

# **NLO Lecture 8: Sum- & Difference Frequency Generation, OPA and OPO**

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## 4.6 Manley-Rowe relations

Three plane waves propagating in  $z$ -direction with frequencies  $\omega_1$ ,  $\omega_2$ , and  $\omega_3$ , and interacting via a  $\chi^{(2)}$  nonlinearity, can be described by the coupled equations

$$\frac{d\hat{E}(\omega_1)}{dz} = -j\kappa_1 \hat{E}(\omega_3) \hat{E}^*(\omega_2) e^{-j\Delta kz}, \quad (4.92)$$

$$\frac{d\hat{E}(\omega_2)}{dz} = -j\kappa_2 \hat{E}(\omega_3) \hat{E}^*(\omega_1) e^{-j\Delta kz}, \quad (4.93)$$

$$\frac{d\hat{E}(\omega_3)}{dz} = -j\kappa_3 \hat{E}(\omega_1) \hat{E}(\omega_2) e^{+j\Delta kz}, \quad (4.94)$$

with coupling coefficients and difference wave number

$$\kappa_i = \omega_i d_{eff} / n_i c_0, \quad \text{and} \quad \Delta k = k_3 - k_1 - k_2. \quad (4.95)$$

We multiply Eq. (4.92) by  $n_1 c_0 \varepsilon_0 \hat{E}^*(\omega_1) / 2$ , and add the complex conjugated part, thus obtaining

$$\left( \frac{1}{\omega_1} \right) \frac{dI(\omega_1)}{dz} = \frac{j\varepsilon_0 d_{eff}}{2} \hat{E}(\omega_3) \hat{E}^*(\omega_2) \hat{E}^*(\omega_1) e^{-j\Delta kz} + c.c.$$

We again assume a lossless medium, i.e.,  $d_{eff} = d_{eff}^*$ , and treat Eqs. (4.93), (4.94) similar to Eq. (4.92), and obtain

$$\frac{1}{\omega_1} \frac{dI(\omega_1)}{dz} = \frac{1}{\omega_2} \frac{dI(\omega_2)}{dz} = -\frac{1}{\omega_3} \frac{dI(\omega_3)}{dz}. \quad (4.96)$$

I.e., for each photon, that is created (annihilated) at frequency  $\omega_3$ , one photon at frequency  $\omega_1$  and one photon at frequency  $\omega_2$  must be annihilated (created). The corresponding spatial variations of the intensities  $\frac{dI(\omega_i)}{dz}$  scale with the frequencies  $\omega_i$ . This is an interesting result, because no quantum-mechanical treatment has been used to obtain it. Nevertheless, this classical nonlinear electro-dynamical treatment already strongly suggests a photon hypothesis  $E = n \cdot h\nu$ .

## 4.7 Sum-frequency generation (SFG)

If the nonlinear medium is irradiated by two input fields with frequencies  $\omega_1$  and  $\omega_2$ , it is possible to generate the sum frequency  $\omega_3 = \omega_1 + \omega_2$  via a  $\chi^{(2)}$ -process. The corresponding coupled-wave equations describing the amplitudes at the three frequencies are

$$\frac{\partial \hat{E}(\omega_3)}{\partial z} = -j\kappa_3 \hat{E}(\omega_1) \hat{E}(\omega_2) e^{j\Delta kz} \quad (4.97)$$

$$\frac{\partial \hat{E}(\omega_2)}{\partial z} = -j\kappa_2 \hat{E}(\omega_3) \hat{E}^*(\omega_1) e^{-j\Delta kz} \quad (4.98)$$

$$\frac{\partial \hat{E}(\omega_1)}{\partial z} = -j\kappa_1 \hat{E}(\omega_3) \hat{E}^*(\omega_2) e^{-j\Delta kz}, \quad (4.99)$$

with  $\Delta k = k(\omega_3) - k(\omega_1) - k(\omega_2)$  and  $\kappa_i = \omega_i d_{eff} / n_i c_0$ . In the special case  $\omega_1 = \omega_2$ , we again obtain frequency doubling. In the low-conversion case, we can solve Eqs. (4.97)-(4.99) in an analogous manner to the case of frequency doubling. Assuming  $\hat{E}(\omega_2)$  and  $\hat{E}(\omega_1)$  to be constant, we obtain

$$I(\omega_3, \ell) = \frac{2\kappa_3^2 n_3}{n_1 n_2 c_0 \epsilon_0} \cdot \ell^2 I(\omega_1) I(\omega_2) \left\{ \frac{\sin \Delta k \ell / 2}{\Delta k \ell / 2} \right\}^2. \quad (4.100)$$

