2021 Nov 17

NLO #12

- Nonlinear Schrödinger Equation
  - Examples
  - Adding more physics
- Chi3 resulting from molecular orientation
- Non-instantaneous nonlinearities

https://www.youtube.com/watch?v=hfc3IL9gAts

Nonlinear Schrödinger equation and solitons

Nonlinear Schrödinger Equation (NLSE)

$$\boxed{\frac{\partial}{\partial z}A'(z,\tau) + \tfrac{1}{2} i k_2 \frac{\partial^2}{\partial \tau^2} A'(z,\tau) = i\gamma |A'(z,\tau)|^2 A'(z,\tau)}$$

$k_2 > 0$: normal group velocity dispersion
$k_2 < 0$: anomalous group velocity dispersion

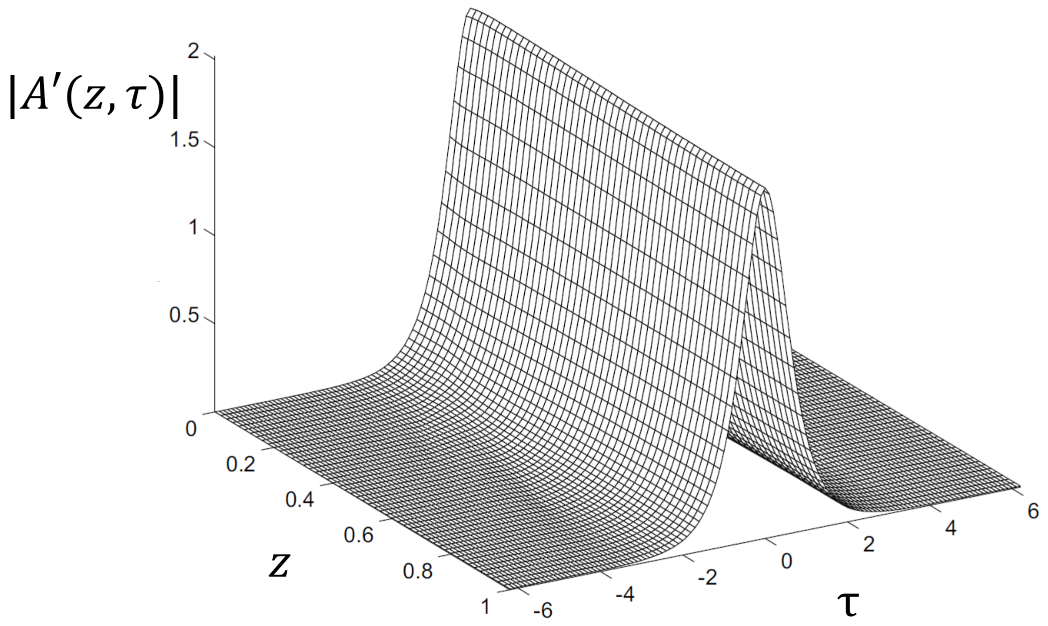
$$k(\omega) = k_0 + k_1(\omega - \omega_0) + \tfrac{1}{2}k_2(\omega-\omega_0)^2 + \tfrac{1}{6}k_3(\omega-\omega_0)^3 + \ldots$$

Solitons are solution to the NLSE:

