2021 Dec 13

NLO #18

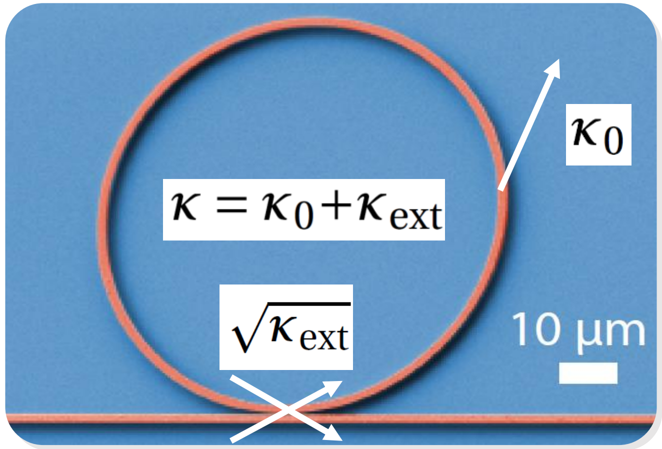
- **High-Q microresonators** (linear regime)
- **Nonlinear Physics in microresonators**
  - Nonlinear interactions and coupled mode equations
  - Light coupling and effective detuning
  - Three-mode system and parametric threshold (modulation instability MI)
  - Cascaded four-wave mixing (FWM)
  - Lugiato-Lefer-Equation and dissipative solitons
  - Some applications

Coupling light to a resonator

$$\tilde{A} = A(t)\,\mathrm{e}^{-i\omega_0 t}$$ ($|A|^2$ is number of photons in cavity)

($|s_{\mathrm{in}}|^2$ is number of photons per second)

$s_{\mathrm{in}}(t)$



Time evolution:

$$\frac{\mathrm{d}A(t)}{\mathrm{d}t} = -\frac{\kappa}{2}A(t) + \sqrt{\kappa_{\mathrm{ext}}}\,s_{\mathrm{in}}(t)\,\mathrm{e}^{-i(\omega_{\mathrm{p}}-\omega_0)t}$$

Quality-factor

$$\boxed{Q = \omega/\kappa = \omega\tau}$$

Transform into rotating frame of pump:

$$a = A\,e^{i(\omega_{\mathrm{p}}-\omega_0)t} \longrightarrow \frac{\mathrm{d}a(t)}{\mathrm{d}t} = -(i\underbrace{(\omega_0-\omega_{\mathrm{p}})}_{\text{Detuning}} + \frac{\kappa}{2})a(t) + \sqrt{\kappa_{\mathrm{ext}}}\,s_{\mathrm{in}}(t)$$

Assume continuous wave laser:

$$s_{\mathrm{in}}(t) = s_{\mathrm{in}} \xrightarrow[\text{Steady-state}]{} a = \frac{\sqrt{\kappa_{\mathrm{ext}}}}{i(\omega_0-\omega_{\mathrm{p}}) + (\frac{\kappa}{2})}\cdot s_{\mathrm{in}} \longrightarrow$$

Steady-state number of photons in cavity

$$\boxed{|a|^2 = \frac{\kappa_{\mathrm{ext}}}{(\omega_0-\omega_{\mathrm{p}})^2 + (\frac{\kappa}{2})^2}\cdot|s_{\mathrm{in}}|^2}$$

Coupling light to a resonator

from previous page

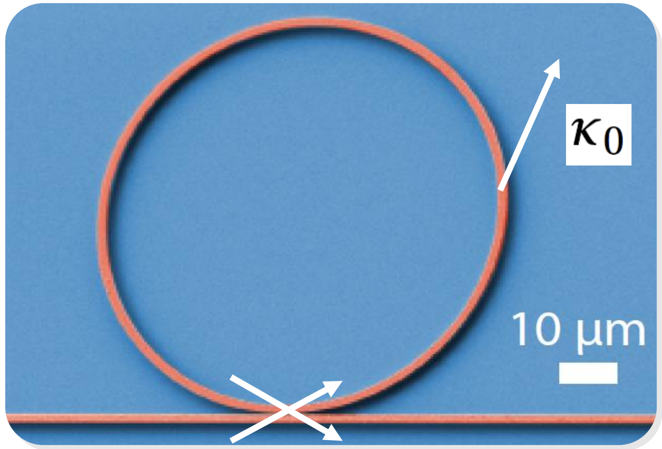
$$|a|^2 = \frac{\kappa_{\mathrm{ext}}}{(\omega_0 - \omega_{\mathrm{p}})^2 + (\frac{\kappa}{2})^2} \cdot |s_{\mathrm{in}}|^2$$

Finesse

$$F = \frac{\mathrm{FSR}}{\kappa/2\pi}$$

($|s_{\mathrm{in}}|^2$ is number of photons per second)

$s_{\mathrm{in}}(t)$



$$P_{\mathrm{cav}} = \mathrm{FSR} \cdot \frac{\kappa_{\mathrm{ext}}}{(\omega_0 - \omega_{\mathrm{p}})^2 + (\frac{\kappa}{2})^2} \cdot P_{\mathrm{in}} = 2\eta \cdot \frac{F}{\pi} \cdot \frac{1}{\left(\frac{4(\omega_0-\omega_{\mathrm{p}})^2}{\kappa^2}\right) + 1} \cdot P_{\mathrm{in}}$$

$$\boxed{P_{\mathrm{cav}} = \frac{F}{\pi} \cdot P_{\mathrm{in}}}$$

Power enhancement is proportional to Finesse

Coupling ratio

$$\boxed{\eta = \kappa_{\mathrm{ext}}/\kappa}$$

Coupling regimes

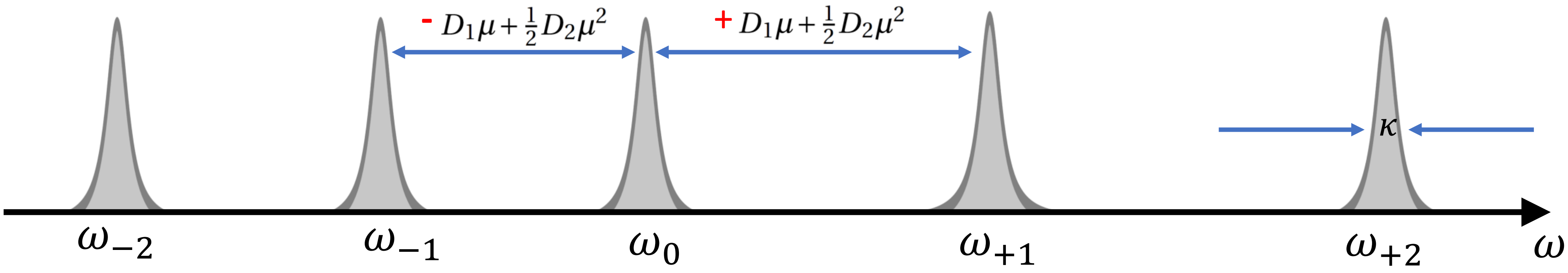
$\kappa_0 > \kappa_{\mathrm{ext}}$	undercoupled
$\kappa_0 = \kappa_{\mathrm{ext}}$	critically coupled
$\kappa_0 < \kappa_{\mathrm{ext}}$	overcoupled