One important spectroscopic application of sum-frequency generation is the up-conversion of a weak signal at frequency  $\omega_1$  with a strong signal at frequency  $\omega_2$ . This can be used to time-resolve weak light-emission dynamics or to convert a signal in the far infrared into the visible spectral range, where much better photodetectors are available. By using a strong signal at  $\omega_2$ , the weak signal can even significantly be enhanced. If the pump signal at frequency  $\omega_2$  is a short pulse, a short slice can temporally be “gated” out of the input signal. Under the assumption of a strong pump signal at  $\omega_2$  and phase matching, we obtain

$$\frac{\partial \hat{E}(\omega_3)}{\partial z} = - \left\{ j\kappa_3 \hat{E}(\omega_2) \right\} \hat{E}(\omega_1), \quad (4.101)$$

$$\frac{\partial \hat{E}(\omega_1)}{\partial z} = - \left\{ j\kappa_1 \hat{E}^*(\omega_2) \right\} \hat{E}(\omega_3). \quad (4.102)$$

In addition, the boundary conditions  $E(\omega_3, z = 0) = 0$  and  $E(\omega_1, z = 0) = E_0(\omega_1)$  apply. Since the system (4.101)-(4.102) is linear, we try an exponential ansatz of the form  $\hat{E}(\omega_{1,3}, z) = \hat{E}_0(\omega_{1,3}) e^{\pm j\gamma z}$ . With this ansatz we obtain from (4.101)-(4.102)

$$\pm j\gamma \hat{E}_0(\omega_3) + \left\{ j\kappa_3 \hat{E}_0(\omega_2) \right\} \hat{E}_0(\omega_1) = 0$$

$$\left\{ j\kappa_1 \hat{E}_0^*(\omega_2) \right\} \hat{E}_0(\omega_3) \pm j\gamma \hat{E}_0(\omega_1) = 0.$$

For solutions to exist (the determinant of the coefficient matrix must vanish), it must hold

$$\gamma^2 = \kappa_1 \kappa_3 |\hat{E}_0(\omega_2)|^2 \quad (4.103)$$

or

$$\gamma = \left\{ \frac{2\kappa_1 \kappa_3 I(\omega_2)}{n_2 c_0 \epsilon_0} \right\}^{1/2}. \quad (4.104)$$

The fundamental solutions are cosine and sine functions, and together with the boundary conditions it follows

$$\hat{E}(\omega_3) = \hat{A} \sin \gamma z \quad (4.105)$$

and

$$\hat{E}(\omega_1) = \hat{E}_0(\omega_1) \cos \gamma z. \quad (4.106)$$

Substitution into Eq.(4.101) yields

$$\hat{A} = -j \sqrt{\frac{\omega_3 n_1}{\omega_1 n_3}} \hat{E}_0(\omega_1) e^{j\phi(\omega_2)}, \quad (4.107)$$

where

$$e^{j\phi(\omega_2)} = \frac{\hat{E}_0(\omega_2)}{|\hat{E}_0(\omega_2)|}$$

contains the phase of the pump wave. The factor  $-j$  in Eq. (4.107) again implies, that the driving polarization advances the generated electric field by  $\pi/2$ . For the intensities we then obtain

$$I(\omega_3) = \frac{\omega_3}{\omega_1} I_0(\omega_1) \sin^2 \gamma z \quad (4.108)$$

$$I(\omega_1) = I_0(\omega_1) \cos^2 \gamma z \quad (4.109)$$

We realize that for  $\gamma z > \pi/2$  again backconversion into  $\omega_1$  occurs. As in the best case each photon at frequency  $\omega_1$  is converted into a photon at frequency  $\omega_3$ , it follows for the maximum enhancement in intensity

$$\frac{I_{\max}(\omega_3)}{I_0(\omega_1)} = \frac{\omega_3}{\omega_1}, \quad (4.110)$$

that again corresponds to the Manley-Rowe relation.