$$A'(z,\tau) = A'_0 \mathrm{sech}(\tau/\tau_0)\mathrm{e}^{i\kappa z}$$

$$|A'_0|^2 = \frac{-k_2}{\gamma \tau_0^2}$$

$$\kappa = \frac{-k_2}{2\tau_0}$$

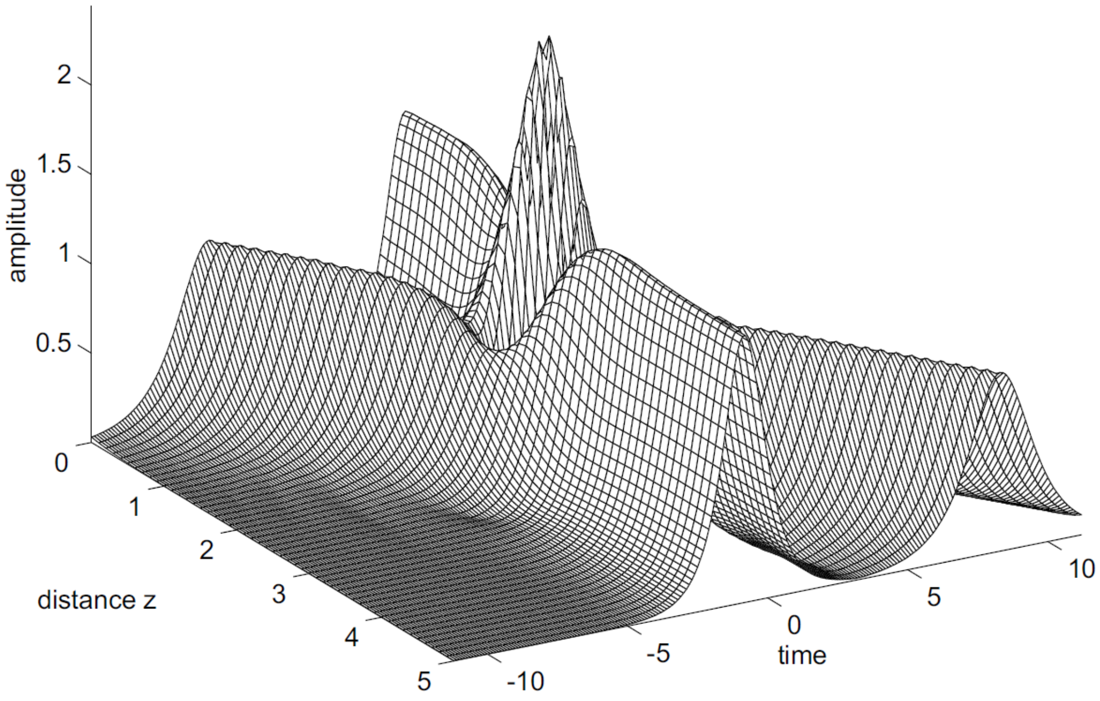


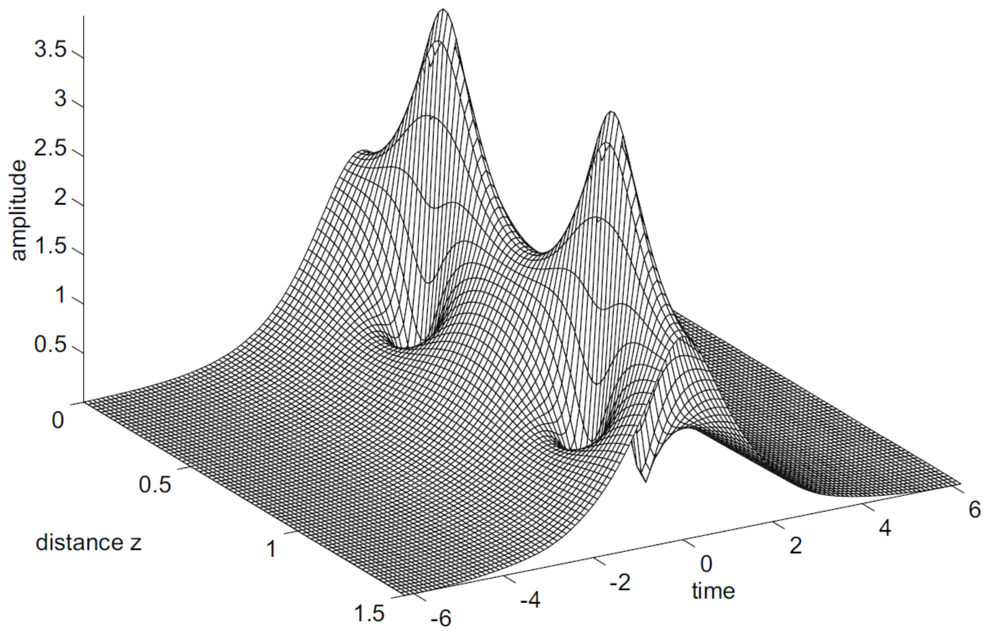
$$w = \int |A'(z,\tau)|^2 \mathrm{d}\tau = 2\tau_0 |A'_0|^2$$

$$\text{area} = \int |A'(z,\tau)| \mathrm{d}\tau = \pi A'_0 \tau_0 = \pi\sqrt{\frac{|k_2|}{\gamma}}$$

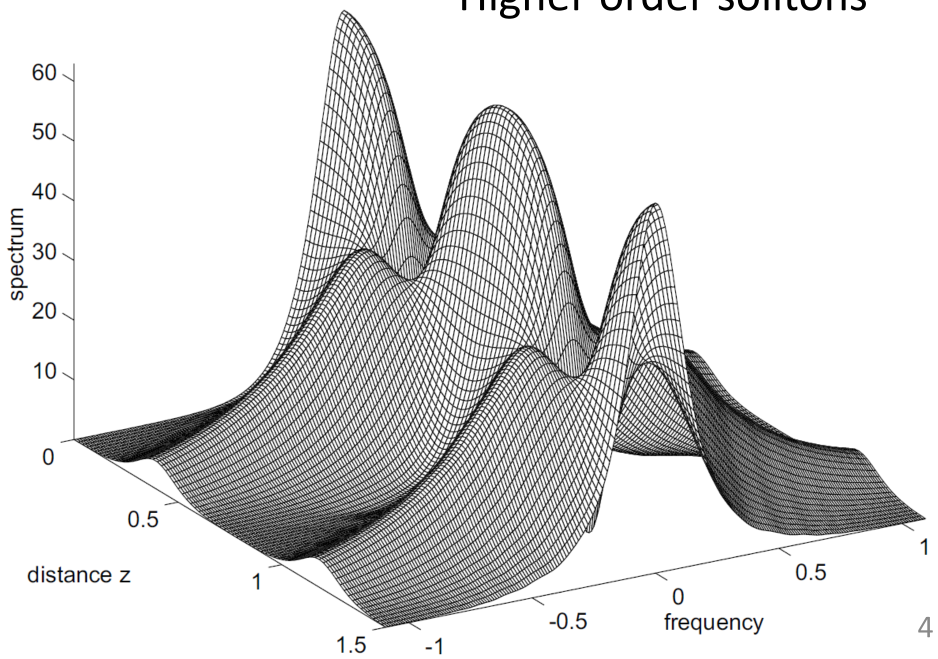
Soliton phenomena

Collision





Higher order solitons



Numeric solution of the NLSE (e.g. in Python)

This code is neither elegant nor efficient but hopefully easy to read

```
import numpy as np
import scipy.integrate as scint
```

```
# discretized tau axis
tau = np.linspace(-tau_span/2, tau_span/2, ntau)
```

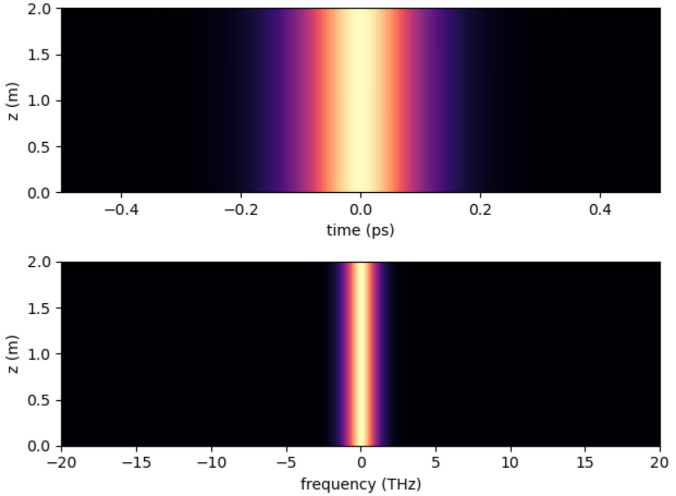
```
# soliton
a0 =np.sqrt(-k2/gamma/tau0**2)
a_init = a0/np.cosh(tau/tau0)
```

```
for z_idx, z in enumerate (z_save):
    solver.integrate(z)
    a_save[z_idx] = solver.y
```

