2021 Nov 15

NLO #11

- **Chromatic Dispersion**
- **Reminder SPM/XPM**
- **SPM of Pulses**
- **Pulse propagation (NLSE)**
- **Temporal solitons**

To the people joining remotely:
Feel free to turn on your webcam

Chromatic dispersion of the refractive index

$$k(\omega) = \frac{\omega}{c} n(\omega)$$

$$k(\omega) = k_0 + k_1(\omega - \omega_0) + \tfrac{1}{2} k_2 (\omega - \omega_0)^2 + \tfrac{1}{6} k_3 (\omega - \omega_0)^3 + \dots$$

Inverse group velocity

$$k_1 = \frac{\partial k(\omega)}{\partial \omega}\Big|_{\omega=\omega_0} = \frac{1}{v_g}$$

Group velocity dispersion (GVD)

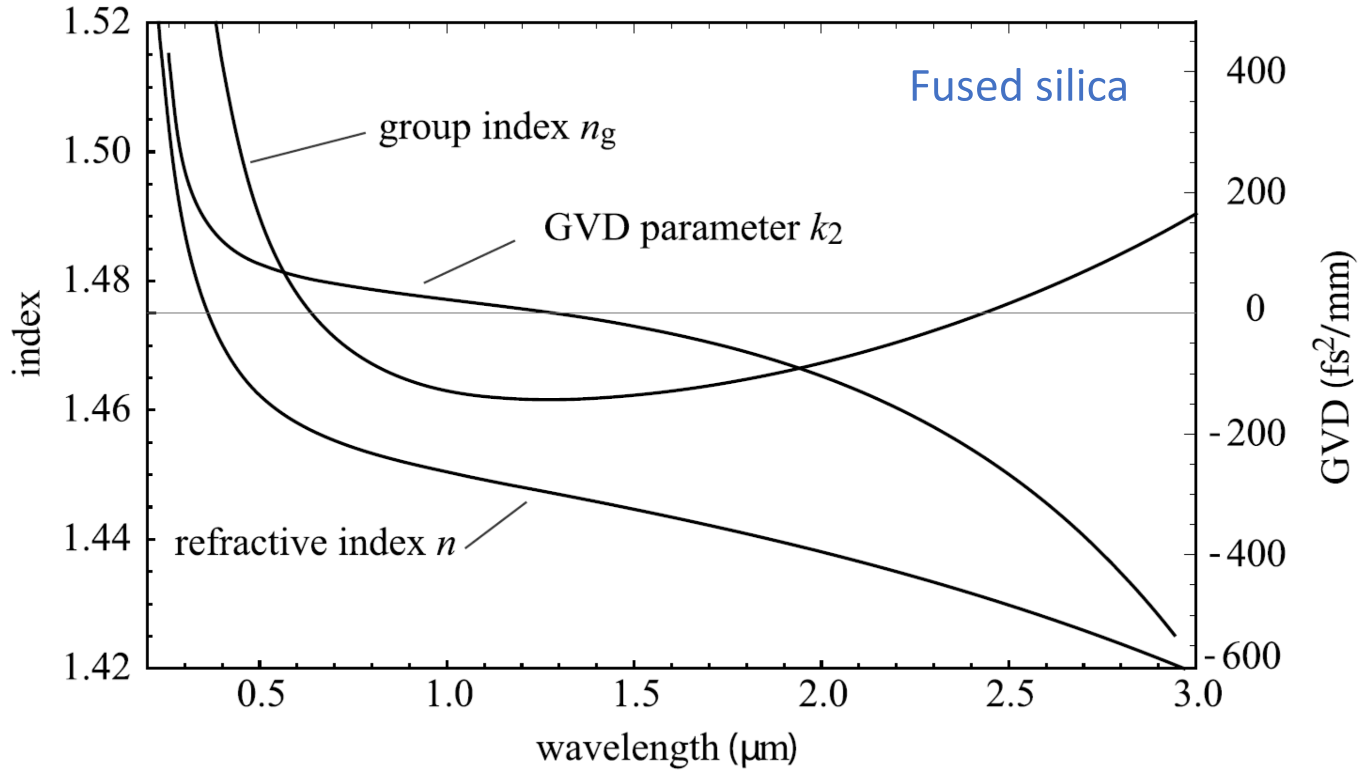
$$k_2 = \frac{\partial^2 k(\omega)}{\partial \omega^2}\Big|_{\omega=\omega_0} = \left(-\frac{1}{v_g^2}\frac{d v_g}{d\omega}\right)_{\omega=\omega_0}$$

Chromatic dispersion of the refractive index

Sellmeier Equation

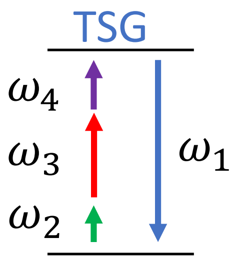
$$n^2(\lambda) = 1 + \sum_i \frac{B_i \lambda^2}{\lambda^2 - C_i}$$

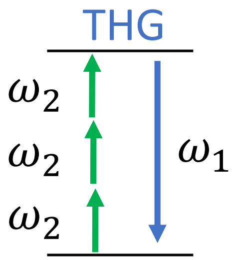
Useful source of data:
https://refractiveindex.info/



Overview $\chi^{(3)}$ processes

$$E(z,t) = \sum_n E_n(z) \mathrm{e}^{-i(\omega_n t - k_n z)} + \text{c.c.}$$



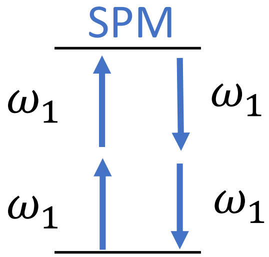


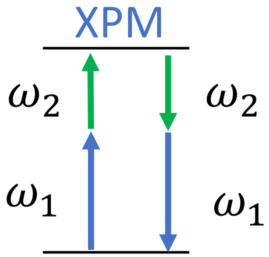
Triple-sum (TSG)
Third harmonic (THG)

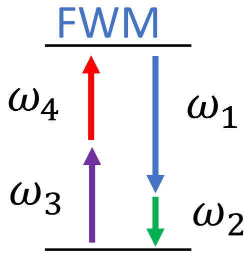
$$\frac{\partial}{\partial z} E_m = -i \frac{\omega_m}{2cn_m} \chi^{(3)} \sum_{n,p,q} \underline{\delta(\omega_n + \omega_p + \omega_q - \omega_m)} E_n E_p E_q \mathrm{e}^{i(k_n + k_p + k_q - k_m)z}$$

$$- i \frac{3\omega_m}{2cn_m} \chi^{(3)} \sum_{n,p,q} \underline{\delta(\omega_n + \omega_p - \omega_q - \omega_m)} E_n E_p E_q^* \mathrm{e}^{i[(k_n + k_p - k_q - k_m)z}$$