## 4.8 Difference-frequency generation (DFG)

Next we look at the generation of the difference frequency  $\omega_1 = \omega_3 - \omega_2$ ,  $\Delta k = k(\omega_1) + k(\omega_2) - k(\omega_3)$ . Again in the low-conversion limit with  $\hat{E}(\omega_3)$  and  $\hat{E}(\omega_2)$  assumed to be constant, we obtain

$$I(\omega_1, \ell) = \frac{2\kappa_1^2 n_1}{n_2 n_3 c_0 \epsilon_0} \cdot \ell^2 I(\omega_3) I(\omega_2) \left\{ \frac{\sin \Delta k \ell / 2}{\Delta k \ell / 2} \right\}^2. \quad (4.111)$$

This can, e.g., be used to generate infrared light from two visible light fields for spectroscopic applications. Another case is, if a strong pump wave at  $\omega_3$  converts light from  $\omega_2$  to  $\omega_1$ . For a strong pump wave,  $\hat{E}(\omega_3)$  is constant,

**$E^*$  in contrast to SFG**

$$\frac{\partial \hat{E}(\omega_1)}{\partial z} = -j\kappa_1 \hat{E}(\omega_3) \hat{E}^*(\omega_2) e^{j\Delta k z} \quad (4.112)$$

$$\frac{\partial \hat{E}(\omega_2)}{\partial z} = -j\kappa_2 \hat{E}(\omega_3) \hat{E}^*(\omega_1) e^{j\Delta k z}. \quad (4.113)$$

The boundary conditions are now  $E(\omega_2, z=0) = E_0(\omega_2)$  and  $E(\omega_1, z=0) = 0$ . With the ansatz  $\hat{E}(\omega_{1,2}, z) = \hat{E}_0(\omega_{1,2}) e^{\pm\gamma z}$ , we obtain when phase-matched

$$\pm\gamma \hat{E}_0(\omega_1) + j \left\{ \kappa_1 \hat{E}(\omega_3) \right\} \hat{E}_0^*(\omega_2) = 0$$

$$\left\{ -j\kappa_2 \hat{E}^*(\omega_3) \right\} \hat{E}_0(\omega_1) \pm \gamma \hat{E}_0^*(\omega_2) = 0,$$



which enforces

$$\gamma^2 = \kappa_1 \kappa_2 |\hat{E}(\omega_3)|^2. \quad (4.114)$$

In comparison to sum-frequency generation, the fundamental solutions are now hyperbolic functions

$$\hat{E}(\omega_1) = \hat{A} \sinh \gamma z$$

and

$$\hat{E}(\omega_2) = \hat{E}_0(\omega_2) \cosh \gamma z + \hat{B} \sinh \gamma z$$

After back substitution, it again follows  $\hat{B} = 0$  and

$$\hat{A} = -j \sqrt{\frac{\omega_1 n_2}{\omega_2 n_1}} e^{j\phi(\omega_3)} \hat{E}_0^*(\omega_2).$$

Thus it follows

$$I(\omega_1) = (\omega_1/\omega_2) I_0(\omega_2) \sinh^2 \gamma z,$$

$$I(\omega_2) = I_0(\omega_2) \cosh^2 \gamma z.$$

This is indeed a different behavior compared to sum-frequency generation. **Both waves grow!** At first glimpse, it seems that energy is not conserved. Of course, this can not be the case. The required energy actually comes from the pump wave, however, this can not be observed because of the assumption  $\hat{E}(\omega_3) = \text{const.}$  For high conversion, we obtain for the intensity ratio of the low-frequency waves

$$\frac{I(\omega_1)}{I(\omega_2)} = \frac{\omega_1}{\omega_2}.$$

## 4.9 Optical parametric amplification (OPA)

Already for difference-frequency generation it became obvious, that in this frequency mixing process, gain can be achieved. We inspect this process now more generally taking into account losses and without the assumption of perfect phase matching. This will allow us to derive the underlying equations governing the amplification process, its gain bandwidth and associated amplifier noise. We keep the assumption of a constant pump wave at  $\omega_3$ . Then it holds

symmetric  
in 1 and 2

$$\frac{\partial \hat{E}(\omega_1)}{\partial z} + \alpha_1 \hat{E}(\omega_1) = -j\kappa_1 \hat{E}(\omega_3) \hat{E}^*(\omega_2) e^{j\Delta kz} \quad (4.115)$$

$$\frac{\partial \hat{E}(\omega_2)}{\partial z} + \alpha_2 \hat{E}(\omega_2) = -j\kappa_2 \hat{E}(\omega_3) \hat{E}^*(\omega_1) e^{j\Delta kz}. \quad (4.116)$$