Resonance spectrum and chromatic dispersion

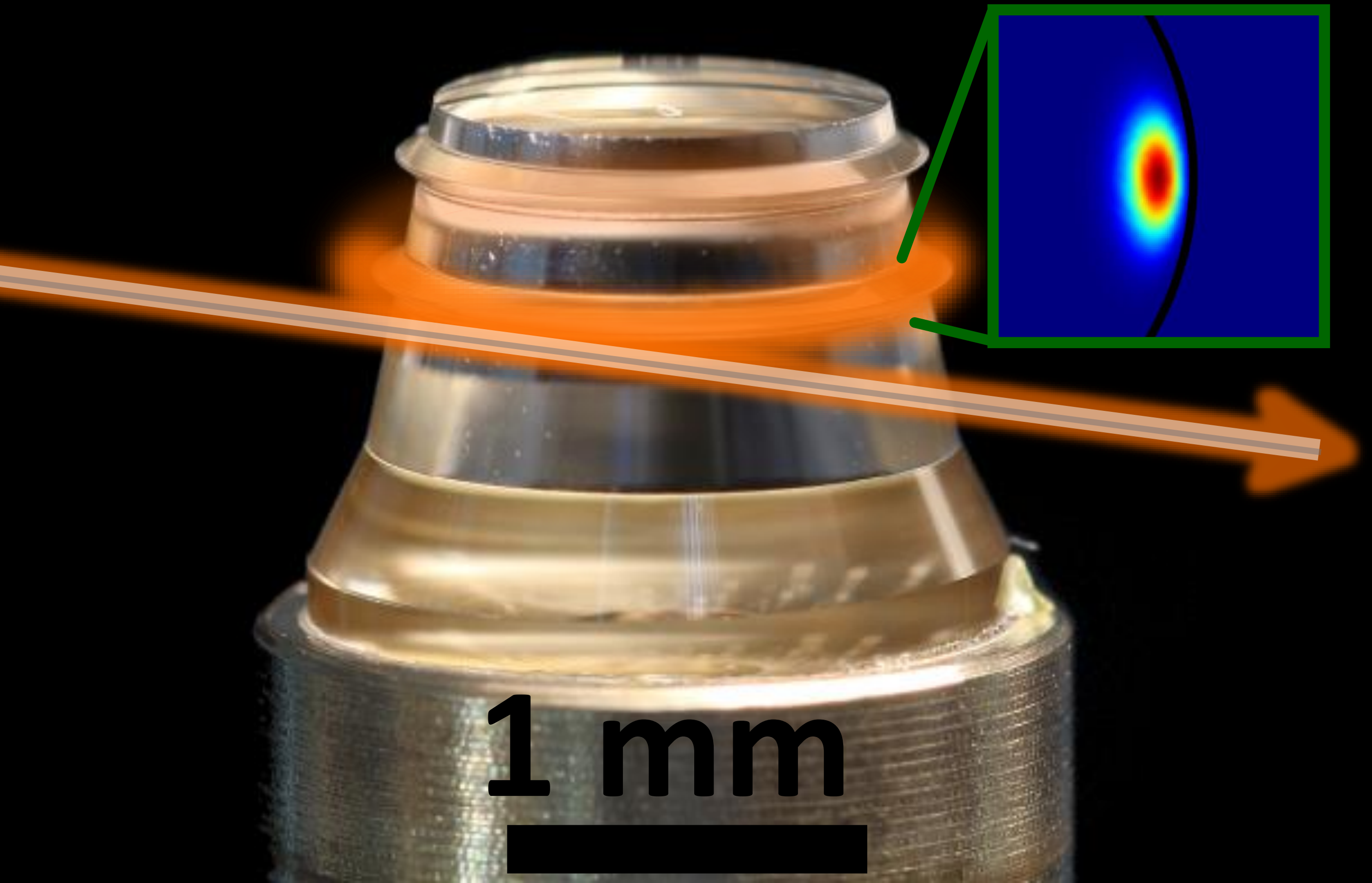
$$\omega_\mu = \omega_0 + D_1\mu + \underline{\tfrac{1}{2}D_2\mu^2}$$

Dispersion leads to non-equidistant resonance frequencies

$$D_1 = 2\pi\ \mathrm{FSR}(\mu) = 2\pi \frac{c}{n_g L} \qquad D_2 = 2\pi\ \mathrm{FSR}(\mu) = -\frac{c}{n_g} D_1^2 \beta_2 \qquad d_2 = D_2/\kappa$$

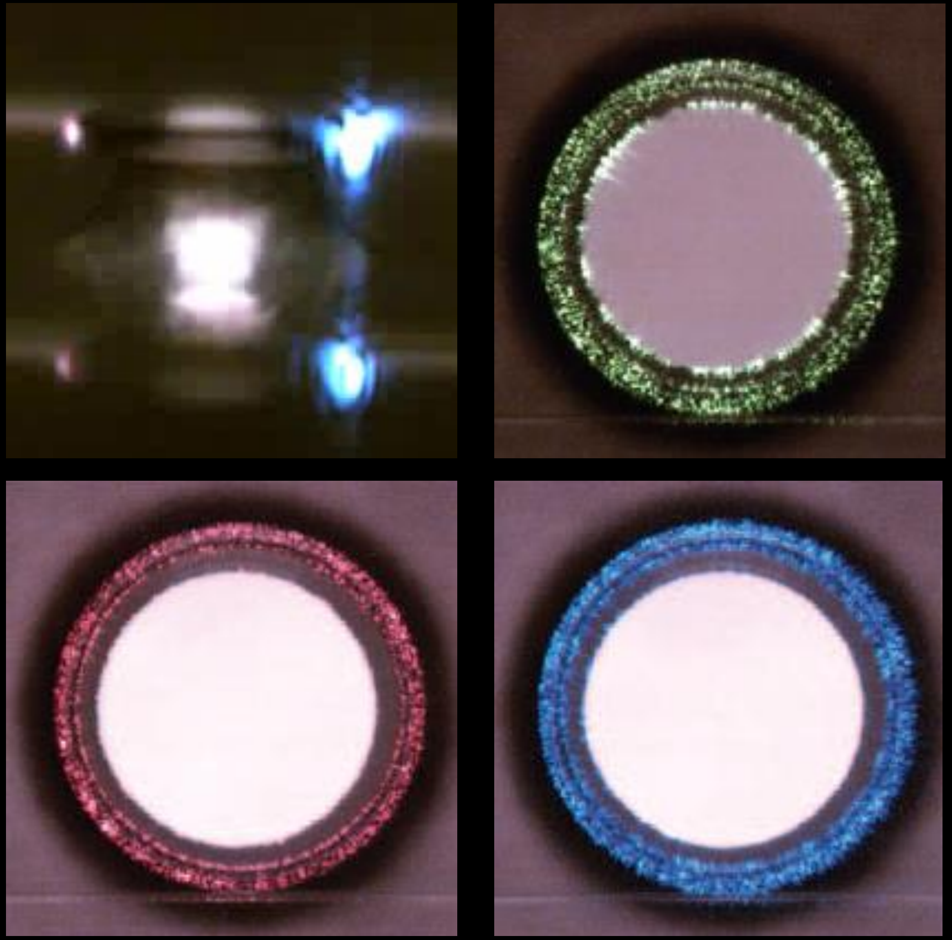


Nonlinear Microresonators



Polarization of the medium:

$$\tilde{P}(t) = \epsilon_0\left[\chi^{(1)}\tilde{E}(t) + \chi^{(2)}\tilde{E}^2(t) + \chi^{(3)}\tilde{E}^3(t) + \cdots\right]$$

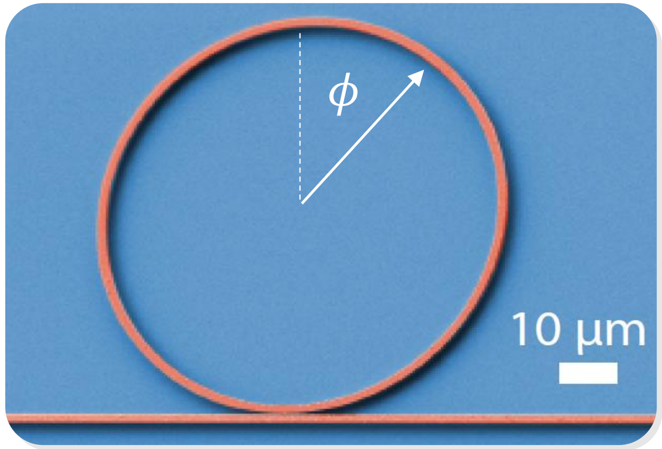


Nonlinear coupling in a resonator

Wave equation in radial coordinates:

$$\frac{1}{R^2}\frac{\partial^2}{\partial\phi^2}E - \frac{n^2}{c^2}\frac{\partial^2}{\partial t^2}E = \frac{1}{\varepsilon_0 c^2}\frac{\partial^2}{\partial t^2}P_{\mathrm{NL}}$$

$$n_\nu \approx n$$



$$E(t,\phi) = \sum_\nu \underbrace{E_\nu(t)\, e^{i(\nu\phi - \omega_\nu t)} + \mathrm{c.c.}}_{\text{Resonantor eigenmodes}}$$

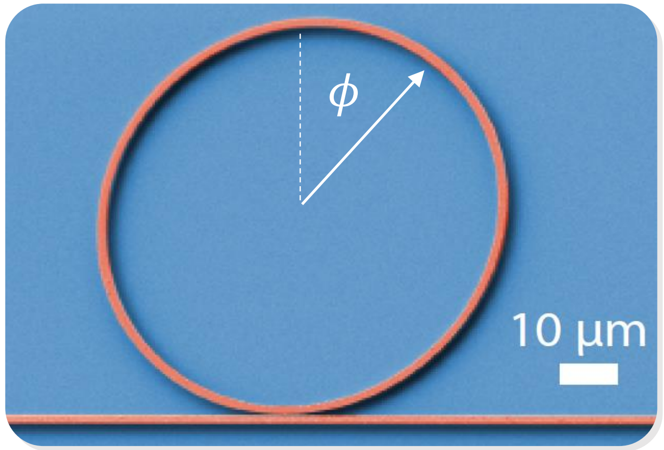
$$P_{\mathrm{NL}}(t,\phi) = \varepsilon_0 \chi^{(3)} E^3(t,\phi)$$

$$\frac{\nu^2}{R^2} - \frac{n^2}{c^2}\omega_\nu^2 = 0$$

$$-\frac{n^2}{c^2}\sum_\nu \left(\frac{\partial^2}{\partial t^2}E_\nu - 2i\omega_\nu \frac{\partial}{\partial t}E_\nu\right) e^{i(\nu\phi - \omega_\nu t)} + \mathrm{c.c.}$$

$$= \frac{3\chi^{(3)}}{c^2}\frac{\partial^2}{\partial t^2}\sum_{\mu',\mu'',\mu'''} E_{\mu'}E_{\mu''}E^*_{\mu'''}\, e^{i((\mu'+\mu''-\mu''')\phi - (\omega_{\mu'}+\omega_{\mu''}-\omega_{\mu'''})t)} + \mathrm{c.c.}$$