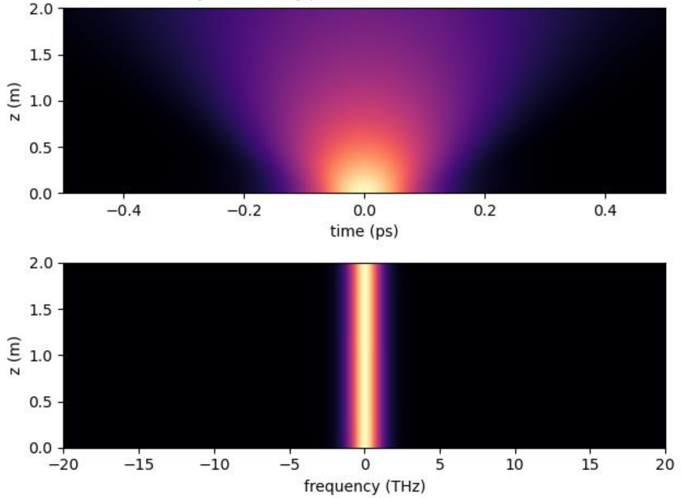
```
def NLSE(z, a):

    # This function return the RHS ofthe NLSE according to:
    # d/dz a = -i/2 k_2 (d/dtau)^2 a + i gamma |a|^2 a
    # d/dz does not depend on z, so the variable z is not used here
    # (but z is required for formal reasons to interface with the solver )

    # linear evolution (dispersion)
    da_dtau = np.gradient(a, dtau)
    d2a_dtau2 = np.gradient(da_dtau, dtau)
    da_dz_L = -1j*0.5*k2*d2a_dtau2

    # nonlinear evolution
    da_dz_NL = 1j*gamma*a*np.conj(a)*a

    # combined linear and nonlinear evolution
    da_dz = da_dz_L + da_dz_NL

    return da_dz


solver = scint.complex_ode(NLSE)
solver.set_integrator('dopri5', rtol = 1e-19)
solver.set_initial_value(a_init)
```

Simulation of soliton dynamics

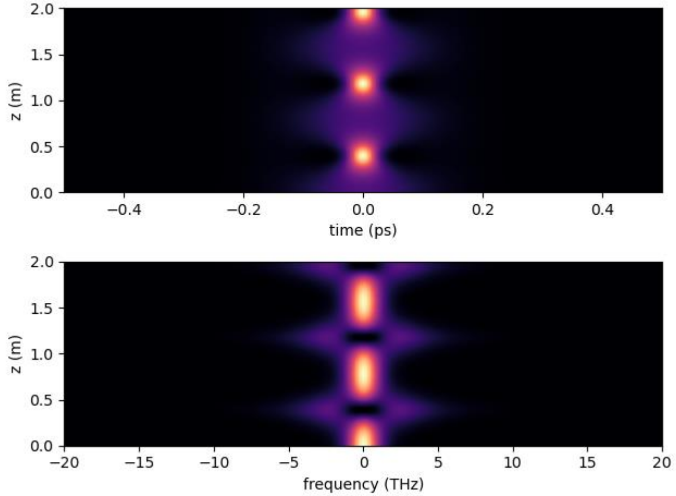
Fundamental soliton (soliton order = 1)