Self-phase modulation (SPM)
Cross-phase modulation (XPM)
Four-wave mixing (FWM)



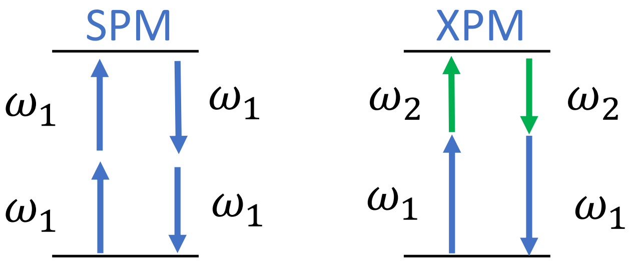




Self-Cross phase and modulation (SPM/XPM)

$$E(z,t) = \sum_n E_n(z)\mathrm{e}^{-i(\omega_n t - k_n z)} + \text{c.c.}$$

$$\frac{\partial}{\partial z}E_m = -i\frac{3\omega_m}{2cn_m}\chi^{(3)}(E_m E_m^*)E_m - \underline{2}\,i\frac{3\omega_m}{2cn_m}\chi^{(3)}(E_n E_n^*)E_m$$



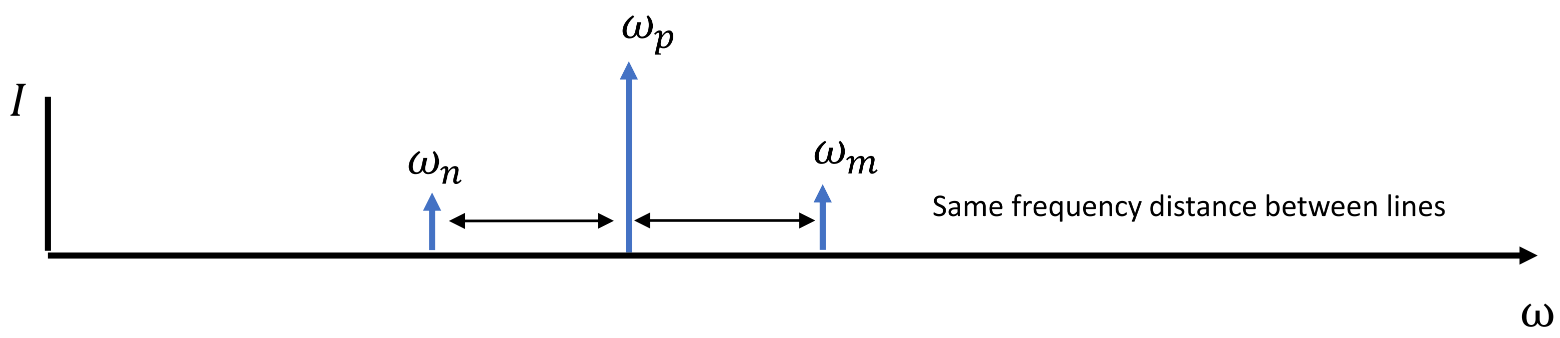
Solution:

$$\boxed{E_m(z) = E_m(0)\exp\left[-i\underbrace{\left(\frac{3\omega_1\chi^{(3)}}{2cn_m}|E_m|^2 + \underline{2}\frac{3\omega_1\chi^{(3)}}{2cn_m}|E_n|^2\right)}_{\Delta k_m} z\right]}$$

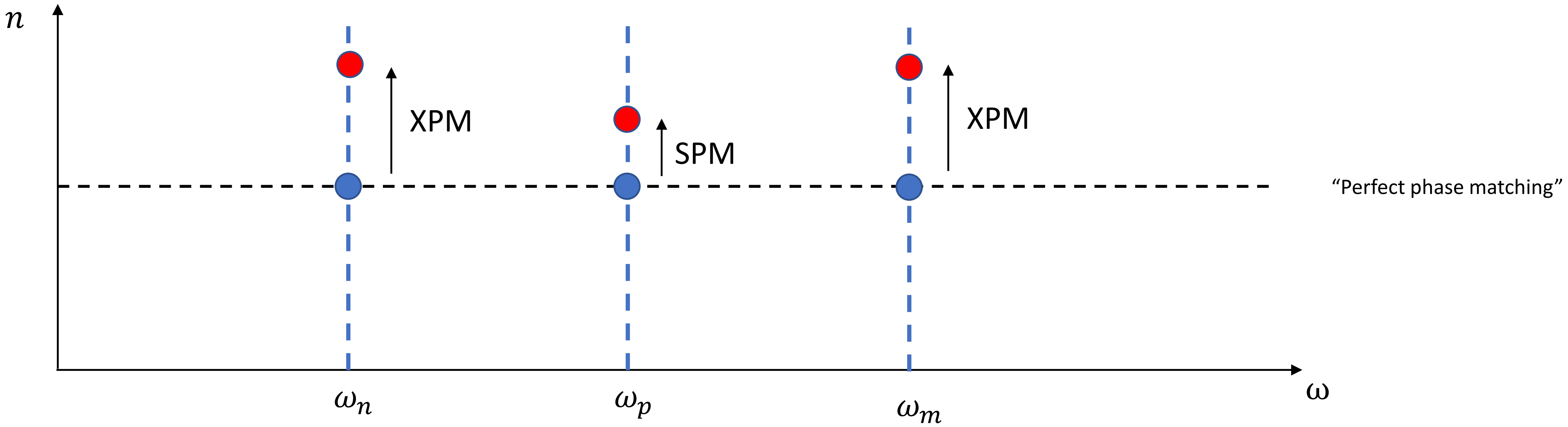
$$\boxed{\Delta n_m = n_2 I_m + 2n_2 I_n}$$

