delta \omega$

$$\Delta k_{par} = -\frac{dk_s}{d\omega_s} \Delta \omega + \frac{dk_i}{d\omega_i} \cos \Omega \Delta \omega - k_i \sin \Omega \frac{d\Omega}{d\omega_i} \Delta \omega = 0$$

$$\Delta k_{perp} = \frac{dk_i}{d\omega_i} \sin \Omega \Delta \omega + k_i \cos \Omega \frac{d\Omega}{d\omega_i} \Delta \omega = 0$$

And addition

$$\frac{dk_i}{d\omega_i} - \cos \Omega \frac{dk_s}{d\omega_s} = 0$$

Correct index

$$v_{gs} - v_{gi} \cos \Omega = 0$$

Only possible if: $v_{gi} > v_{gs}$

$$\alpha = \arcsin \left[ \frac{1 - \frac{v_{gs}^2}{v_{gi}^2}}{1 + 2v_s n_s \lambda_i/v_i n_i \lambda_s + (n_s \lambda_i/n_i \lambda_s)^2} \right]$$
Figure 12.40: Phase-matching curves for a noncollinear type I BBO OPA pumped at $\lambda_p=0.4 \mu m$, as a function of the pump-signal angle $\alpha$. [5]
**Figure 12.41**: Scheme of a noncollinear visible OPA. BS: beam splitter; VA: variable attenuator; S: 1-mm-thick sapphire plate; DF: dichroic filter; M1, M2, M3, spherical mirrors.[5]
Figure 12.42: a) Solid line: NOPA spectrum under optimum alignment conditions; dashed line: sequence of spectra obtained by increasing the white light chirp; b) points: measured GD of the NOPA pulses; dashed line: GD after ten bounces on the ultrabroadband chirped mirrors.
Figure 12.43: Reconstructed temporal intensity of the compressed NOPA pulse measured by the SPIDER technique. The inset shows the corresponding pulse spectrum.[5]
12.8.10 Optical Parametric Chirped Pulse Amplifier (OPCPA)

2-µm OPCPA

Nd:YLF regen amp @1047nm
120ps, 1 mJ @1kHz

2 Nd:YLF-MPS modules
120ps, 6mJ @1kHz

Grating Pair Compressor

100µJ
3.5mJ

800 nm OPCPA

YDFA

Circulator

CFBG stretcher

YDFA

MgO:PPLN 13.1µm

120ps

Si

30 mm

AOPDF

Si

30 mm

BBO

suprasil

300 mm

10 nJ, 6.7 ps

OPA 2

MgO:PPSLT 31.4µm

Ti:S oscillator

DCM

800 nm OPCPA
Optical Synthesis from OPAs

Combination of light from broadband Optical Parametric Amplifiers.

$\chi^{(2)}$ optical process in nonlinear crystals

Broadband phase-matching

D. Brida et al., Journal of Optics A 12, 013001 (2010)
Multi-millijoule Pulse Synthesis with OPAs

- **Two-octave-wide** waveform synthesis from OPAs at the **multi-mJ energy level**
- **passively CEP-stable** WLG seed [G. Cerullo *et al.*, Laser Photonics Rev. 5, 323 (2010)]
- WLG seed split into 3 wavelength channels and amplified in 3 OPA stages each
- Three channels are **individually** compressed and coherently recombined
- relative timing is tightly locked using balanced optical cross-correlators (BOCs)
Optical Pulse Synthesizer

### VIS NOPA
- 0.17 mJ signal
- 20% (0.8 mJ pump) pump-signal conversion efficiency
- TL 5.6 fs
- 2.9 optical cycles @ $\lambda_c=573$nm

### NIR DOPA
- 0.20-0.25 mJ signal
- 12-15% (1.7 mJ pump) pump-signal conversion efficiency
- TL 5.2 fs
- 2.1 optical cycles @ $\lambda_c=750$nm

### IR DOPA
- 1.7 mJ octave-spanning signal
- 22% (7.7 mJ pump) pump-signal conversion efficiency
- TL 5.2 fs
- 1.1 optical cycle @ $\lambda_c=1.4$um
13 High Harmonic Generation