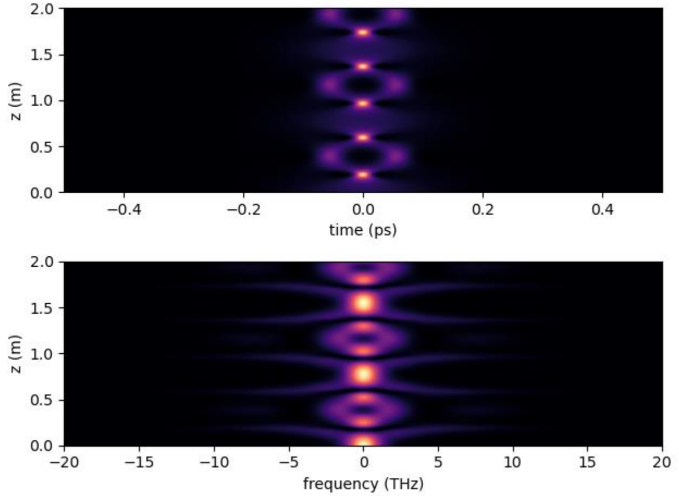
Not enough energy



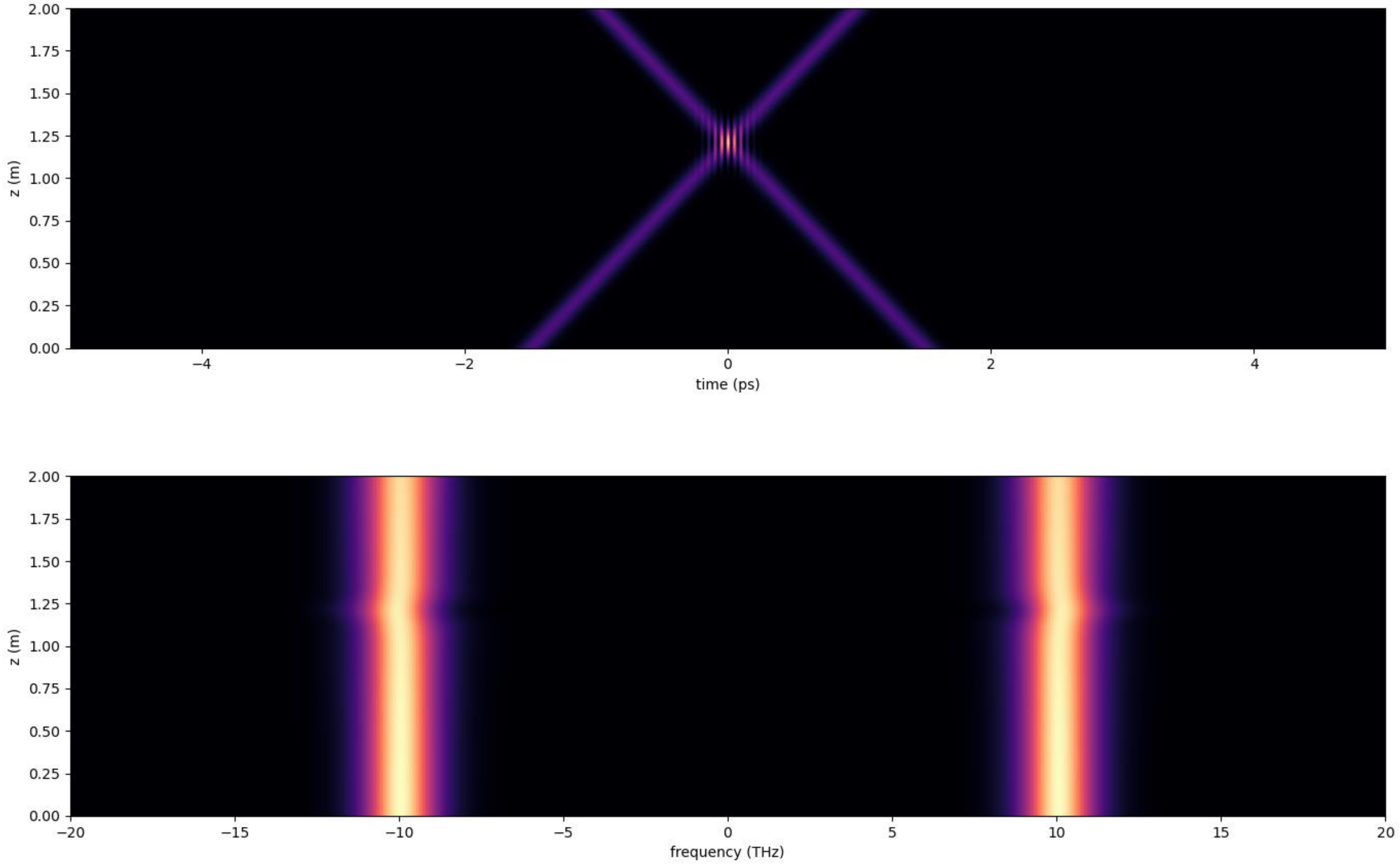
Soliton order = 2



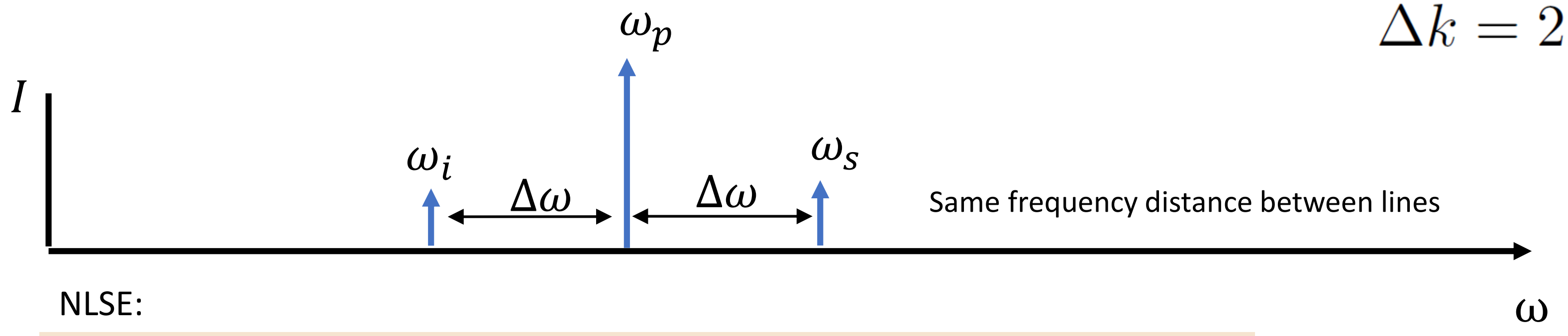
Soliton order = 3



Soliton Collision



Modulation instability (can we do things other then solitons with the NLSE?)



$$\Delta k = 2k_p - k_s - k_i = -2\gamma P - k_2 \Delta\omega^2$$

$\Delta k = 0$:

$$\boxed{\Delta\omega = \pm\sqrt{\frac{-2\gamma P}{k_2}}}$$

NLSE:

$$\frac{\partial}{\partial z} A(z,\tau) + \tfrac{1}{2} i k_2 \frac{\partial^2}{\partial \tau^2} A(z,\tau) = i\gamma |A(z,\tau)|^2 A(z,\tau)$$

Ansatz for a small perturbation:

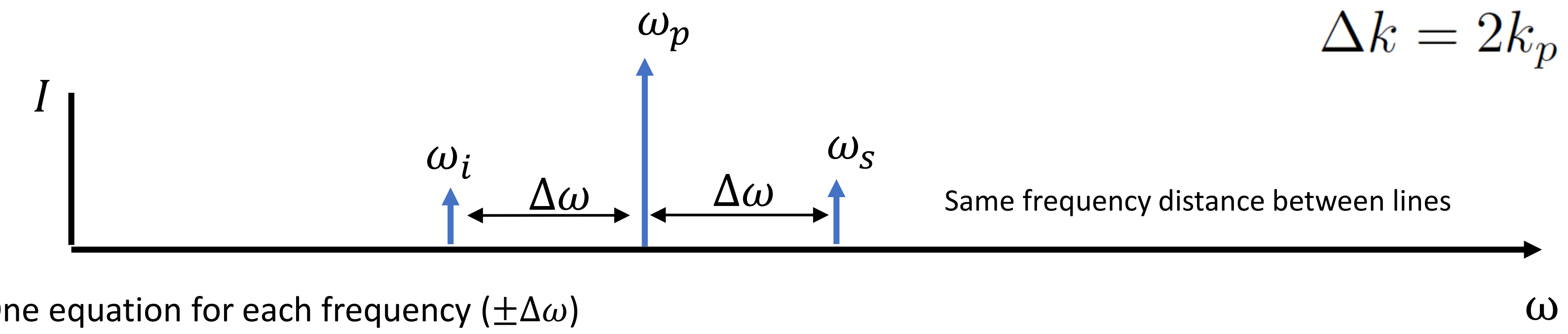
$$A(z,\tau) = a_p(z) + \underbrace{a_s^0(z)\, \mathrm{e}^{i\Delta\omega\tau}}_{a_s(z)} + \underbrace{a_i^0(z)\, \mathrm{e}^{-i\Delta\omega\tau}}_{a_i(z)}$$

Here no factor 2. As we took out the pump field a_p, this term does not represent XPM but the difference XPM and SPM. Formally this is done by applying to each field a phase factor $\exp(i\gamma|a_p|^2 z)$.