Example: Importance of SPM/XPM and Dispersion

Parametric Amplifier

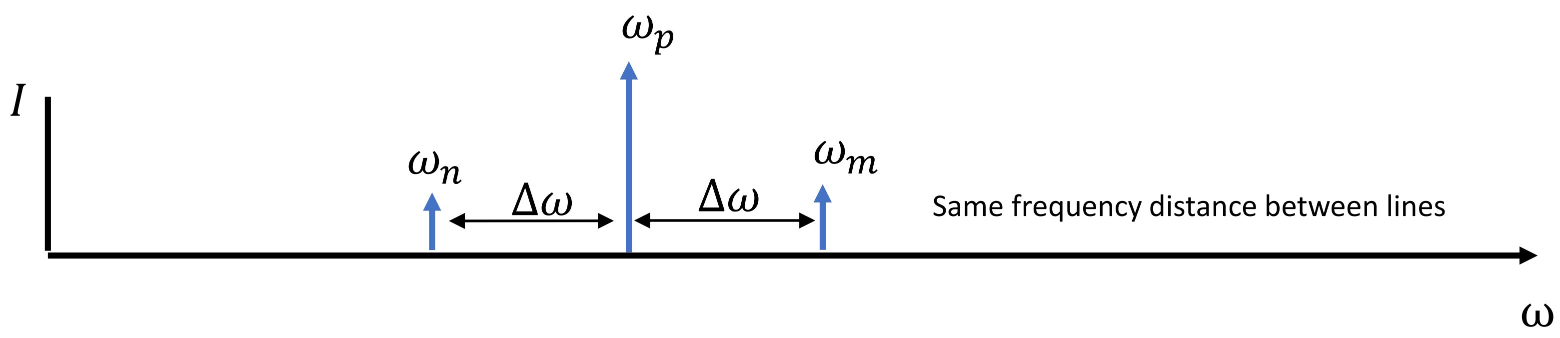


$$\frac{\partial}{\partial z}E_m = -i\frac{3\omega_m}{2cn_m}\chi^{(3)}\delta(\omega_p + \omega_p - \omega_n - \omega_m)E_pE_pE_n^*\mathrm{e}^{i[(2k_p - k_n - k_m)z}$$



Example: Importance of SPM/XPM and Dispersion

Parametric Amplifier



$$k_p = k_0 + \gamma P$$

$$k_m = k_0 + k_1 \Delta\omega + \tfrac{1}{2} k_2 \Delta\omega^2 + 2\gamma P$$

$$k_n = k_0 - k_1 \Delta\omega + \tfrac{1}{2} k_2 \Delta\omega^2 + 2\gamma P$$

$$\Delta k = 2k_p - k_m - k_n = -2\gamma P - k_2 \Delta\omega^2$$

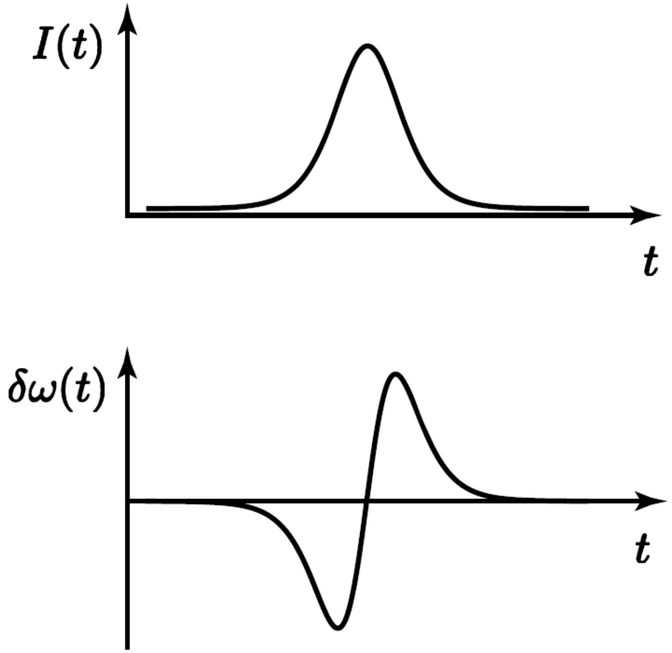
$\Delta k = 0$: $\boxed{\Delta\omega = \pm\sqrt{\frac{-2\gamma P}{k_2}}}$

Self-phase modulation of a pulse

Optical pulse

$$E(z,t) = A(z,t)e^{i(k_0 z - \omega_0 t)} + \text{c.c.}$$

$$I(t) = I_0\,\text{sech}^2(t/\tau_0)$$



$$n(t) = n_0 + n_2 I(t)$$

Nonlinear phase

$$\phi_{\text{NL}}(t) = -n_2 I(t)\omega_0 L/c$$

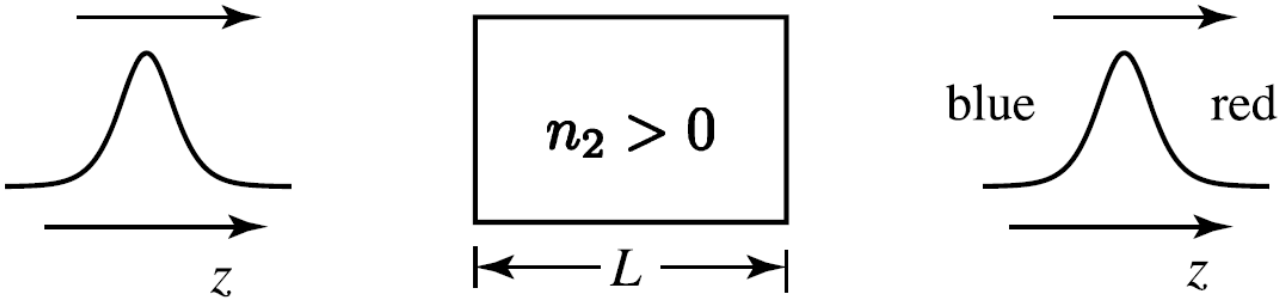
$$\delta\omega(t) = \frac{d}{dt}\phi_{\text{NL}}(t)$$

Instantaneous frequency

$$\omega(t) = \omega_0 + \delta\omega(t)$$

Inserting the intensity:

$$\boxed{\delta\omega(t) = 2n_2\frac{\omega_0}{c\tau_0}L\,I_0\,\text{sech}^2(t/\tau_0)\tanh(t/\tau_0)}$$



Self-phase modulation of pulse

$$\phi_{\mathrm{NL}}(t) = -n_2 I(t)\omega_0 L/c$$

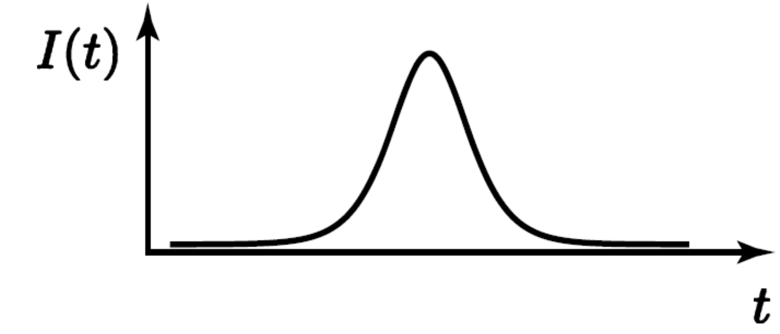
$$\Delta\phi_{\mathrm{NL}}^{(\max)} = -n_2 \frac{\omega_0}{c} I_0 L$$