Nonlinear coupling in a resonator



From previous page:

SVEA

$$-\frac{n^2}{c^2}\sum_{\nu}\left(\cancel{\frac{\partial^2}{\partial t^2}E_\nu} - 2i\omega_\nu \frac{\partial}{\partial t}E_\nu\right) e^{i(\nu\phi-\omega_\nu t)} + \text{c.c.}$$

$$= \frac{3\chi^{(3)}}{c^2}\frac{\partial^2}{\partial t^2}\sum_{\mu',\mu'',\mu'''} E_{\mu'}E_{\mu''}E^*_{\mu'''}\, e^{i((\mu'+\mu''-\mu''')\phi-(\omega_{\mu'}+\omega_{\mu''}-\omega_{\mu'''})t)} + \text{c.c.}$$

Projection onto mode with index μ by multiplication with $e^{-i(\mu\phi-\omega_\mu t)}$ and integration over ϕ.

$$\frac{\partial}{\partial t}E_\mu = i\frac{3\chi^{(3)}\omega_\mu}{2n^2}\sum_{\mu',\mu'',\mu'''} E_{\mu'}E_{\mu''}E^*_{\mu'''}\, e^{-i(\omega_{\mu'}+\omega_{\mu''}-\omega_{\mu'''}-\omega_\mu)t}$$

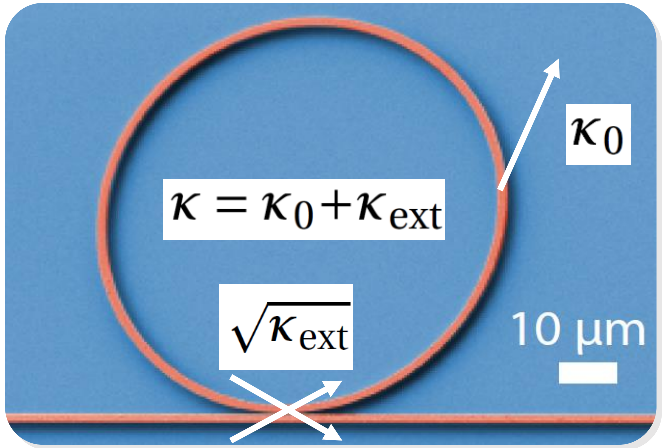
$$\boxed{\mu'+\mu''-\mu'''-\mu=0}$$

Normalizing again such that $|A_\mu|^2$ is number of photons in mode μ.

$$\boxed{\frac{\partial A_\mu}{\partial t} = ig\sum_{\mu',\mu'',\mu'''} A_{\mu'}A_{\mu''}A^*_{\mu'''}\,\mathrm{e}^{-i(\omega_{\mu'}+\omega_{\mu''}-\omega_{\mu'''}-\omega_\mu)}}$$

$$\boxed{g = \frac{\hbar\omega_0^2 c n_2}{n_0^2 V_{\text{eff}}}}$$

Full coupled mode equations



loss input pump detuning

$$\frac{\partial A_\mu}{\partial t} = \underbrace{-\frac{\kappa}{2} A_\mu}_{\text{loss}} + \underbrace{\delta_{\mu 0} \sqrt{\kappa_{ext}}\, s_{in}\, e^{\overbrace{-i(\omega_p - \omega_0)t}^{\text{pump detuning}}}}_{\text{input}}$$
$$\underbrace{+ ig \sum_{\mu', \mu'', \mu'''} A_{\mu'} A_{\mu''} A^*_{\mu'''} e^{-i(\omega_{\mu'} + \omega_{\mu''} - \omega_{\mu'''} - \omega_\mu)t}}_{\text{nonlinear coupling}}$$

nonlinear coupling

$$\mu' + \mu'' - \mu''' - \mu = 0$$

$$s_{in} = \sqrt{P_{in}/\hbar\omega_p}$$

$|A_\mu|^2$ is number of photons in mode μ

$$g = \frac{\hbar \omega_0^2 c n_2}{n_0^2 V_{eff}}$$

- Coupled eigenmode equations
- One equation for each eigenmode of the resonator

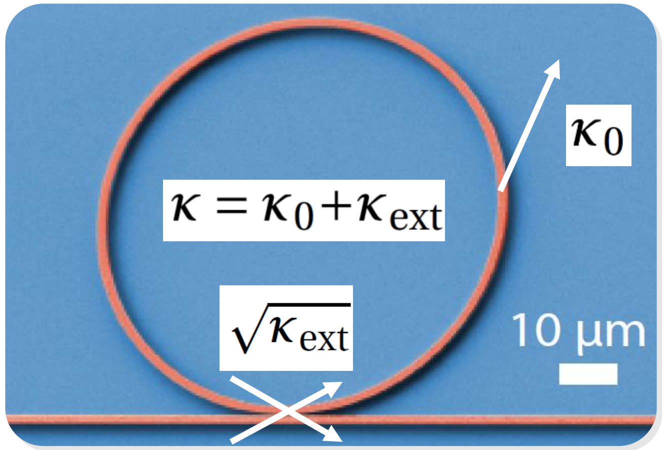
Full coupled mode equations

From previous page:

$$\frac{\partial A_\mu}{\partial t} = -\frac{\kappa}{2} A_\mu + \delta_{\mu 0}\sqrt{\kappa_{\text{ext}}}\, s_{\text{in}}\, e^{-i(\omega_p-\omega_0)t} + ig \sum_{\mu',\mu'',\mu'''} A_{\mu'} A_{\mu''} A^*_{\mu'''} \underline{e^{-i(\omega_{\mu'}+\omega_{\mu''}-\omega_{\mu'''}-\omega_\mu)t}}$$

Oscillating term for non-zero dispersion

$$\mu' + \mu'' - \mu''' - \mu = 0$$



Instead of using resonator eigenmodes to describe system one can use an equidistant frequency grid with frequencies $\omega_p + \mu D_1$:

$$a_\mu = A_\mu \sqrt{2g/\kappa}\, e^{-i(\omega_\mu - \omega_p - \mu D_1)t}$$

$$\boxed{\frac{\partial}{\partial \tau} a_\mu = -(1 + i\zeta_\mu)\, a_\mu + i \sum_{\mu',\mu'',\mu'''} a_{\mu'} a_{\mu''} a^*_{\mu'''} + \delta_{0\mu} f}$$

$$\mu' + \mu'' - \mu''' - \mu = 0$$

$$\tau = \kappa t/2$$

$$\zeta_\mu = 2(\omega_\mu - \omega_p - \mu D_1)/\kappa$$

$$f = \sqrt{8\eta g/\kappa^2}\, s_{\text{in}}$$

Coupling light to a nonlinear resonator

From previous page (consider only mode $\mu = 0$):

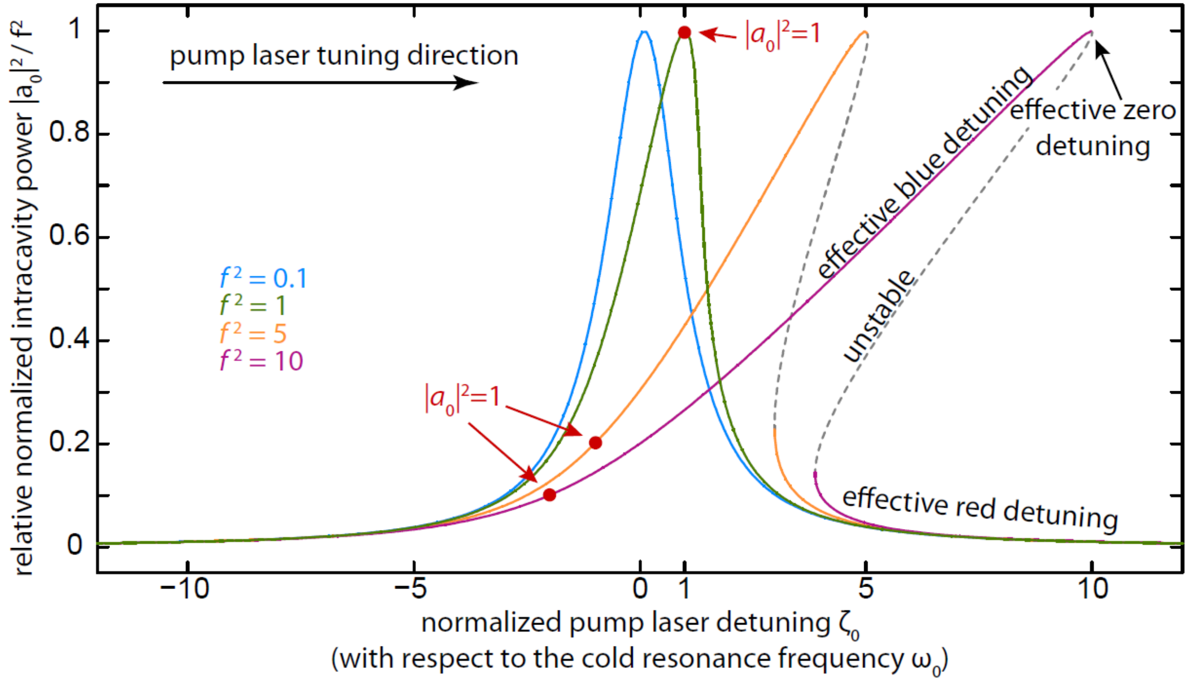
$$\frac{\partial a_0}{\partial \tau} = -[1 + i\zeta_0 - i|a_0|^2]a_0 + f$$