Only keep terms linear in the weak field:

$$\frac{\partial}{\partial z} a_s + \frac{\partial}{\partial z} a_i - \tfrac{1}{2} i k_2 \Delta\omega^2 (a_s + a_i) = i\gamma |a_p|^2 (a_s + a_i) + i\gamma a_p^2 (a_s^* + a_i^*)$$

Modulation instability (can we do things other then solitons with the NLSE?)



$$\Delta k = 2k_p - k_s - k_i = -2\gamma|a_p|^2 - k_2\Delta\omega^2$$

$\Delta k = 0$:

$$\boxed{\Delta\omega = \pm\sqrt{\frac{-2\gamma|a_p|^2}{k_2}}}$$

One equation for each frequency ($\pm\Delta\omega$)

$$\frac{\partial}{\partial z}a_s - \tfrac{1}{2}ik_2\Delta\omega^2 a_s = i\gamma|a_p|^2 a_s + i\gamma a_p^2 a_i^*$$

$$\frac{\partial}{\partial z}a_i^* + \tfrac{1}{2}ik_2\Delta\omega^2 a_i^* = -i\gamma|a_p|^2 a_i^* - i\gamma a_p^{*2} a_s$$

Same thing in matrix form:

$$\begin{pmatrix} \frac{\partial}{\partial z}a_s \\ \frac{\partial}{\partial z}a_i^* \end{pmatrix} = \begin{pmatrix} \frac{1}{2}ik_2\Delta\omega^2 + i\gamma|a_p|^2 & i\gamma a_p^2 \\ -i\gamma a_p^{*2} & -\frac{1}{2}ik_2\Delta\omega^2 - i\gamma|a_p|^2 \end{pmatrix} \begin{pmatrix} a_s \\ a_i^* \end{pmatrix}$$

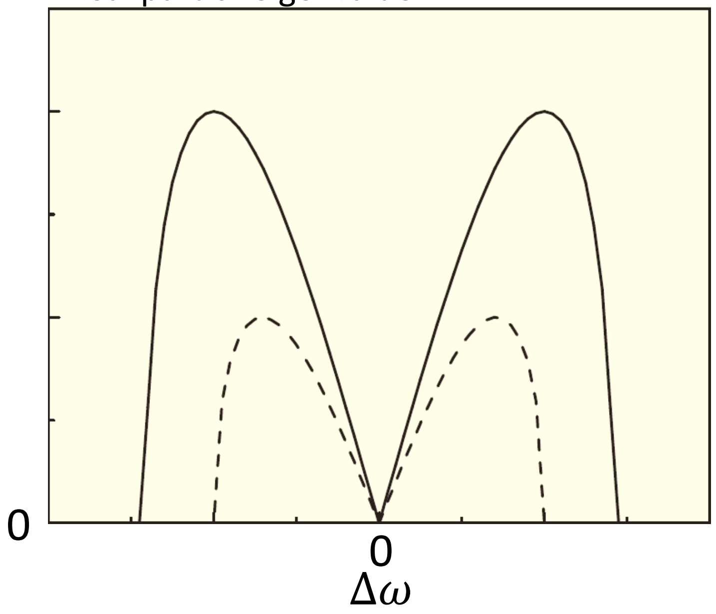
Eigenvalues (describe growth of weak sidebands):

$$\boxed{\lambda_{1,2} = \pm\sqrt{\gamma^2|a_p|^4 - \tfrac{1}{4}\left(k_2\Delta\omega^2 + 2\gamma|a_p|^2\right)^2}}$$