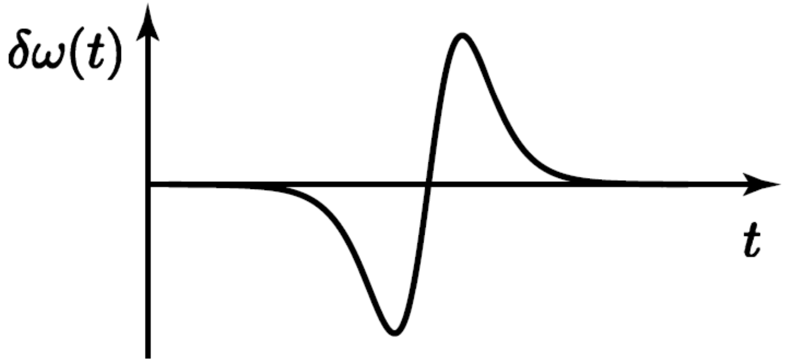
$$\boxed{\delta\omega_{\max} = \frac{-0.77\Delta\phi_{\mathrm{NL}}^{(\max)}}{\tau_0}}$$

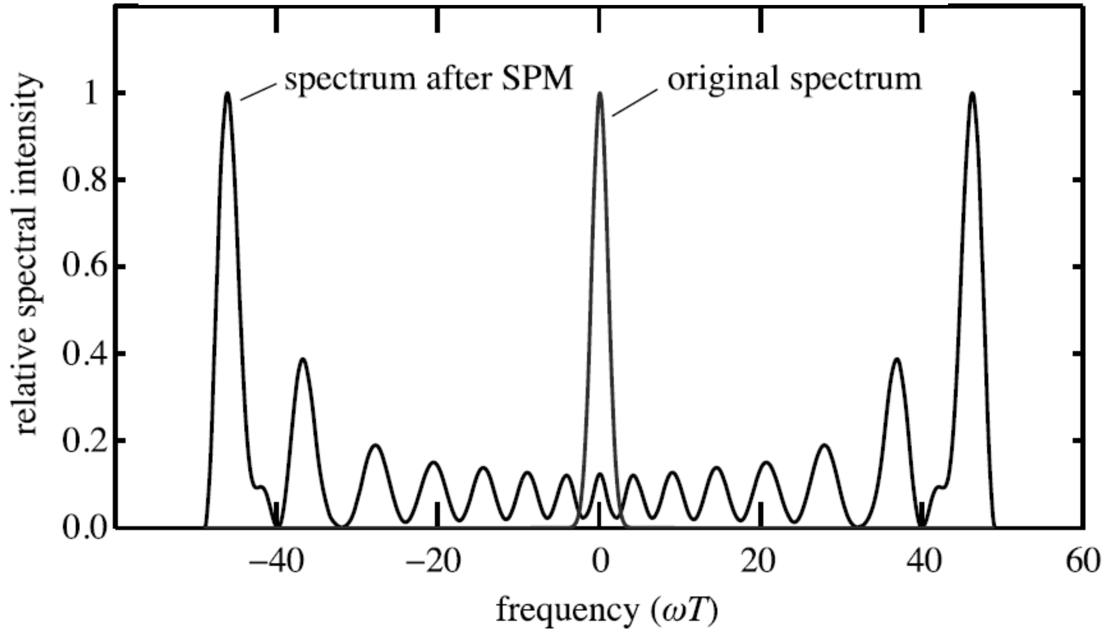
Spectral broadening

Optical Spectrum

$$S(\omega) = \left| \int_{-\infty}^{\infty} \tilde{A}(t) e^{-i\omega_0 t - i\phi_{\mathrm{NL}}(t)} e^{i\omega t}\, dt \right|^2$$

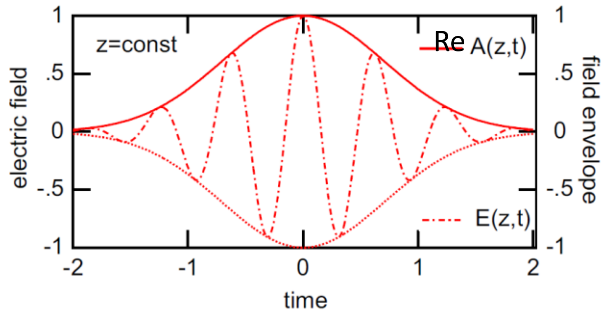






Pulse propagation

Pulse with center frequency ω_0:



Wave equation

$$\frac{\partial^2}{\partial z^2}E(z,t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}E(z,t) = \frac{1}{\epsilon_0 c^2}\frac{\partial^2}{\partial t^2}P(z,t)$$

+

$$D(z,t) = \epsilon_0 E(z,t) + P(z,t)$$

$$\frac{\partial^2}{\partial z^2}E(z,t) - \frac{1}{\epsilon_0 c^2}\frac{\partial^2}{\partial t^2}D(z,t) = 0$$

+

$$E(z,t) = \int \hat{E}(z,\omega)\, \mathrm{e}^{-i\omega t}\,\frac{\mathrm{d}\omega}{2\pi}$$

$$D(z,t) = \int \hat{D}(z,\omega)\, \mathrm{e}^{-i\omega t}\,\frac{\mathrm{d}\omega}{2\pi}$$

Wave equation in the frequency domain:

$$\frac{\partial^2}{\partial z^2}\hat{E}(z,\omega) - \frac{1}{\epsilon_0 c^2}\frac{\partial^2}{\partial t^2}\hat{D}(z,\omega) = 0$$

$$E(z,t) = \underbrace{A(z,t)}_{?}\,\mathrm{e}^{i(k_0 z - \omega_0 t)} + \text{c.c.}$$

+

$$\hat{D}(z,\omega) = \epsilon_0 \underline{\epsilon(\omega)} E(z,\omega)$$

chromatic dispersion

Helmholtz equation

$$\frac{\partial^2}{\partial z^2}\hat{E}(z,\omega) + \frac{\omega^2}{c^2}\epsilon(\omega)\hat{E}(z,\omega) = 0$$

+

$$\frac{\omega^2}{c^2}\epsilon(\omega) = k^2(\omega)$$

$$\frac{\partial^2}{\partial z^2}\hat{E}(z,\omega) + k^2(\omega)\hat{E}(z,\omega) = 0$$