Steady state:

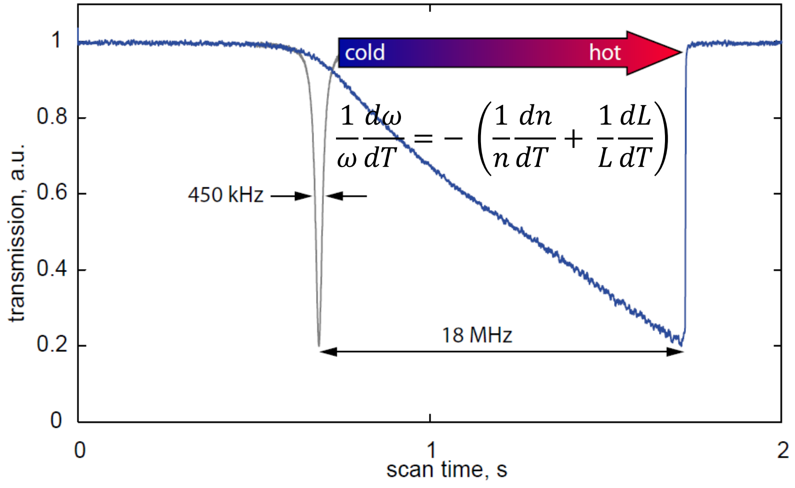
$$\boxed{(\zeta_0 - |a_0|^2)^2 |a_0|^2 + |a_0|^2 = f^2}$$

Effective nonlinear detuning

$$\boxed{\zeta_0^{\mathrm{eff}} = \zeta_0 - |a_0|^2 = \frac{2}{\kappa}\left((\omega_0 - g|A_0|^2) - \omega_p\right)}$$

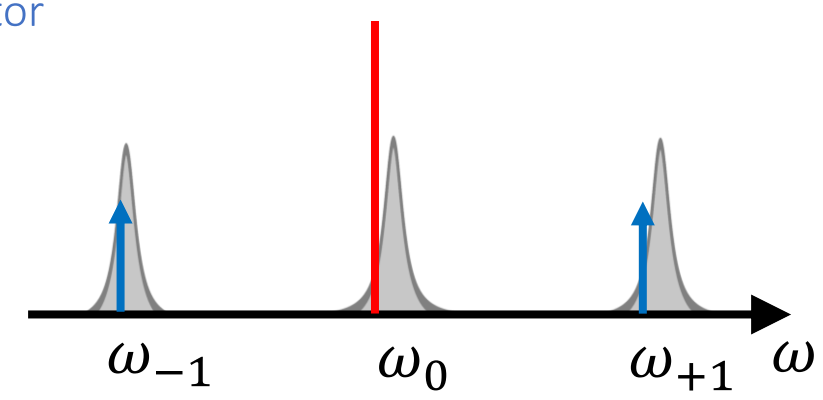


Similar to thermal effect



Three-mode system Modulation instability (MI) in a resonator

Derivation similar to Modulation instability in non-resonant systems



$$\frac{\partial a_0}{\partial \tau} = -[1 + i\zeta_0 - i|a_0|^2]a_0 + f \longrightarrow \boxed{(\zeta_0 - |a_0|^2)^2|a_0|^2 + |a_0|^2 = f^2}$$

$$\frac{\partial a_{+\mu}}{\partial \tau} = -[1 + i\zeta_\mu - 2i|a_0|^2]a_{+\mu} + i a_0^2 a_{-\mu}^*,$$

$$\frac{\partial a_{-\mu}^*}{\partial \tau} = -[1 - i\zeta_\mu + 2i|a_0|^2]a_{-\mu}^* - i a_0^{*2} a_{+\mu}.$$

$$\longrightarrow \begin{pmatrix} \partial a_{+\mu}/\partial \tau \\ \partial a_{-\mu}^*/\partial \tau \end{pmatrix} = \begin{pmatrix} -[1 + i\zeta_\mu - 2i|a_0|^2] & i a_0^2 \\ -i a_0^{*2} & -[1 - i\zeta_\mu + 2i|a_0|^2] \end{pmatrix} \cdot \begin{pmatrix} a_{+\mu} \\ a_{-\mu}^* \end{pmatrix}$$

Eigenvalues

$$\lambda = -1 \pm \sqrt{|a_0|^4 - (\zeta_\mu - 2|a_0|^2)^2}$$

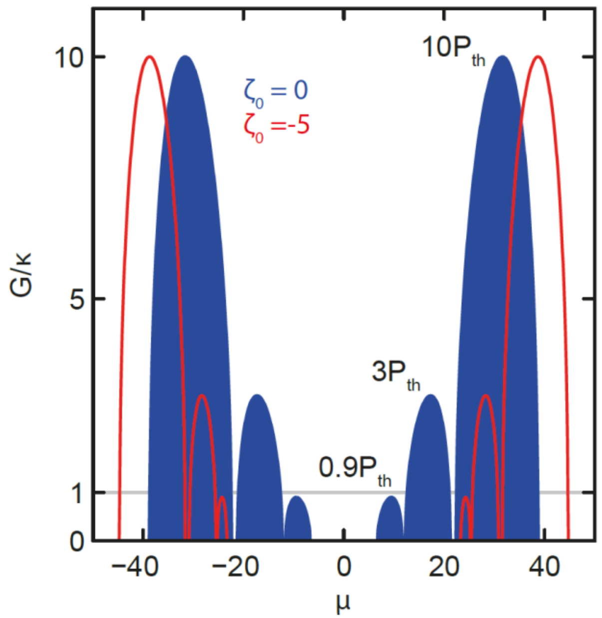
Parametric gain: $G = Re[1 + \lambda]\ \kappa$

$$G = \sqrt{\kappa^2|a_0|^4 - 4(\omega_0 - \omega_p + \mu^2 D_2 - \kappa|a_0|^2)^2}$$

Threshold: $G > \kappa$ (i.e. gain compensates loss)

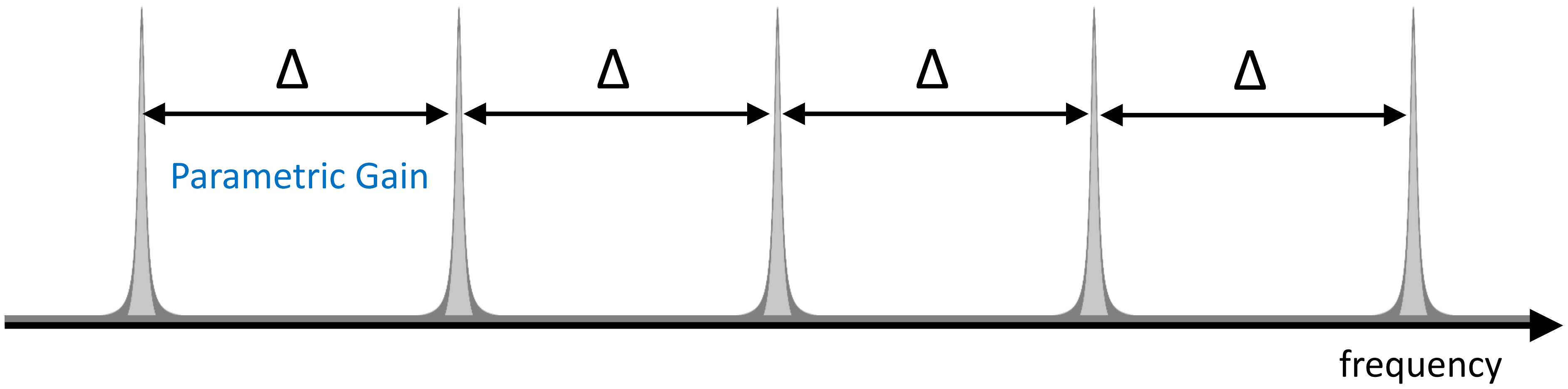
$$\boxed{P_{\mathrm{in}}^{\mathrm{th}} = \frac{\kappa^2 n_0^2 V_{\mathrm{eff}}}{8\eta\omega_0 c n_2}}$$