k_2 must be negative

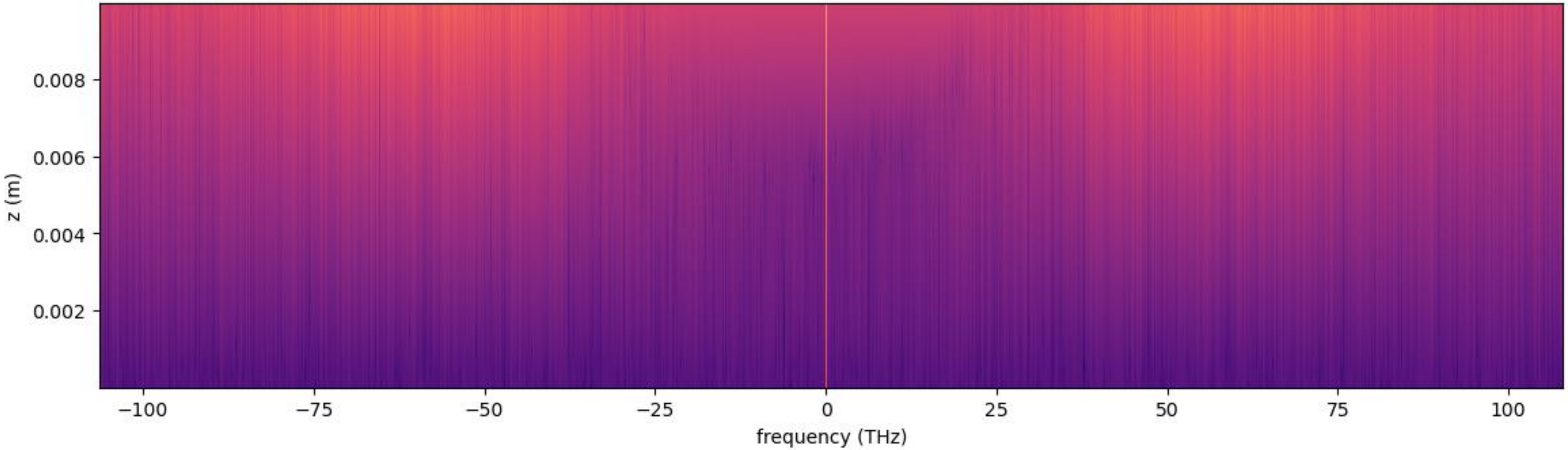
Fastest growth when …

Real part of eigenvalue



$$\text{Max.}\Delta\omega = \sqrt{\frac{-4\gamma|a_p|^2}{k_2}}$$

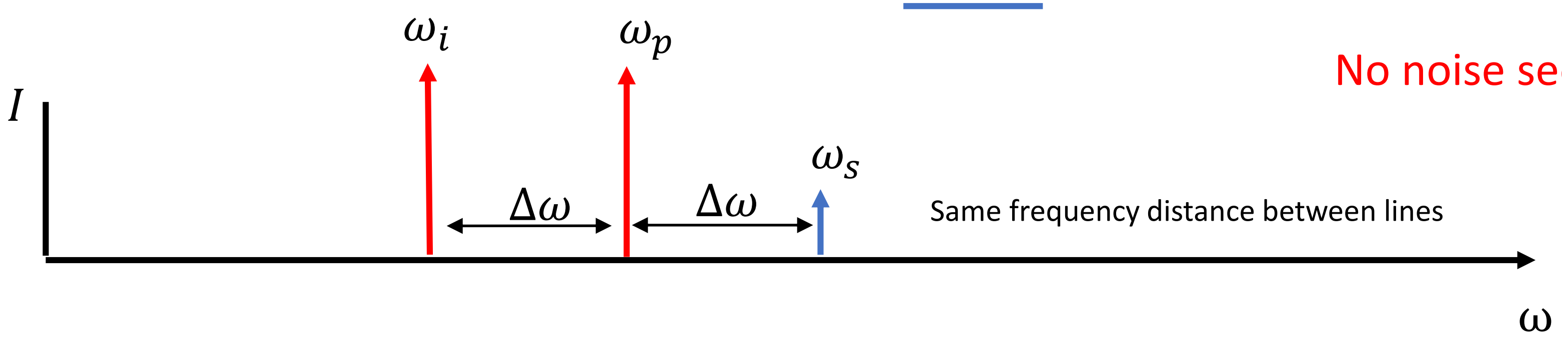
Simulation of modulation instability

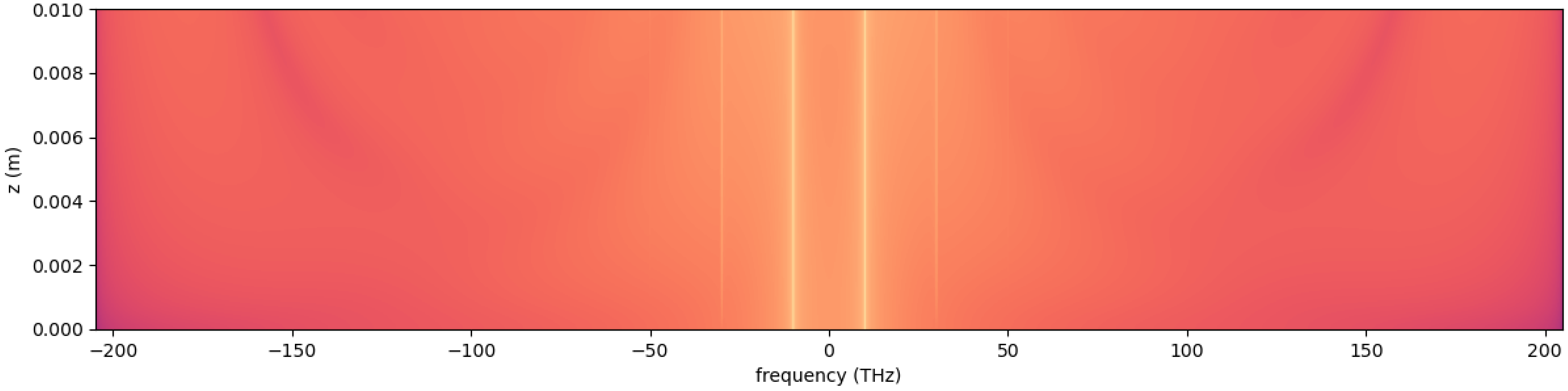


Non-degenerate FWM

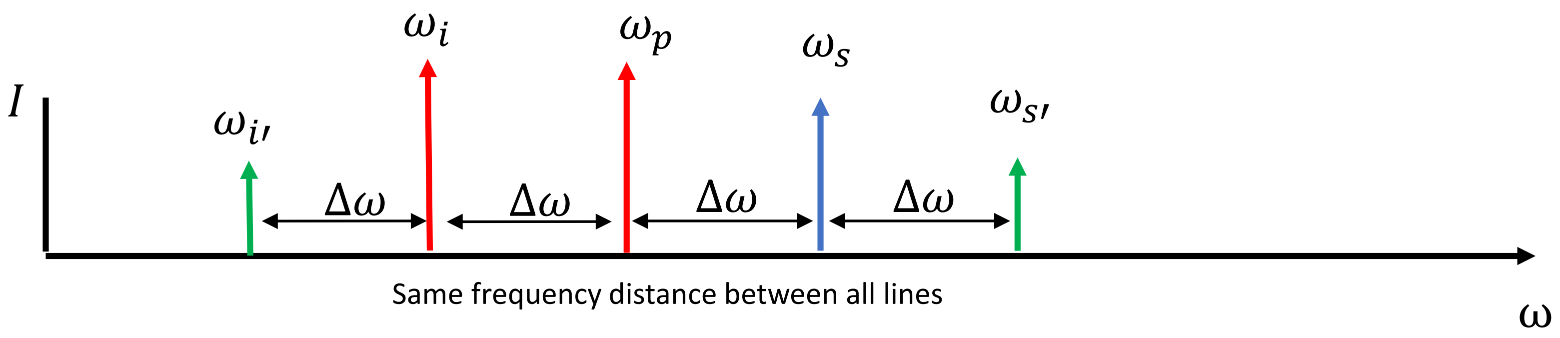
$$\frac{\partial}{\partial z} a_s - \frac{1}{2} i k_2 \Delta\omega^2 a_s = i\gamma |a_p|^2 a_s + \underline{i\gamma a_p^2 a_i^*}$$