Pulse propagation

From previous page…

$$\frac{\partial^2}{\partial z^2}\hat{E}(z,\omega) + k^2(\omega)\hat{E}(z,\omega) = 0$$

+

$$\begin{aligned}\hat{E}(z,\omega) &= \int E(z,t)\,\mathrm{e}^{i\omega t}\,\mathrm{d}t \\ &= \int A(z,t)\,\mathrm{e}^{ik_0 z}\,\mathrm{e}^{i(\omega-\omega_0)t}\,\mathrm{d}t + \int A^*(z,t)\,\mathrm{e}^{-ik_0 z}\,\mathrm{e}^{i(\omega+\omega_0)t} \\ &= \hat{A}(z,\omega-\omega_0)\,\mathrm{e}^{ik_0 z} + \hat{A}^*(z,\omega+\omega_0)\,\mathrm{e}^{-ik_0 z} \\ &\approx \hat{A}(z,\omega-\omega_0)\,\mathrm{e}^{ik_0 z}\end{aligned}$$

↓ also assume SVEA ("$\frac{\partial^2}{\partial z^2} = 0$")

$$2ik_0\frac{\partial}{\partial z}\hat{A}(z,\omega-\omega_0) + (k(\omega)^2 - k_0^2)\hat{A}(z,\omega-\omega_0) = 0$$

+

$$k(\omega)^2 - k_0^2 \approx 2k_0(k(\omega) - k_0)$$

$$k(\omega) = \underbrace{k_0 + k_1(\omega-\omega_0) + \tfrac{1}{2}k_2(\omega-\omega_0)^2}_{\text{chromatic dispersion}} + \underbrace{\Delta k_{\mathrm{NL}}(z,t)}_{\text{nonlinearity SPM}}$$

↓

$$\left(\frac{\partial}{\partial z} + ik_1(\omega-\omega_0) + \tfrac{1}{2}ik_2(\omega-\omega_0)^2 - i\Delta k_{\mathrm{NL}}(z,t)\right)\hat{A}(z,\omega-\omega_0) = 0$$

↓

<u>Back to time domain:</u>
multiply by $(\omega-\omega_0)$
and $\int ..\mathrm{d}(\omega-\omega_0)$

$$\left(\frac{\partial}{\partial z} + k_1\frac{\partial}{\partial t} + +\tfrac{1}{2}ik_2\frac{\partial^2}{\partial t^2} - i\Delta k_{\mathrm{NL}}(z,t)\right)A(z,t) = 0$$

Pulse propagation

From previous page…

$$\left(\frac{\partial}{\partial z} + k_1\frac{\partial}{\partial t} + +\tfrac{1}{2}ik_2\frac{\partial^2}{\partial t^2} - i\Delta k_{\mathrm{NL}}(z,t)\right) A(z,t) = 0$$

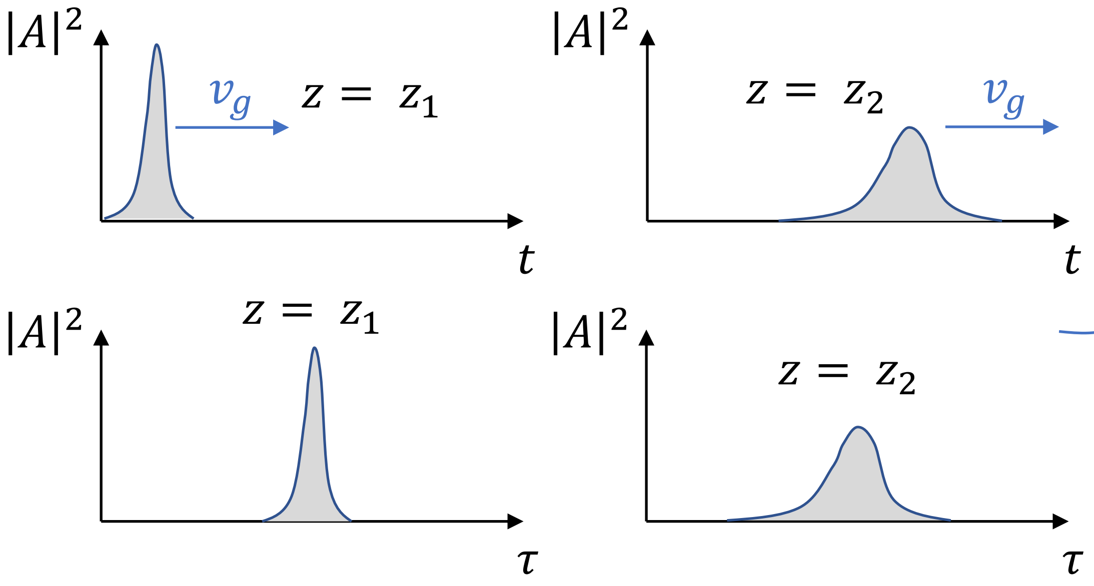
+

$$\Delta k_{\mathrm{NL}} = k_0 n_2 I = 2n_0\epsilon_0\omega_0 n_2 |A'(z,\tau)|^2 = \gamma |A'(z,\tau)|^2$$

+

Transform to comoving frame

$$\tau = t - \frac{z}{v_g} = t - k_1 z$$



+

$$\frac{\partial}{\partial t}A(z,t) = \frac{\partial}{\partial \tau}A'(z,\tau)$$

$$\frac{\partial^2}{\partial t^2}A(z,t) = \frac{\partial^2}{\partial \tau^2}A'(z,\tau)$$

$$\frac{\partial}{\partial z}A(z,t) = \left(\frac{\partial}{\partial z} - \overset{k_1\frac{\partial}{\partial \tau}}{\cancel{k_1}}\right)A'(z,\tau)$$

Nonlinear Schrödinger Equation (NLSE)

$$\boxed{\frac{\partial}{\partial z}A'(z,\tau) + \tfrac{1}{2}ik_2\frac{\partial^2}{\partial \tau^2}A'(z,\tau) = i\gamma|A'(z,\tau)|^2 A'(z,\tau)}$$

