Universität Hamburg
Physics Department
Ultrafast Sources
SoSe 2021

Problem Set 3

Issued: April 27, 2021 Due: May 4, 2021

*Instruction: Please submit your answers to the problemset in digital form (e.g. " *.pdf " or similar) to your tutor. If you collaborated on a problem, please note down their names as well. When submitting your answer to a numerical problem, please attach your code as well. The code should be executable without errors and ideally on button press.*

Problem 3.1: The Nonlinear Schrödinger Equation (NSE) and optical solitons (15 points in total)

The Nonlinear Schrödinger Equation is written as follows, (here we assume $D_2 < 0$)

$$\frac{\partial A(z,t)}{\partial z} = jD_2 \frac{\partial^2 A(z,t)}{\partial t^2} - j\delta |A(z,t)|^2 A(z,t).$$

As a reminder, this equation is obtained in optics from Maxwell's equations applied to a non-magnetic class of materials that exhibit primarily a four-wave mixing nonlinearity, i.e. where the polarization is proportional to the cube of the electric field. It dictates the physics of a large area of ultrafast optics to a good approximation (fiber optics, waveguide optics, …). We will first look at the analytical solutions of the NSE.

(a) Show by using the following transform

$$\xi = \frac{z}{L_D}, \tau = \frac{t}{\tau_0},$$

$$L_D = \frac{\tau_0^2}{|D_2|}, u = \sqrt{\frac{\delta}{|D_2|}}\,\tau_0 A$$

that the NSE can be rewritten in the normalized form

$$\frac{\partial u(\xi,\tau)}{\partial \xi} = -j\frac{\partial^2 u(\xi,\tau)}{\partial \tau^2} - j|u(\xi,\tau)|^2 u(\xi,\tau).$$ **(5 points)**

(b) Prove that $u(\xi,\tau) = \mathrm{sech}(\frac{\tau}{\sqrt{2}}) \cdot \exp(-\mathrm{j}\frac{\xi}{2})$ is a solution to the normalized NSE. **(5 points)**

(c) Ultrafast optical solitons have been generated in optical fiber SMF-28 at $\lambda = 1.55\ \mu\mathrm{m}$. Using pulses from mode-locked lasers, hyperbolic-secant pulses with $\tau_0 = 4\ \mathrm{ps}$ were obtained after 700 m propagation in the optical fiber. The nonlinear index coefficient of the fiber is $\mathrm{n}_2 = 3 \times 10^{-20}\mathrm{m}^2/\mathrm{W}$. Additionally, suppose that the effective area is given by the fiber core area, which has a diameter of 10 microns. Calculate the corresponding peak intensity and the corresponding peak power of the soliton. **(5 points)**
Hint: the relationship between fiber dispersion $D(\lambda)$ characterized by fiber communication community and group velocity dispersion $\beta_2\ or\ D_2$ can be found at RP-Photonics website, http://www.rp-photonics.com/group_velocity_dispersion.html .

Problem 3.2: The Split-Step Fourier method (15 points in total)

The normalized Nonlinear Schrödinger Equation (NSE) can be numerically solved using the Split-Step Fourier transform. Firstly, the NSE can be rewritten as

$$\frac{\partial u(\xi,\tau)}{\partial \xi} = \left(\hat{D} + \hat{N}\right)u(\xi,\tau)$$

as the simultaneous action of a dispersion operator $\hat{D} = -j\frac{\partial^2 u(\xi,\tau)}{\partial \tau^2}$ (assume the dispersion is negative), and a nonlinear operator $\hat{N} = -j|u(\xi,\tau)|^2$. Over a very small distance $\Delta\xi$, both operators are assumed to stay constant, thus allowing one to write

$$u(\xi + \Delta\xi,\tau) = e^{(\hat{D}+\hat{N})\Delta\xi}u(\xi,\tau).$$

By recursive application of this equation from the initial condition, one solves the NSE. However, although mathematically well defined, the operators $\hat{D}$ and $\hat{N}$ do not commute. This implies that the exponential can only be represented faithfully by an infinite product of exponential of their operators (Baker–Campbell–Hausdorff formula - Wikipedia). Numerically, this product can be truncated at any given order by losing some accuracy, and here we propose to implement the symmetric split-step equation:

$$u(\xi + \Delta\xi,\tau) = e^{\frac{1}{2}\hat{D}\Delta\xi}e^{\hat{N}\Delta\xi}e^{\frac{1}{2}\hat{D}\Delta\xi}u(\xi,\tau).$$

One can show that iterative application of this propagation step only leads to an error of order $\Delta\xi^3$. Since the linear operator can be easily applied in the Fourier domain and the nonlinear operator (self-phase modulation only) in the time domain, one can simulate the NSE over one propagation step $\Delta\xi$ using the following algorithm

$$u(\xi + \Delta\xi,\tau) = F^{-1}[e^{\frac{1}{2}j\Omega^2\Delta\xi}F[e^{-j|u(\xi,\tau)|^2\Delta\xi}F^{-1}[e^{\frac{1}{2}j\Omega^2\Delta\xi}F[u(\xi,\tau)]]]].$$

where $\Omega = \omega - \omega_0$ is the difference between the considered frequency and the carrier frequency.

(a) Dispersion effect on pulse evolution. The electric field of an unchirped Gaussian pulse is written as $E(0,t) = \mathrm{A}_0 \cdot \exp(-t^2/2\tau_0^2) \cdot \exp(i\omega_0 t)$, where $\tau_0 = 100\ fs$ and the center wavelength of such pulse locates at 1550 nm. We let the pulse propagate inside optical fiber SMF-28 and we do NOT consider the nonlinear effect. Moreover, we ONLY consider second order dispersion effect. What is the characteristic dispersion length L_D? Numerically compute and plot the intensity distribution of such pulse at 0, $1L_D$, $2L_D$, $3L_D$ and $4L_D$ (You can use a normalized scale. Indicate proper axis labels, i.e. t/τ_0). Is the pulse positively chirped or negatively chirped after propagation? **(5 points)**

(b) Write a program to simulate the normalized NSE. You can get inspiration from the following pseudo-code.

```
### NSE simulation

# Simulation parameters
time = linspace(t_min, t_max, t_pts)
omega = linspace( -… defined by time…- )
z = linspace(0, z_max, z_pts)
dz = mean(diff(z))

data_storage = zeros(t_pts, z_pts)

# Initial condition
field = some_function(time)
data_storage[:, 0] = field

# Dispersion operator
def dispersion(field, omega, dist)
        new_field = fft(field)
        new_field *= exp(j*omega**2*dist)
        return ifft(new_field)

# Nonlinear operator
def nonlinearity(field, dist)
        return field*exp(-j*abs(field)**2*dist)

# Simulation
for i in length z_pts-1
        field = dispersion(field, omega, dz/2)
        field = nonlinearity(field, dz)
        field = dispersion(field, omega, dz/2)

        data_storage[:,  i+1] = field

# Plots
# Use some mesh plots both in frequency and time domain for best representation
```

Run the simulation for the following initial pulses

$$u(0,\tau) = N \cdot \mathrm{sech}(\tau/\sqrt{2})$$

for N = 1,2 and 3. Plot the pulse shape (in the time domain) and corresponding amplitude spectra (in the frequency domain) as a function of propagation distance. What happens to the soliton if you change the sign of the dispersion? **(10 points)**