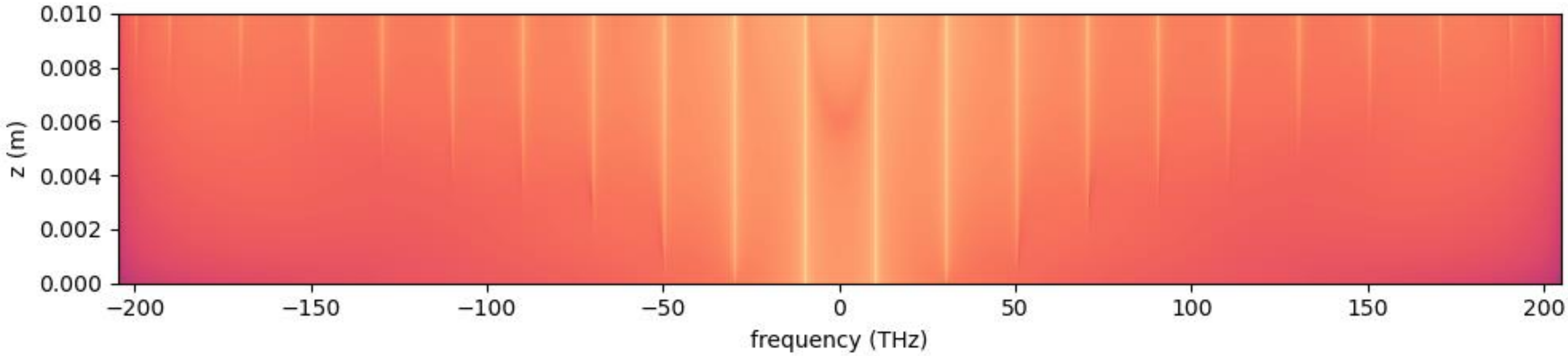
No noise seeding required





Cascaded non-degenerate FWM





Upgrading the NLSE

$$\frac{\partial}{\partial z}A + \tfrac{1}{2}ik_2\frac{\partial^2}{\partial \tau^2}A = i\gamma|A|^2A$$

including loss and higher order dispersion

$$\frac{\partial A}{\partial z} + \frac{\alpha}{2}A - \sum_{j\geq 2}\frac{i^{j+1}}{j!}k_j\frac{\partial^j}{\partial \tau^j}A = i\gamma|A|^2A$$

<u>NLSE is limited:</u>

- Neglects losses
- Chromatic dispersion only valid in small frequency range
- Neglects dispersion in the nonlinearity
- Assumes the nonlinearity is instantaneous

$\chi^{(3)}$ arising from oriented anisotropic molecules (e.g. in liquids)

As defined:

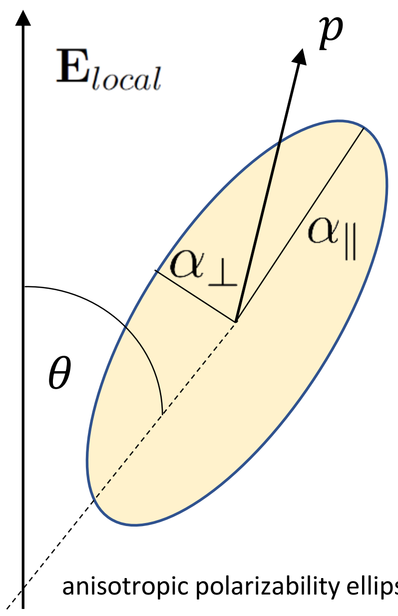
$$\mathbf{P} = \varepsilon_0 \underbrace{(n^2 - 1)}_{\chi^{(1)}} \mathbf{E}$$

$$\mathbf{P} = \varepsilon_0 N \langle \alpha \rangle \mathbf{E}_{local}$$

N: number density of molecules

$$\mathbf{E}_{local} = \mathbf{E} + \frac{1}{3\varepsilon_0}\mathbf{P}$$

(assume void in medium and apply Gauss law on the polarization)



anisotropic polarizability ellipsoid

Lorentz-Lorenz /Clausius-Mossotti

$$\frac{N\langle \alpha \rangle}{3} = \frac{n_0^2 - 1}{n_0^2 + 2}$$

$$E_{local} = \left(\frac{n_0^2 + 2}{3}\right) E$$

E-field enhancement in medium

Mean polarization of medium in direction of E_{local}

$$P = N\left[\alpha_\parallel \langle \cos^2\theta \rangle + \alpha_\perp \langle \sin^2\theta \rangle\right] E_{local}$$

Mean molecular polarizability

$$\langle \alpha \rangle = (\alpha_\parallel - \alpha_\perp)\langle \cos^2\theta \rangle + \alpha_\perp$$