Required $D_2 > 0$ (anomalous GVD)



Cascaded FWM in a system with large FSR

Optical spectrum:



1. Degenerate FWM / MI

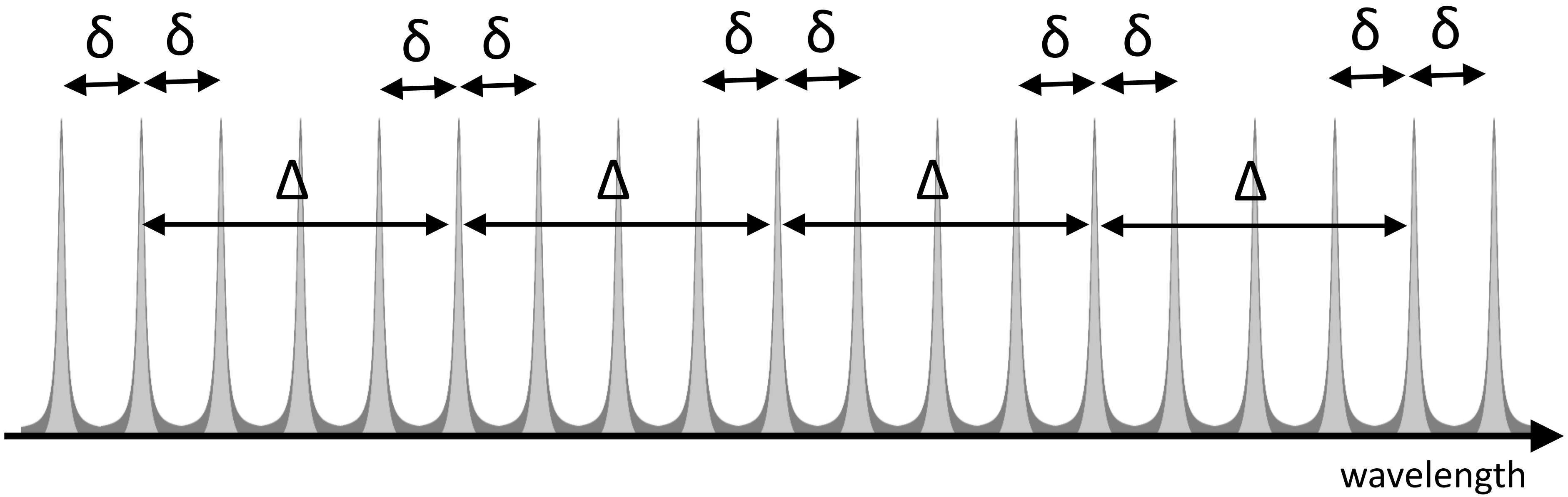
requires anomalous group velocity dispersion

2. Cascaded Non-Degenerate FWM

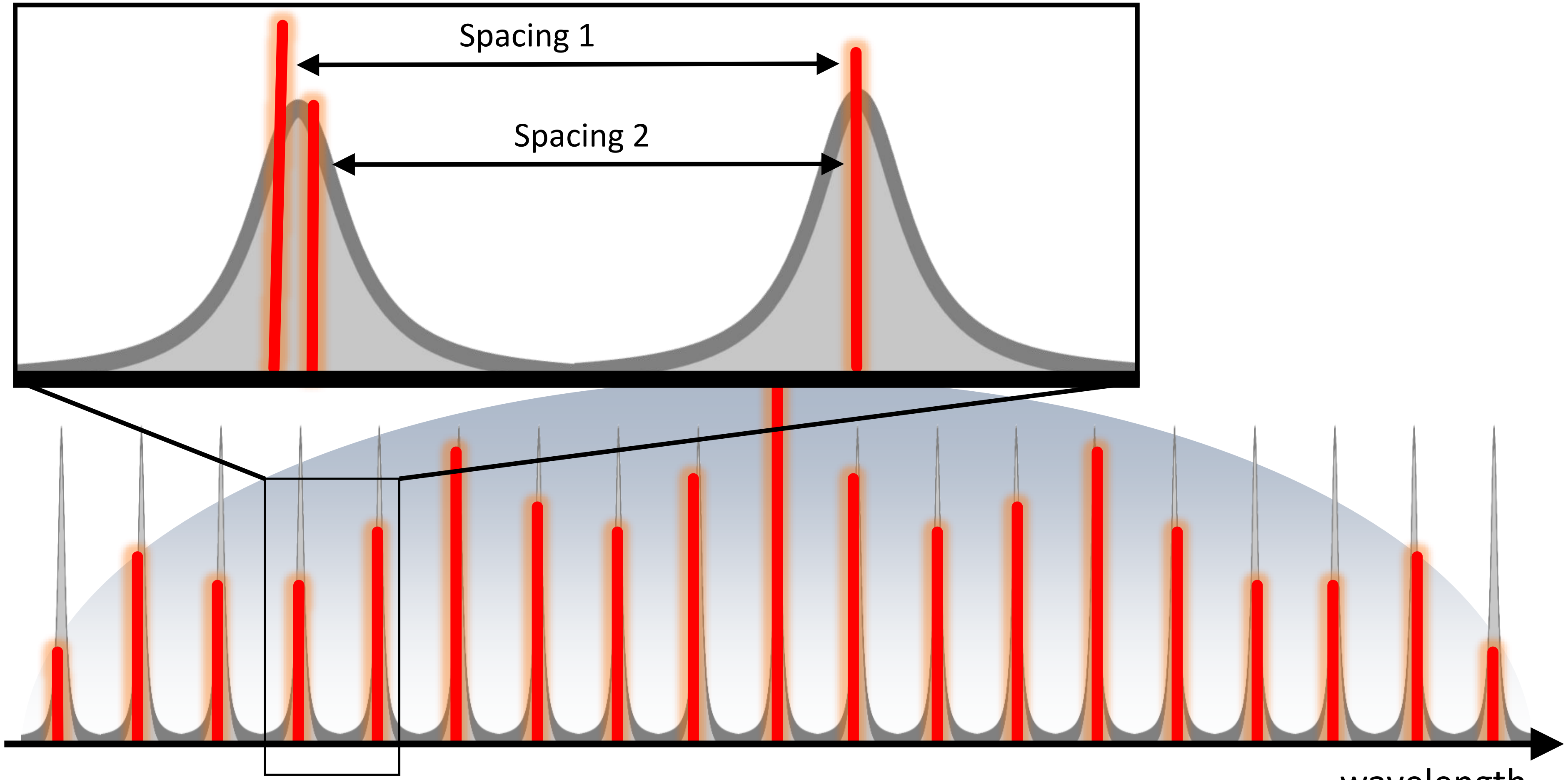
Cascaded FWM in a system with small FSR

$$\boxed{\Delta \neq n\cdot\delta \quad (n=1,2,..)}$$

Primary spacing Δ
Secondary spacing δ



Comb generation via cascaded FWM in a system with small FSR



Need to carefully tune system such that Δ = n·δ

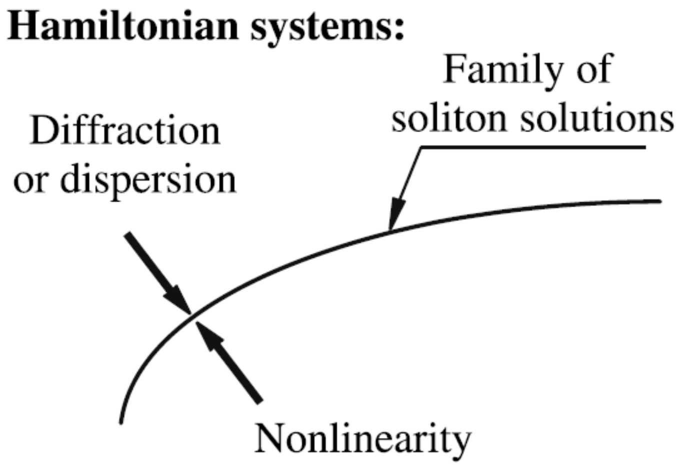
Lugiato-Lefever-equation and soliton solution

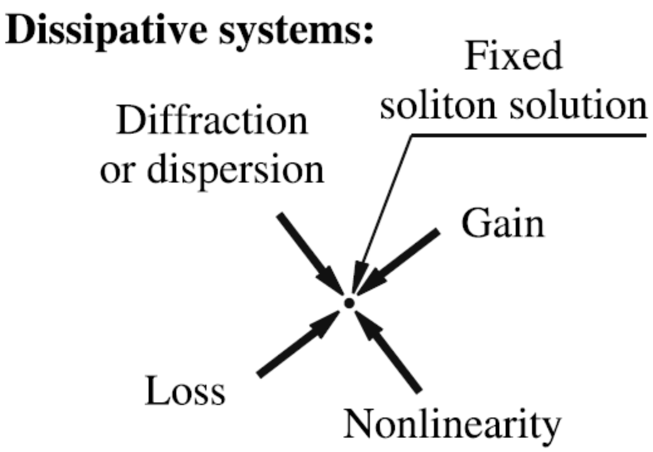
The coupled mode equations can be transformed into the time domain
(similar to what was done in the context of the NLSE)