13.1 Atomic units

13.2 The three step model
  13.2.1 Ionization
  13.2.2 Propagation
  13.2.3 Recombination

13.3 Attosecond pulses

13.4 The intensity challenge
  13.4.1 The necessity of short drive pulses
  13.4.2 Quantum diffusion
  13.4.3 Propagation effects — phase matching

14 Synchrotron Radiation

15 Free-Electron Lasers
13. High Harmonic Generation

HHG: The three step model

13.1 Atomic units

Hydrogen Atom

Radius = 1

and

$\hbar = 1$
Perturbative Nonlinear Optics:

\[ P = \varepsilon_0 \left( \chi^{(1)} E + \chi^{(2)} E^2 + \chi^{(3)} E^3 + \ldots \right) \]

E-field in units of \( E_a \): Susceptibilities of order 1

\[ \chi^{(n)} = \frac{\chi^{(n-1)}}{E_a} \quad \chi^{(2)} \sim 10^{12}/(\text{V/m}) \quad \chi^{(3)} \sim 10^{23}/(\text{V/m})^2 \]
A typical field amplitude for HHG in helium is $0.3 \text{ au} \approx 1.7 \times 10^9 \text{V/cm}$. The corresponding intensity is $0.5 \cdot E^2/377\Omega \approx 4 \times 10^{15} \text{W/cm}^2$. For a Ti:sapphire beam (800 nm wavelength) focused to a $25 \mu\text{m}^2$ spot and for a pulse duration of 10 fs we find that the pulses should carry about 0.1 mJ of energy.

13.2 The Three Step Model

13.2.1 First Step - Ionization

Multiphoton Ionization

$$\omega < I_p \ (\hbar \omega < I_p)$$

Keldysh Parameter:

$$\gamma = \omega \sqrt{2I_p/E} > 1$$

\[ I_p \]
Tunneling Regime

Dominant when: \( U_p \gtrsim I_p \)

Ponderomotive energy:

\[
U_p = \left( \frac{qE}{2m\omega} \right)^2 = \left( \frac{E}{2\omega} \right)^2
\]

Quasi-static approximation:

\[
w(E) \sim \exp\left( -\frac{2(2I_p)^{3/2}}{3E} \right)
\]

|a(t)|^2 = \exp \left( -\int_0^t w(E(t'))dt' \right)

Barrier-Suppression Regime

Emission probability linear in field
Ionization rates

Figure 13.5: Static ionization rate for Hydrogen on a linear and logarithmic scale.

Figure 13.6: Static ionization rate for Helium on a linear and logarithmic scale.
13.2.2 Propagation

\[ \ddot{x}(t) = E_0 \cos \omega t \]
\[ \dot{x}(t) = \frac{E_0}{\omega} \sin \omega t - \frac{E_0}{\omega} \sin \omega t_0 \]
\[ x(t) = -\frac{E_0}{\omega^2} \cos \omega t - (t - t_0) \frac{E_0}{\omega} \sin \omega t_0 + \frac{E_0}{\omega^2} \cos \omega t_0, \]

Figure 13.7:
Trajectory with max. kinetic energy

Maximum for $\omega t \approx 0.31$ \quad $E_k = 3.17U_p$ \quad $U_p = \frac{1}{2} \left( \frac{E}{\omega} \right)^2$
13.2.3 Recombination

Schroedinger Equation in Dipole Approximation

\[ i \frac{d}{dt} |\psi\rangle = H |\psi\rangle - E(t)x \]

Atomic Hamiltonian

\[ H = -\frac{1}{2} \nabla + V(\vec{r}), \]

Wavefunction of partially ionized atom

\[ |\psi(t)\rangle = a(t) |0\rangle + |\varphi(t)\rangle . \]