Energy and probability for specific θ:

$$W = -\tfrac{1}{2}\mathbf{P}\cdot\mathbf{E}_{local} = -\tfrac{1}{2}\alpha(\theta)\left|E_{local}\right|^2$$

$$p(\theta) \sim \exp\left[-\frac{(\alpha_\parallel - \alpha_\perp)\left|E_{local}\right|^2 \cos^2\theta}{2k_B T}\right]$$

$$\langle \cos^2\theta \rangle = \frac{1}{2}\int_0^\pi \cos^2\theta\, p(\theta)\sin\theta\, d\theta \Big/ \frac{1}{2}\int_0^\pi p(\theta)\sin\theta\, d\theta$$

$$\langle \cos^2\theta \rangle = \frac{1}{3} + \frac{4}{45}\frac{(\alpha_\parallel - \alpha_\perp)\left|E_{local}\right|^2}{2k_B T}$$

$$\Delta\langle \alpha \rangle = \frac{4}{45}\frac{(\alpha_\parallel - \alpha_\perp)^2\left|E_{local}\right|^2}{2k_B T} = \frac{3}{N}\frac{6n_0\Delta n}{(n_0^2+2)^2}$$

$$\boxed{n_2^E = \left(\frac{n_0^2+2}{3}\right)^2 \frac{N}{45 n_0}\frac{(\alpha_\parallel - \alpha_\perp)^2}{k_B T}}$$

$\chi^{(3)}$ arising from oriented anisotropic molecules (e.g. in liquids)

Non-instantaneous nonlinearity:

$$\ldots = i\gamma A(z,\tau) \int_{-\infty}^{+\infty} R(\tau')|A(z,\tau-\tau')|^2 \mathrm{d}\tau'$$

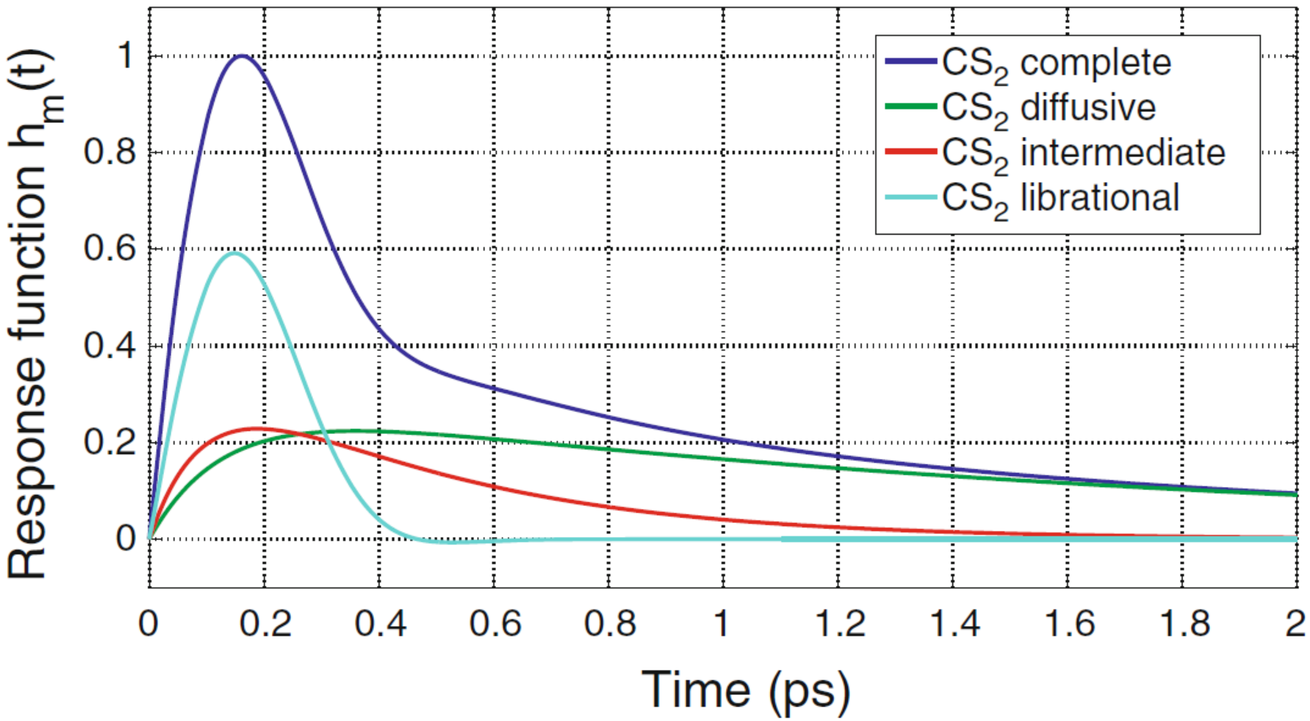
$$R(t) = (1-f)\delta(t) f h(t)$$

$$h(t) = 0 \qquad (t < 0)$$

For CS_2:

$$\begin{aligned} h_{\mathrm{m}}(t) = {} & A_1 e^{(-t/t_{\mathrm{diff}})}(1 - e^{(-t/t_{\mathrm{rise},1})}) \\ & + A_2 e^{(-t/t_{\mathrm{int}})}(1 - e^{(-t/t_{\mathrm{rise},1})}) \\ & + A_3 e^{(-t^2/(2t_{\mathrm{fast}}^2))} \sin(t/t_{\mathrm{rise},2}) \end{aligned}$$

Kedenburg et al., Applied Physics B (2013)



Upgrading the NLSE

$$\frac{\partial A}{\partial z} + \frac{\alpha}{2}A - \sum_{j\geq 2}\frac{i^{j+1}}{j!}k_j\frac{\partial^j}{\partial\tau^j}A = i\gamma|A|^2A$$

Including non-instantaneous nonlinearity

$$\frac{\partial A}{\partial z} + \frac{\alpha}{2}A - \sum_{j\geq 2}\frac{i^{j+1}}{j!}k_j\frac{\partial^j}{\partial\tau^j}A = i\gamma A(z,\tau)\int_{-\infty}^{+\infty}R(\tau')|A(z,\tau-\tau')|^2\mathrm{d}\tau'$$

Including dispersion of nonlinearity

$$\frac{\partial A}{\partial z} + \frac{\alpha}{2}A - \sum_{j\geq 2}\frac{i^{j+1}}{j!}k_j\frac{\partial^j}{\partial\tau^j}A = i\gamma(1+i\tau_{\mathrm{shock}})A(z,\tau)\int_{-\infty}^{+\infty}R(\tau')|A(z,\tau-\tau')|^2\mathrm{d}\tau'$$