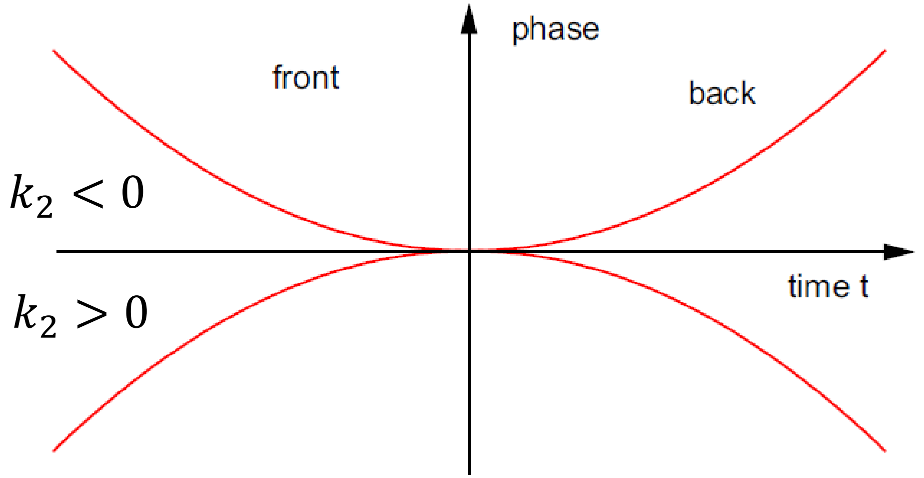
Dispersion of a pulse

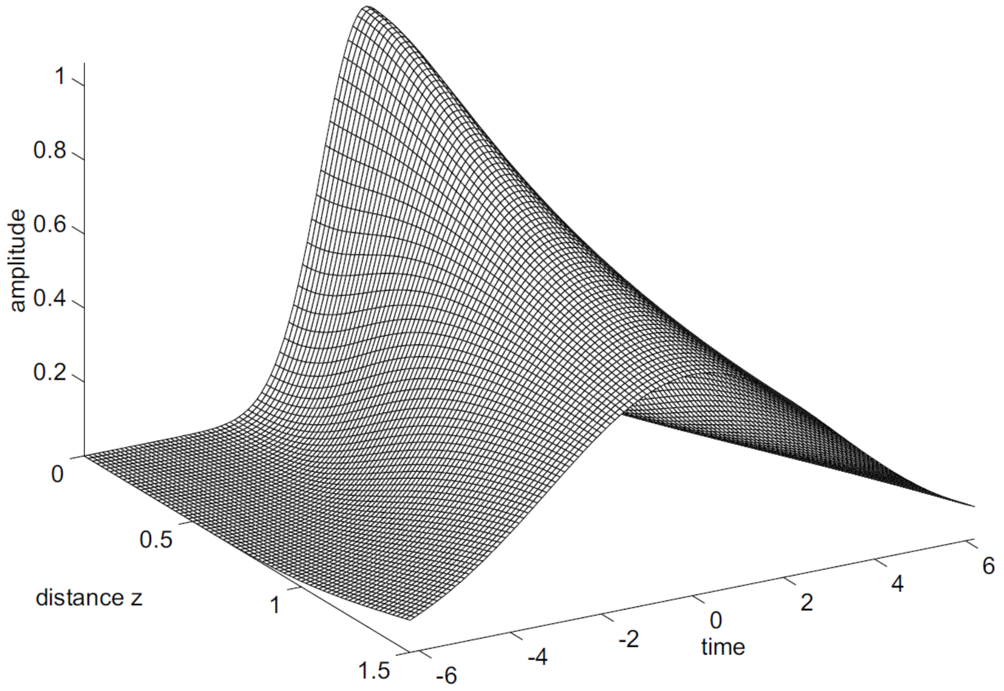
Nonlinear Schrödinger Equation (NLSE)

$$\boxed{\frac{\partial}{\partial z}A'(z,\tau) + \underbrace{\tfrac{1}{2}ik_2\frac{\partial^2}{\partial \tau^2}}_{\text{chromatic dispersion}}A'(z,\tau) = \cancel{i\gamma|A'(z,\tau)|^2 A'(z,\tau)}}$$

reshapes the pulse envelope $|A'|^2$

for Gaussian pulse:





Self-phase modulation (revisited)

Nonlinear Schrödinger Equation (NLSE)

$$\frac{\partial}{\partial z}A'(z,\tau) + \cancel{\tfrac{1}{2}ik_2\frac{\partial^2}{\partial \tau^2}A'(z,\tau)} = i\gamma|A'(z,\tau)|^2A'(z,\tau)$$

nonlinearity
SPM

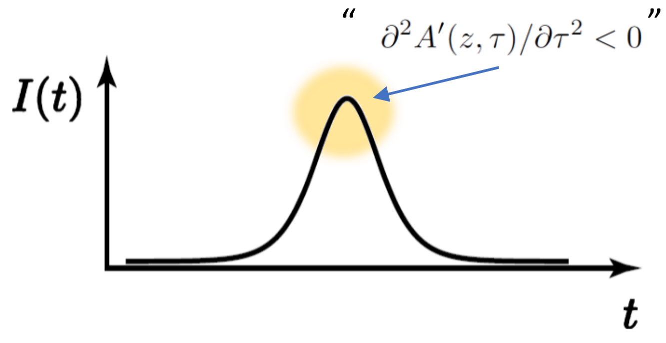
For a pulse

No reshaping of the pulse envelope $|A'|^2$,
But phase is changed.

Temporal solitons

Nonlinear Schrödinger Equation (NLSE)

$$\boxed{\frac{\partial}{\partial z}A'(z,\tau) + \underbrace{\tfrac{1}{2}ik_2\frac{\partial^2}{\partial\tau^2}A'(z,\tau)}_{\text{chromatic dispersion}} = \underbrace{i\gamma|A'(z,\tau)|^2A'(z,\tau)}_{\text{nonlinearity SPM}}}$$



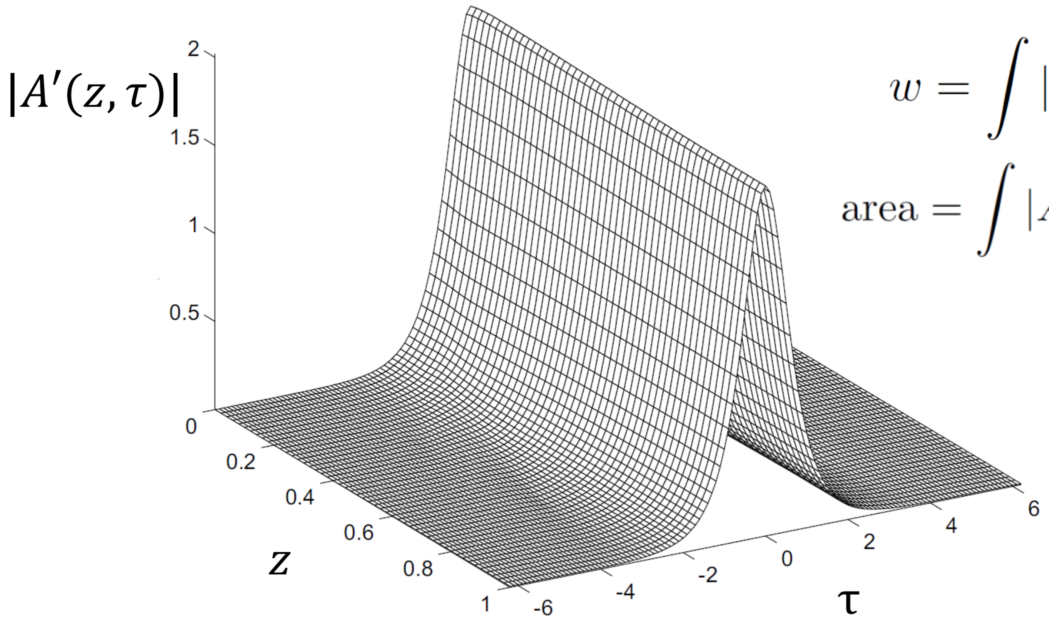
Dispersion and nonlinearity can compensate each other when $\mathrm{sign}(k_2) = -\mathrm{sign}(\gamma)$