We again look for an exponential solution  $\hat{E}(\omega_1) \sim \hat{E}_0(\omega_1) e^{\gamma'z + j\Delta kz/2}$ , and  $\hat{E}(\omega_2) \sim \hat{E}_0(\omega_2) e^{\gamma'z + j\Delta kz/2}$ , with suitable initial values denoted by the index 0. It follows

$$\left\{ \gamma' + \alpha_1 + \frac{j\Delta k}{2} \right\} \hat{E}_0(\omega_1) + \left\{ j\kappa_1 \hat{E}(\omega_3) \right\} \hat{E}_0^*(\omega_2) = 0 \quad (4.117)$$

$$\left\{ j\kappa_2 \hat{E}^*(\omega_3) \right\} \hat{E}_0(\omega_1) + \left\{ \gamma' + \alpha_2 - \frac{j\Delta k}{2} \right\} \hat{E}_0^*(\omega_2) = 0. \quad (4.118)$$

From the determinant condition, it follows

$$\gamma'^2 + \gamma' \{\alpha_1 + \alpha_2\} + \alpha_1\alpha_2 + (\Delta k/2)^2 + (\alpha_2 - \alpha_1) \frac{j\Delta k}{2} - \kappa_1\kappa_2 \left| \hat{E}(\omega_3) \right|^2 = 0$$

$$\Rightarrow \gamma' = -\frac{(\alpha_1 + \alpha_2)}{2} \pm \left\{ \left[ \frac{\alpha_1 - \alpha_2}{2} + \frac{j\Delta k}{2} \right]^2 + \kappa_1\kappa_2 \left| \hat{E}(\omega_3) \right|^2 \right\}^{1/2} \quad (4.119)$$

For the case of phase matching and no losses, we find

$$\gamma^2 = \kappa_1\kappa_2 \left| \hat{E}(\omega_3) \right|^2 \quad (4.120)$$

For the case of equal losses  $\alpha_1 = \alpha_2 = \alpha$ , we obtain

$$\gamma' = -\alpha \pm \left\{ \gamma^2 - (\Delta k/2)^2 \right\}^{1/2} = -\alpha \pm g, \quad (4.121)$$

where

$$g = \left\{ \gamma^2 - (\Delta k/2)^2 \right\}^{1/2}. \quad (4.122)$$

The solutions of the coupled equations (4.115)-(4.116) are of the form

$$\hat{E}(\omega_1, \ell) = e^{-\alpha\ell + j(\Delta k/2)\ell} \left\{ \hat{E}_0(\omega_1) \cosh g\ell + B \sinh g\ell \right\} \quad (4.123)$$

$$\hat{E}(\omega_2, \ell) = e^{-\alpha\ell - j(\Delta k/2)\ell} \left\{ \hat{E}_0(\omega_2) \cosh g\ell + D \sinh g\ell \right\} \quad (4.124)$$

Insertion into the coupled equations (4.115)-(4.116) yields

$$B = -j \frac{\Delta k}{2g} \hat{E}_0(\omega_1) - j \frac{\kappa_1}{g} \hat{E}(\omega_3) \hat{E}_0^*(\omega_2) \quad (4.125)$$

$$D = -j \frac{\Delta k}{2g} \hat{E}_0(\omega_2) - j \frac{\kappa_2}{g} \hat{E}(\omega_3) \hat{E}_0^*(\omega_1). \quad (4.126)$$