Dipolmoment

\[ \vec{d}(t) = \langle \psi(t) | \vec{x} |\psi(t)\rangle \]
Dipol acceleration and Ehrenfest Theorem

\[ \ddot{x} = -\nabla V(\vec{r}) + E(t) \]

\[ \ddot{d}_{\text{HHG}}(t) = -\langle \psi(t) | \nabla V(\vec{r}) | \psi(t) \rangle \]
\[ = -|a(t)|^2 \langle 0 | \nabla V(\vec{r}) | 0 \rangle - a(t) \langle \varphi(t) | \nabla V(\vec{r}) | 0 \rangle \]
\[ - a^*(t) \langle \psi(t) | \nabla V(\vec{r}) | 0 \rangle - \langle \varphi(t) | \nabla V(\vec{r}) | \varphi(t) \rangle. \]
\[ \sim - a^*(t) \langle \varphi(t) | \nabla V(\vec{r}) | 0 \rangle - a(t) \langle 0 | \nabla V(\vec{r}) | \varphi(t) \rangle \]
\[ = \ddot{\xi}(t) + \ddot{\xi}^*(t), \text{ with } \ddot{\xi}(t) = -a^*(t) \langle \varphi(t) | \nabla V(\vec{r}) | 0 \rangle \]

After some calculations

\[ \ddot{\xi}(t) = 2^{3/2} \pi (2I_P)^{1/4} e^{i\pi/4} \sum_n \frac{a(t_n b(t)a(t) \sqrt{w(E t_n b(t))}}{E(t_n b(t))(t - t_n b(t))^{3/2}} \dot{\alpha}_{\text{rec}} e^{-iS_n(t)} \]
13.2.3 Recombination

\[ \omega_{h_{\text{max}}} = I_p + 3.17 U_p \]

Ideal sinusoidal single cycle pulse
\[ E(t) = E_0 \sin \omega t. \]

Secant hyperbolic pulse with 5fs FWHM duration and a max. field amplitude of 0.12 au.

Figure 13.10-11: Simulated HHG spectra for hydrogen excited by Ti:sapphire pulses (800 nm, corresponding to \( \omega = 0.057 \text{au} \)).
Figure 13.11: Kinetic energy of long and short trajectories.
HHG from Multiple Cycle Fields

(a) Electric field

(b) Dipole radiation

(c) Ground state population vs. time (multiples of period)
HHG Experiment and Theory

(a) Experimental results for HHG driven by 400 nm, 1 mJ and 26 fs driver pulses with beam waist 30 μm: top row, efficiencies for Ar (50 mbar), Ne (300 mbar) and He (2 bar) using a 2 mm long nozzle; remaining rows, the respective normalized HHG spectra. b) Simulation results for Ar (2.5×10^{14} W/cm^2), Ne (5.3×10^{14} W/cm^2) and He (8.5×10^{14} W/cm^2) for the same interaction parameters like in (a). c) Experimental results for HHG driven by 800 nm, 35 fs driver pulses with beam waist 40 μm: top row, efficiencies for Ar (50 mbar, 0.6 mJ), Ne (300 mbar, 2 mJ) and He (2 bar, 2 mJ) using a 2 mm long nozzle; remaining rows, the respective normalized HHG spectra. d) Simulation results for Ar (1.2×10^{14} W/cm^2), Ne (3.2×10^{14} W/cm^2) and He (7.4×10^{14} W/cm^2) for the same interaction parameters like in (c).
13.3 Attosecond Pulses

Figure 13.11: (a) Amplitude of radiated HHG field for the same parameters as Fig. 13.9. Note the chirp.

Figure 13.12: Neighborhood of the most energetic trajectory, which is responsible for the highest frequency radiation emitted.
Figure 13.13: The same as Fig. 13.11a, before and after high-pass filtering.
Figure 13.14: Ionization of helium in the presence of a linearly polarized electric field of a laser pulse with 800nm wavelength and a peak intensity $4 \times 10^{15}$W/cm$^2$: (a) electric field; (b) fraction of ionized electrons; (a) instantaneous ionization rate. The thin and the thick lines represent pulses of durations of 50fs and 5fs FWHM, respectively.[1]

Other Difficulties:

Quantum Diffusion

Phase Matching