Solitons are solution to the NLSE:

$$A'(z,\tau) = A'_0\mathrm{sech}(\tau/\tau_0)\mathrm{e}^{i\kappa z}$$

$$|A'_0|^2 = \frac{-k_2}{\gamma\tau_0^2}$$

$$\kappa = \frac{-k_2}{2\tau_0}$$

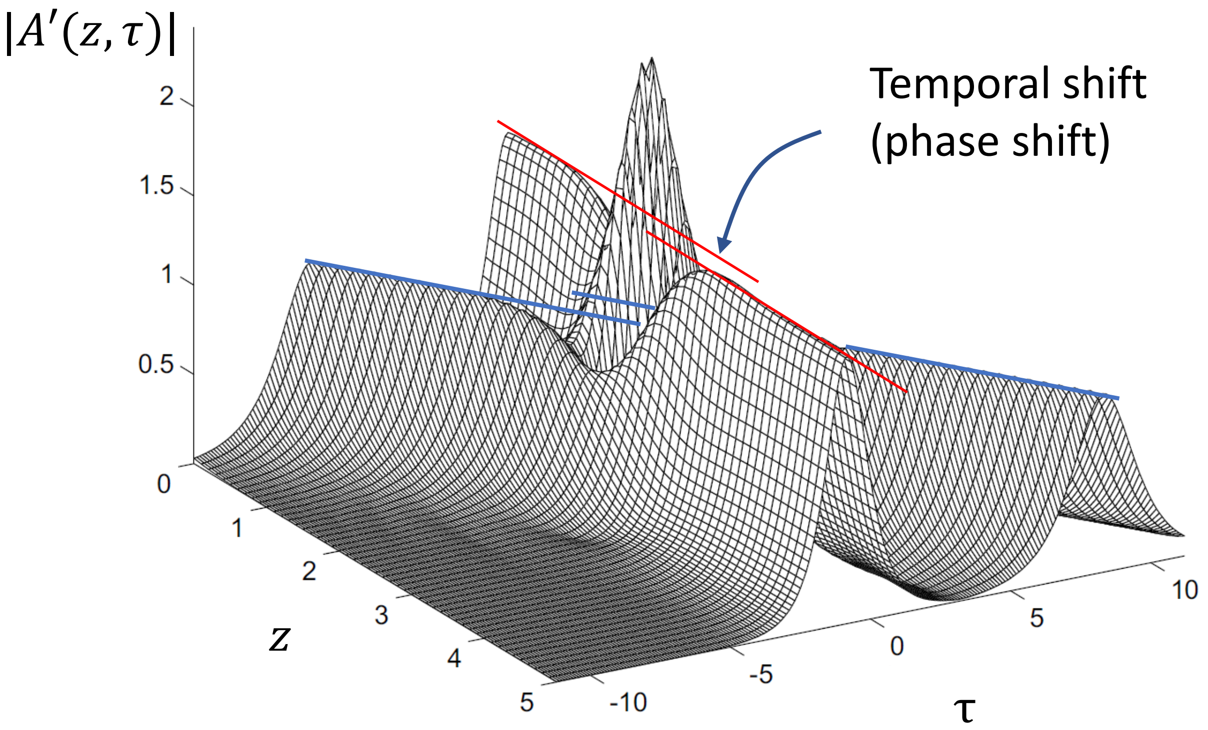


$$w = \int |A'(z,\tau)|^2\mathrm{d}\tau = 2\tau_0|A'_0|^2$$

$$\mathrm{area} = \int |A'(z,\tau)|\mathrm{d}\tau = \pi A'_0\tau = \pi\sqrt{\frac{|k_2|}{\gamma}}$$