Note the symmetry between both solutions. Anyway, the two waves are generally called signal and idler wave. If the optical parametric amplifier (OPA) is seeded by one wave only and if  $\omega_1 \neq \omega_2$ , then this wave is amplified independent of the phase

$$\sinh^2 = \cosh^2 - 1$$

$$\left| \frac{\hat{E}(\omega_1, \ell) e^{\alpha \ell}}{\hat{E}_0(\omega_1)} \right|^2 = \{ \cosh^2 gl + (\Delta k/2g)^2 \sinh^2 gl \} = \frac{1}{g^2} \{ \gamma^2 \cosh^2 gl - (\Delta k/2)^2 \} \quad (4.127)$$

We define the parametric gain as

$$G_1(\ell) = \left| \frac{\hat{E}(\omega_1, \ell) e^{\alpha \ell}}{E_0(\omega_1)} \right|^2 - 1 \Rightarrow G_1(\ell) = (\gamma \ell)^2 \frac{\sinh^2 gl}{(gl)^2} \quad (4.128)$$

Note that for small gain, i.e.,  $\gamma < \Delta k$ , the gain has the form

$$G_1(\ell) = (\gamma \ell)^2 \frac{\sin^2 \left\{ [(\Delta k/2)^2 - \gamma^2]^{1/2} \ell \right\}}{\ell^2 \{ (\Delta k/2)^2 - \gamma^2 \}}, \quad (4.129)$$

$g$  becomes  
imaginary:  
 $\sinh \rightarrow \sin$

that can be further simplified to

$$G_1(\ell) = (\gamma\ell)^2 \frac{\sin^2(\Delta k\ell/2)}{(\Delta k\ell/2)^2}$$

for  $\gamma \ll \Delta k/2$ . In the opposite limit for large gain  $\gamma \gg \Delta k/2$

$$G_1(\ell) = (\gamma\ell)^2 \frac{\sinh^2 g\ell}{(g\ell)^2} \Rightarrow \frac{1}{4} e^{2g\ell}. \quad (4.130)$$

We define the bandwidth of the OPA via

$$\{(\Delta k/2)^2 - \gamma^2\}^{1/2} \ell = \pi, \quad (4.131)$$

that for small gain implies  $\Delta k = 2\pi/\ell$ , as we already found for frequency doubling. In general, we obtain from Eq. (4.131)

$$\frac{2\pi\Delta f}{c} = \Delta k = \left\{1 + \left(\frac{\gamma\ell}{\pi}\right)^2\right\}^{1/2} (2\pi/\ell). \quad (4.132)$$

This means that the bandwidth increases with the gain. For the ratio between the bandwidths at high and low gain, we obtain

$$\frac{\Delta f_{\text{High Gain}}}{\Delta f_{\text{Low Gain}}} = \left\{1 + (\gamma\ell/\pi)^2\right\}^{1/2}. \quad (4.133)$$

Fig. 4.15 shows the gain as function of  $\Delta k\ell$  for various values of  $\gamma\ell$ .

gain bandwidth increases  
with stronger pumping

noncollinear type-I OPAs  
feature ultrabroad  
bandwidth for few-optical-  
cycle pulse generation  
(→ later lecture)

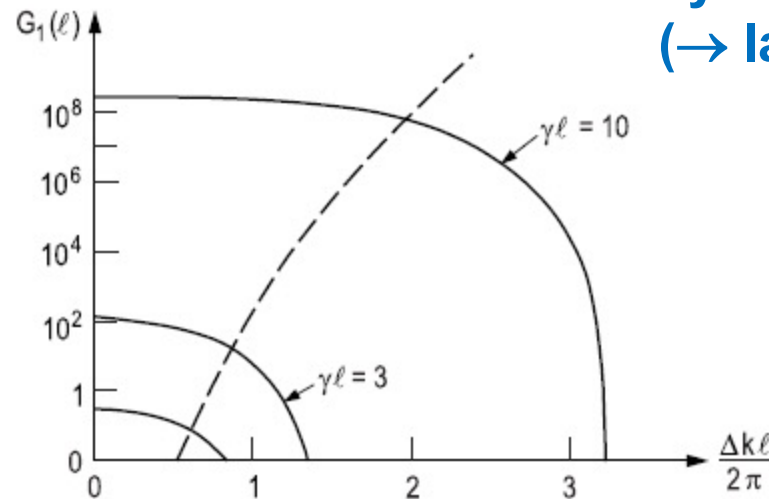


Figure 4.15: Gain of an optical parametric amplifier (OPA) as function of the wave number difference and gain.