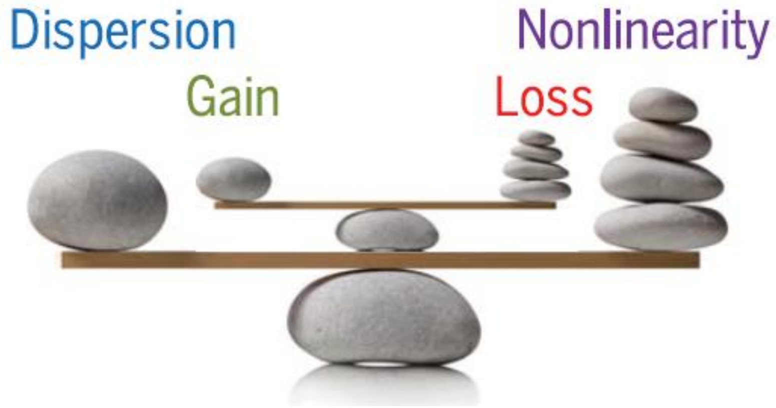
$$\frac{\partial A}{\partial t} - i\frac{1}{2}D_2\frac{\partial^2 A}{\partial \phi^2} - ig|A|^2 A = -\left(\frac{\kappa}{2} + i(\omega_0 - \omega_p)\right)A + \sqrt{\frac{\kappa\eta P_{\mathrm{in}}}{\hbar\omega_0}}$$

- This equation in similar to the NLSE, but includes detuning, loss and external driving
- There is a "solitonic" solution with sech-shape envelope existing on top of a continuous wave background:

→ **Dissipative solitons**







Dissipative temporal cavity soliton properties

Frequency domain field:

$$\boxed{\Psi(\omega-\omega_p) = \sqrt{d_2/2}\,\mathrm{sech}((\omega-\omega_p)/\Delta\omega)}$$

$$\Delta\omega = \frac{2D_1}{\pi}\sqrt{\frac{\zeta_0}{d_2}}$$

$$d_2 = D_2/\kappa$$

Time domain:

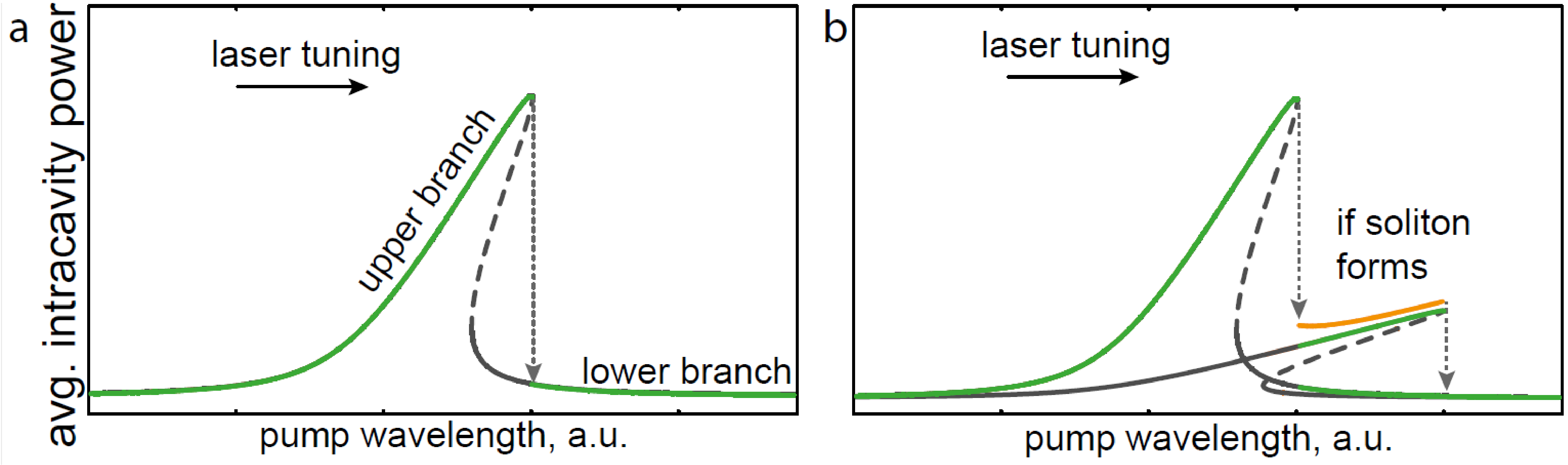
$$\boxed{\Psi(t) = \sqrt{2\zeta_0}\,\mathrm{sech}(t/\Delta t)}$$

$$\Delta t = \frac{1}{D_1}\sqrt{\frac{d_2}{\zeta_0}}$$

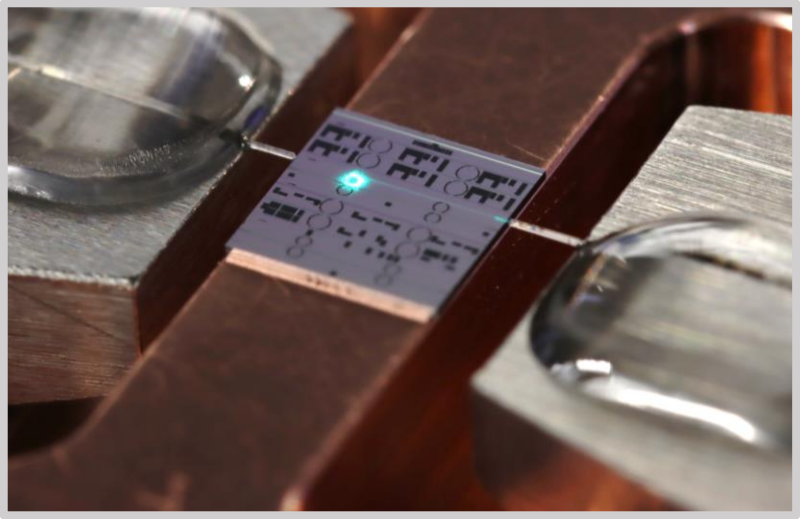
$$\Delta t_{\mathrm{min}} = \frac{2}{\sqrt{\pi}}\sqrt{\frac{-\beta_2}{\gamma F P_{\mathrm{in}}}}$$