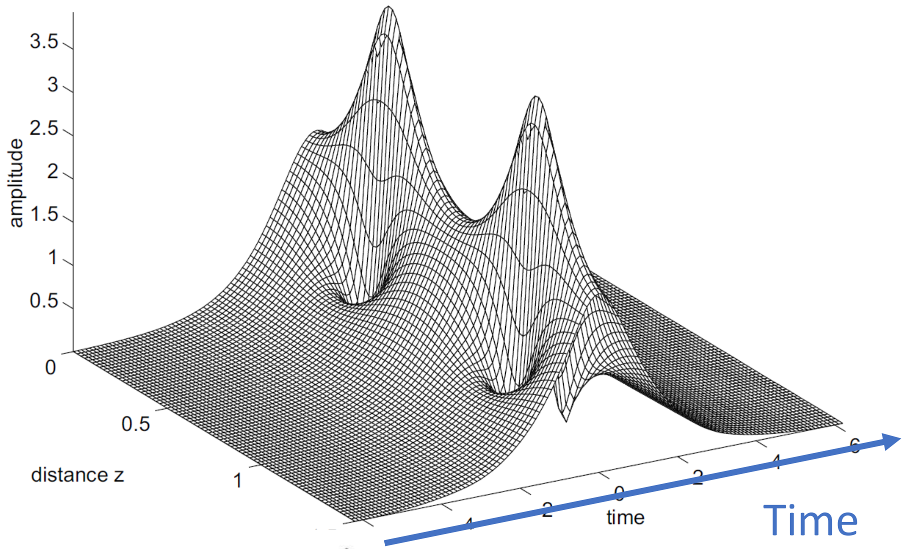
Solitons

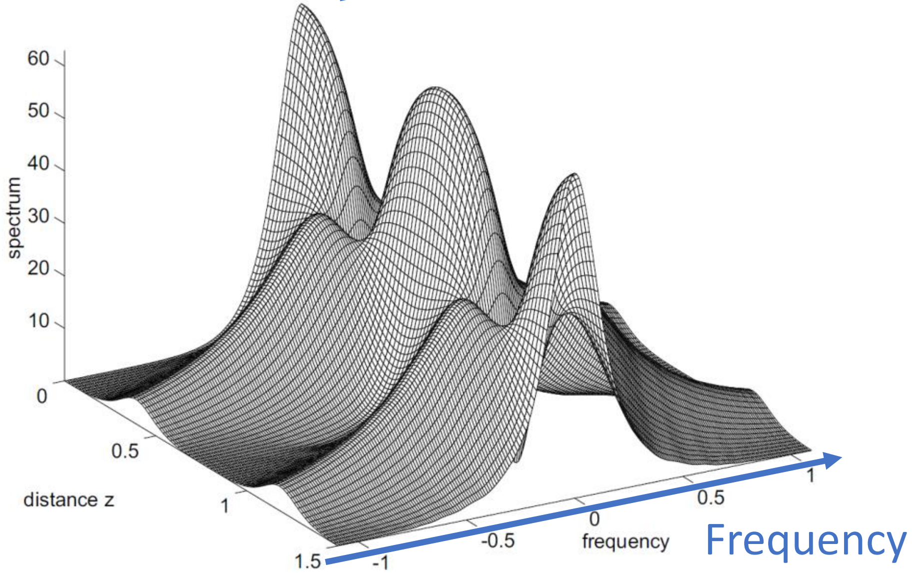
Soliton collision



Solitons recover after collision with small phase shift

Higher order solitons (Breathers)

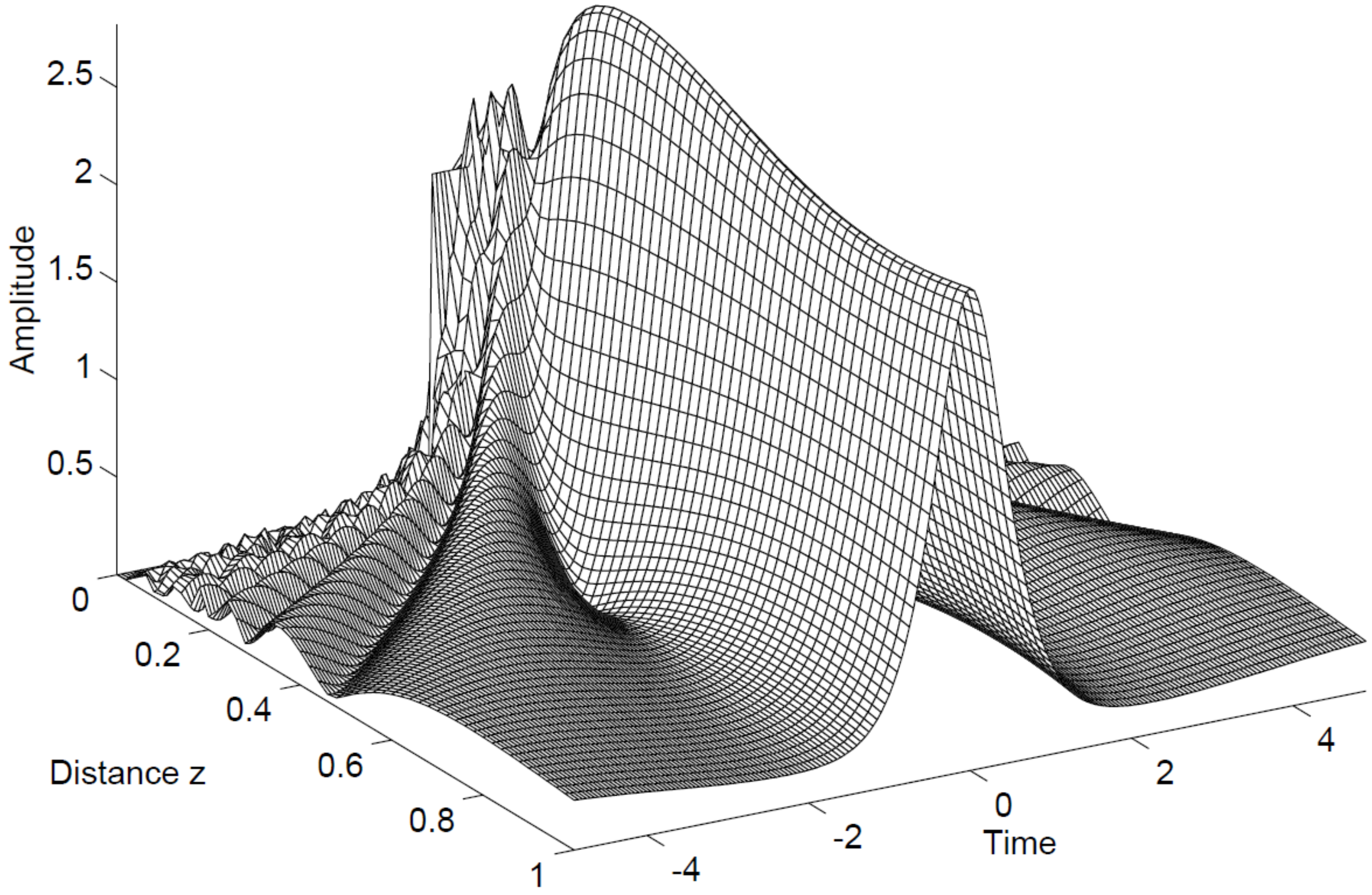




For energies of 4, 16, … times that of a fundamental soliton (-> area theorem)

Soliton

Self-organized Pulse re-shaping into a soliton (+ a dispersing temporal ‘continuum’)



Spatial and temporal solitons

$$\underbrace{\frac{\partial}{\partial z}A'(z,\tau) + \frac{1}{2}ik_2\frac{\partial^2}{\partial\tau^2}A'(z,\tau)}_{\text{Dispersion}} = \underbrace{i\gamma|A'(z,\tau)|^2A'(z,\tau)}_{\text{Nonlinearity}}$$

$$2ik_0\frac{\partial A}{\partial z} + \underbrace{\frac{\partial^2 A}{\partial x^2}}_{\text{Diffraction}} = \underbrace{-3\chi^{(3)}\frac{\omega^2}{c^2}|A|^2A}_{\text{Self-focusing}}$$

Temporal soliton:

$$A'(z,\tau) = A'_0\mathrm{sech}(\tau/\tau_0)\mathrm{e}^{i\kappa z}$$

Spatial soliton

$$A(x,z) = A_0\,\mathrm{sech}(x/x_0)e^{i\gamma z}$$

Light bullets if both soliton conditions fulfilled.