## 4.10 Optical parametric oscillation (OPO)

In a single pass through a parametric amplifier medium, which is described by Eqs. (4.123)-(4.126), the signal and idler waves grow when phase-matched ( $\Delta k = 0$ ) according to

$$\hat{E}(\omega_1, \ell) e^{\alpha_1 \ell} = \hat{E}_0(\omega_1) \cosh \gamma \ell - j \frac{\kappa_1}{\gamma} \hat{E}(\omega_3) \hat{E}_0^*(\omega_2) \sinh \gamma \ell \quad (4.134)$$

$$\hat{E}(\omega_2, \ell) e^{\alpha_2 \ell} = \hat{E}_0(\omega_2) \cosh \gamma \ell - j \frac{\kappa_2}{\gamma} \hat{E}(\omega_3) \hat{E}_0^*(\omega_1) \sinh \gamma \ell. \quad (4.135)$$

Most often parametric amplifiers only permit gain for passage in a single direction. In the other direction, only damping occurs

$$\hat{E}'(\omega_1, \ell) = \hat{E}_0(\omega_1) e^{-\alpha_1 \ell}$$

$$\hat{E}'(\omega_2, \ell) = \hat{E}_0(\omega_2) e^{-\alpha_2 \ell}$$

If feedback of the parametric amplifier is realized by means of a Fabry-Pérot resonator and if the field is larger after a round trip than at the beginning, so the amplifier is turned into a self-starting oscillator. The threshold condition is that the losses must equal the gain

$$\hat{E}'_0(\omega_1) = \hat{E}_0(\omega_1)$$

$$\hat{E}'_0(\omega_2) = \hat{E}_0(\omega_2)$$

or inserted into Eqs. (4.134)-(4.135) it follows with  $e^{-2\alpha \ell} \sim 1 - 2\alpha \ell$

$$\frac{\hat{E}_0(\omega_1)}{1 - 2\alpha_1\ell} = \hat{E}_0(\omega_1) \cosh \gamma\ell - j \frac{\kappa_1}{\gamma} \hat{E}(\omega_3) \hat{E}_0^*(\omega_2) \sinh \gamma\ell$$

$$\frac{\hat{E}_0^*(\omega_2)}{1 - 2\alpha_2\ell} = \hat{E}_0^*(\omega_2) \cosh \gamma\ell + j \frac{\kappa_2}{\gamma} \hat{E}^*(\omega_3) \hat{E}_0(\omega_1) \sinh \gamma\ell.$$

Again the solution of this equation system is only non-zero, if the determinant of the coefficient matrix vanishes, i.e.,

$$\left[ \cosh \gamma\ell - \frac{1}{1 - 2\alpha_1\ell} \right] \left[ \cosh \gamma\ell - \frac{1}{1 - 2\alpha_2\ell} \right] = \frac{\kappa_1\kappa_2}{\gamma^2} \left| \hat{E}(\omega_3) \right|^2 \sinh^2 \gamma\ell = \sinh^2 \gamma\ell$$

so that

$$1 - \cosh \gamma\ell \left( \frac{1}{1 - 2\alpha_1\ell} + \frac{1}{1 - 2\alpha_2\ell} \right) + \left( \frac{1}{1 - 2\alpha_2\ell} \right) \left( \frac{1}{1 - 2\alpha_1\ell} \right) = 0 \quad (4.136)$$

or

$$\cosh \gamma\ell = 1 + \frac{2\alpha_1\alpha_2\ell^2}{1 - \alpha_1\ell - \alpha_2\ell}. \quad (4.137)$$

For  $\alpha_1 \approx \alpha_2 \approx \alpha$  and the case of small losses or small gain  $\alpha\ell$ ,  $\gamma\ell \ll 1$ , it follows

$$(\gamma\ell)^2 \approx 4\alpha\ell. \quad \cosh x = 1 + \frac{x^2}{2!} \quad (4.138)$$

One distinguishes between doubly resonant parametric oscillators (DROs) and singly resonant ones (SRO). In the first case, both signal and idler waves are resonant, in the second case only the signal wave. The threshold for SROs is many times higher than for DRO. Nevertheless, most OPOs are singly resonant, because it is much more difficult to operate a DRO.