After finding the maximal detuning

Solitons and nonlinear resonance shape

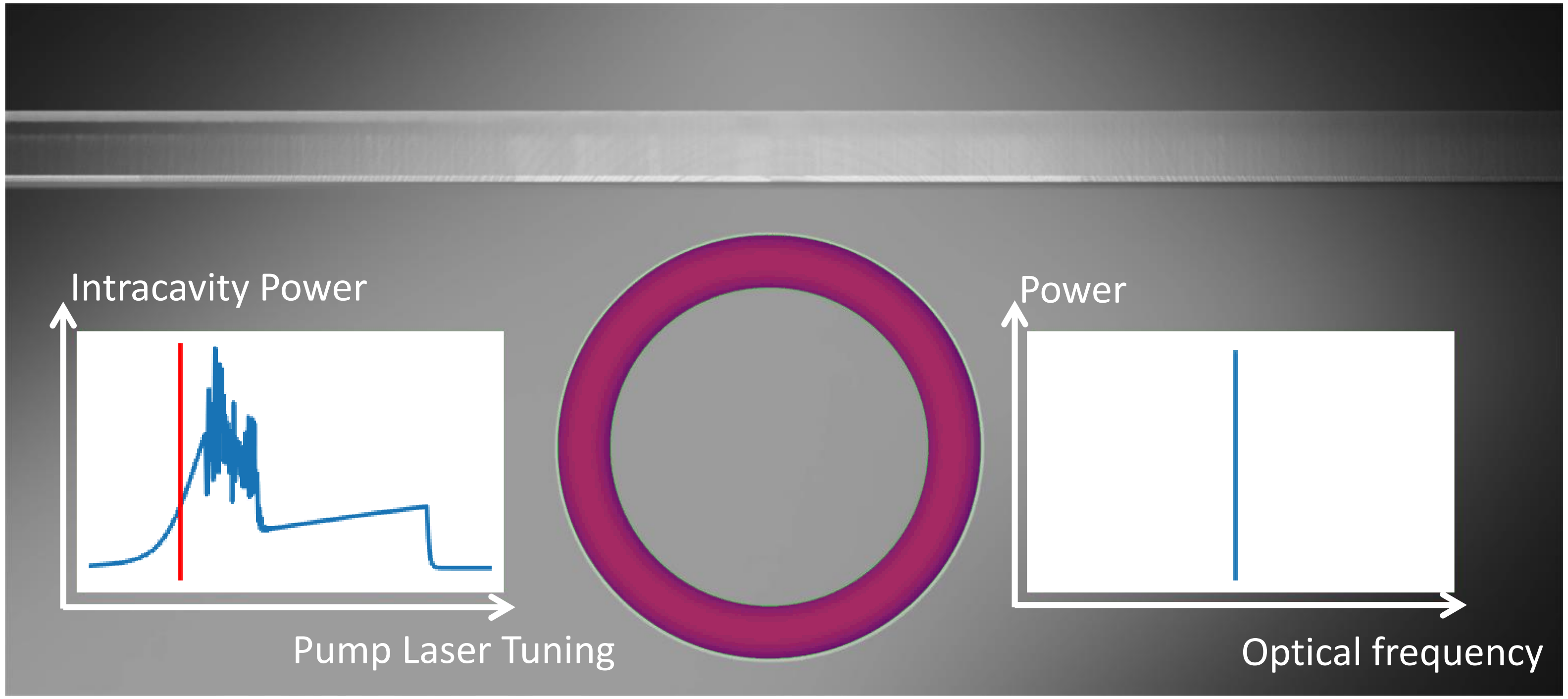


Simulation of nonlinear dynamics

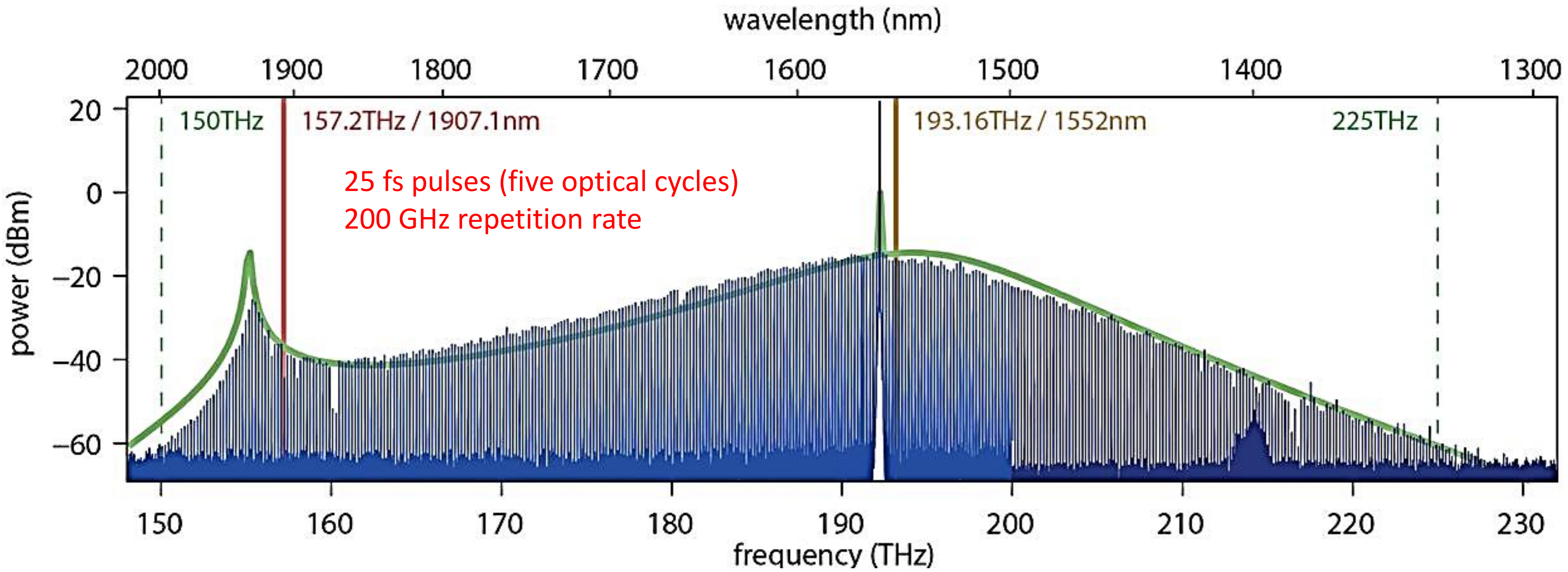


Based on coupled mode equations

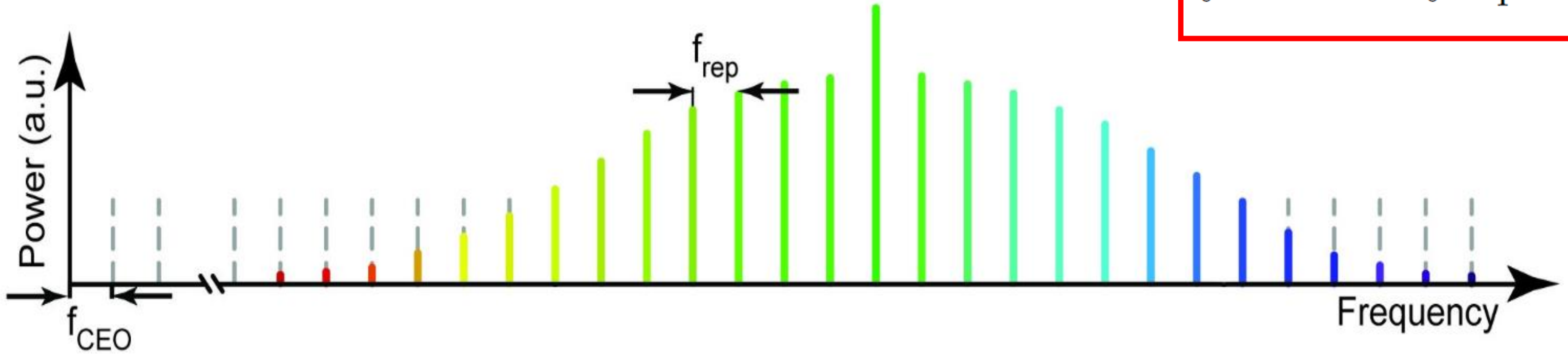
$$\frac{\partial}{\partial \tau} a_\mu = -\left(1 + i\zeta_\mu\right) a_\mu + i \sum_{\mu', \mu'', \mu'''} a_{\mu'} a_{\mu''} a^*_{\mu'''} + \delta_{0\mu} f$$



Soliton frequency combs



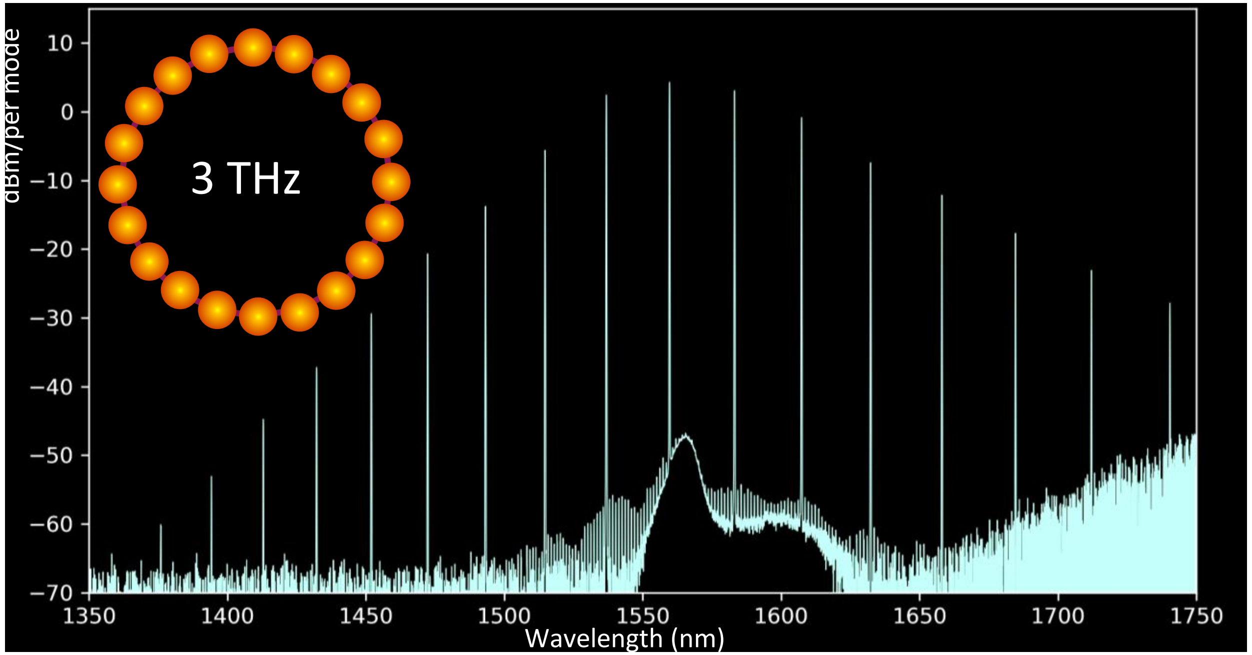
Frequency combs:

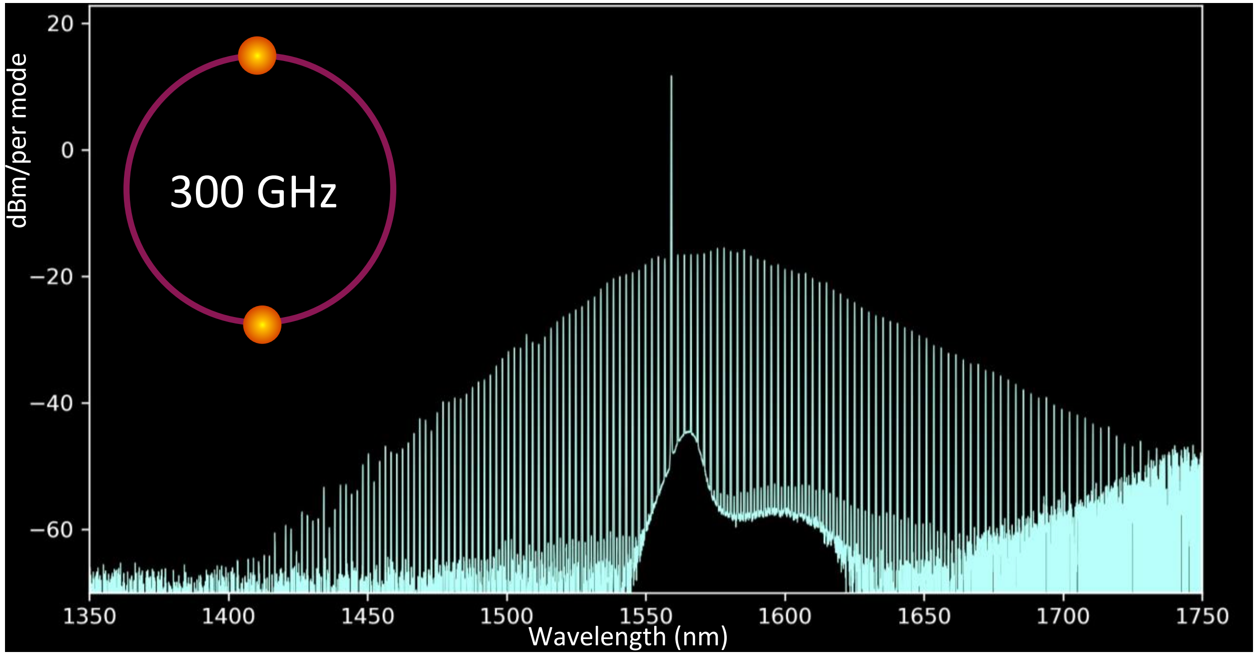


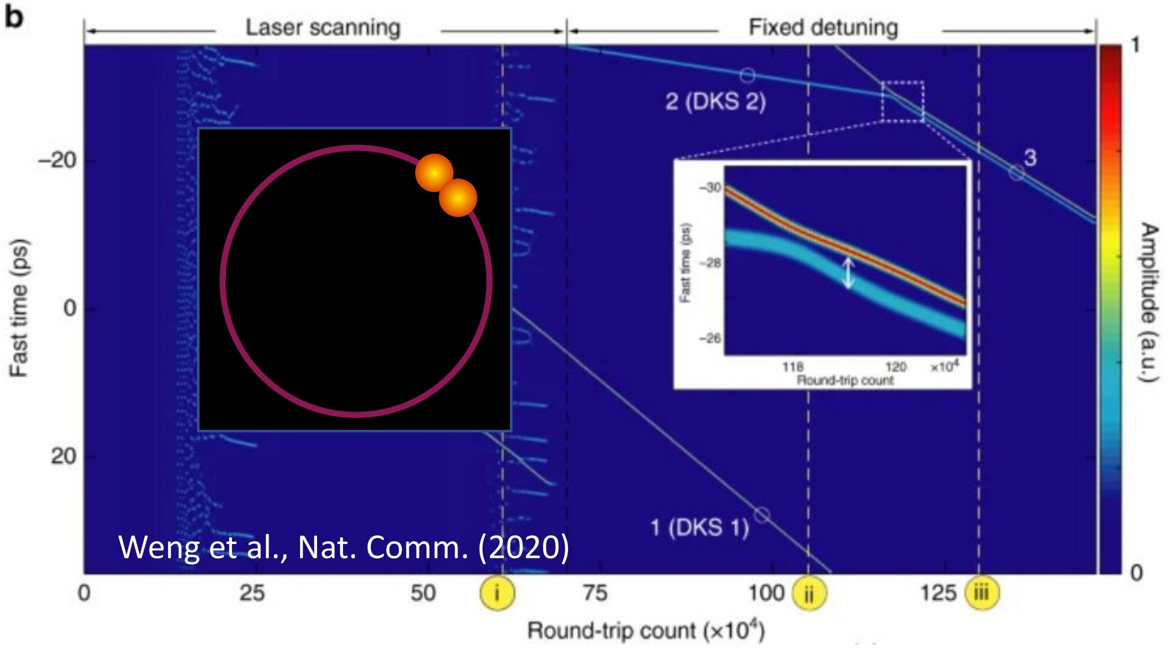
n^{th} Comb Line at

$$\boxed{f_n = n \cdot f_{\mathrm{rep}} + f_{\mathrm{CEO}}}$$

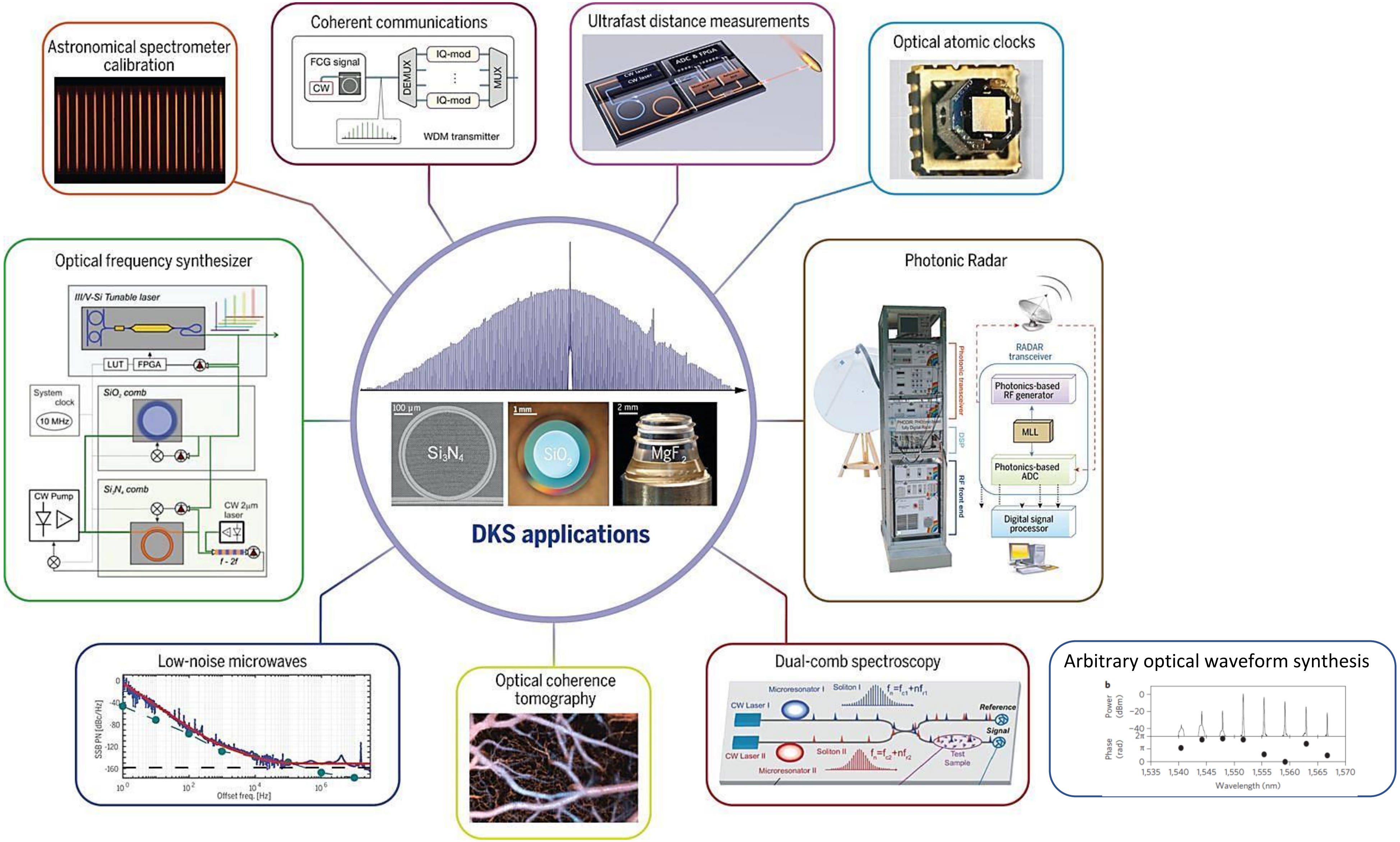
Soliton Crystals
& Soliton Molecules







Microresonator Solitons Applications



Review: Kippenberg, et al. Science (2018)