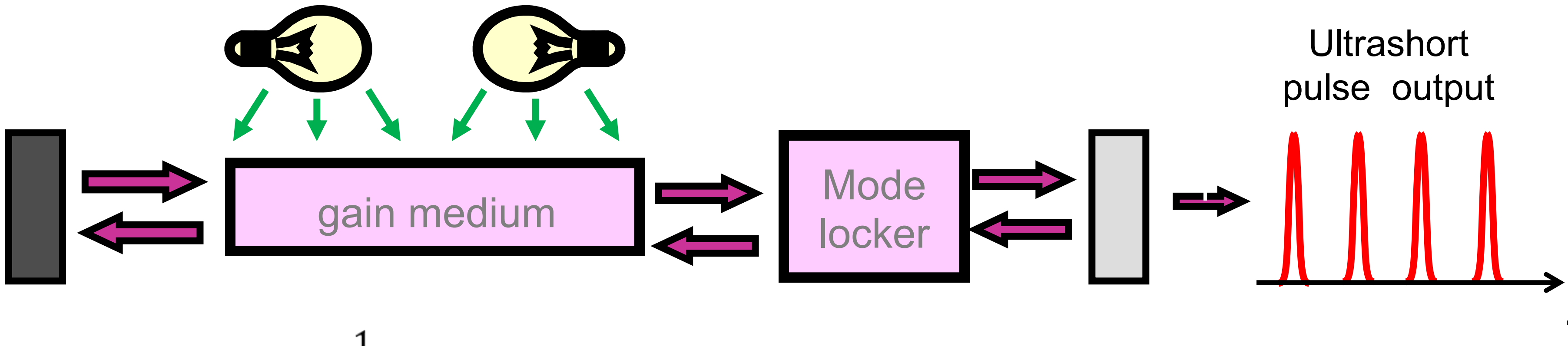
# Ultrafast Optical Physics II (SoSe 2021)
# Lecture 23, June 24

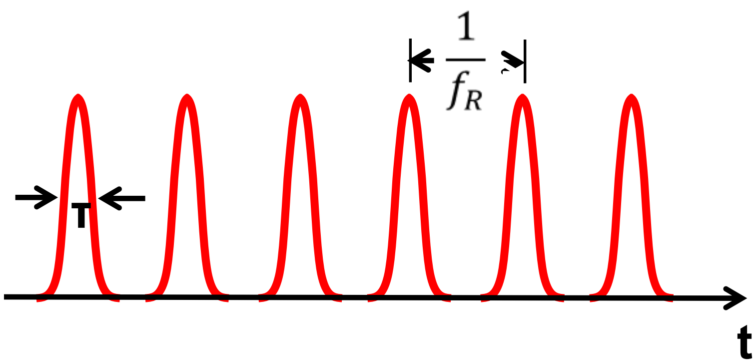
## Pulse Characterization

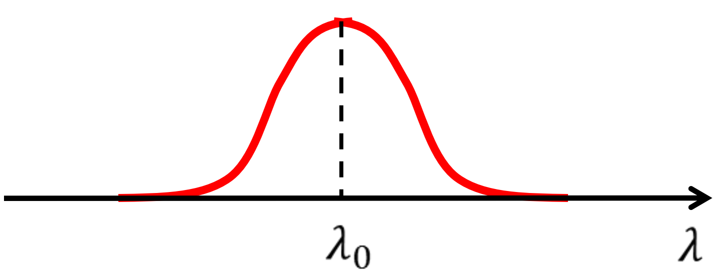
- Intensity Autocorrelation
- Interferometric Autocorrelation (IAC)
- Frequency Resolved Optical Gating (FROG)
- Spectral Shearing Interferometry for Direct Electric Field Reconstruction (SPIDER)
- 2D-Spectral Shearing Interferometry (2DSI)

**Follows partly Rick Trebino's lecture at Georgia Tech**

# Ultrafast laser: the 4th element—mode locker

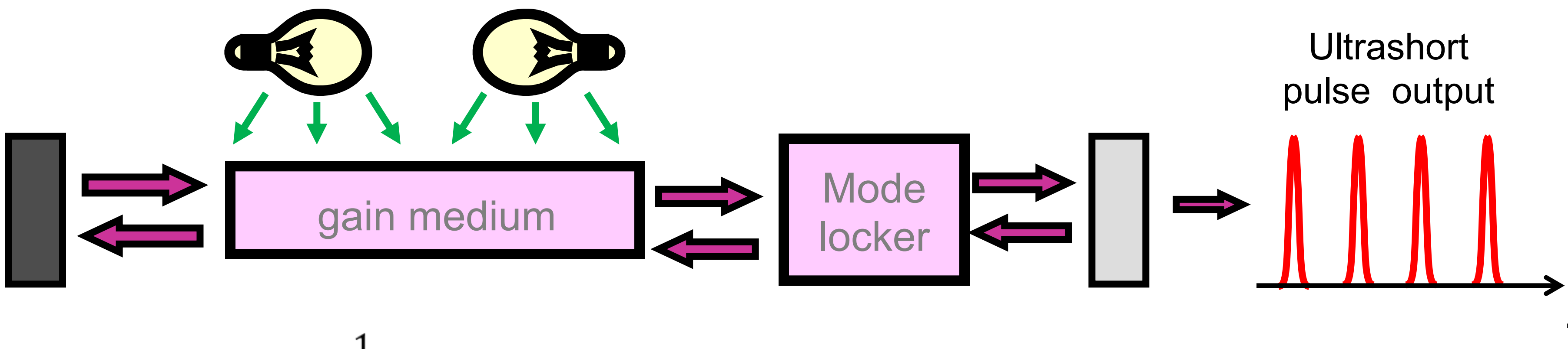


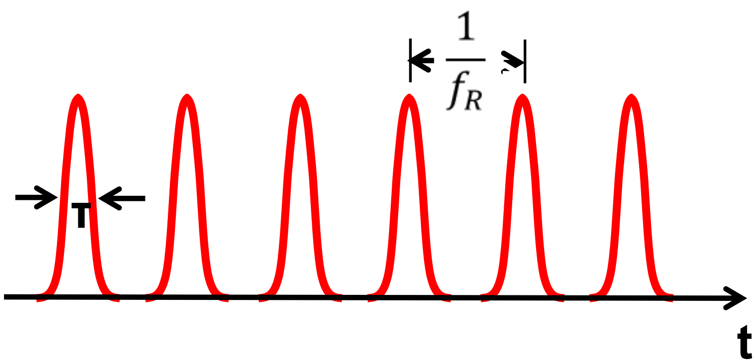



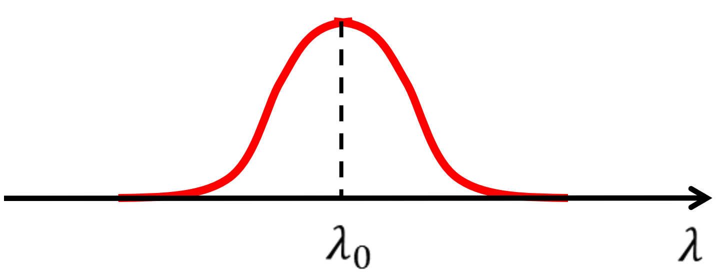




- Pulse duration $T$ (fs – ps)
- Pulse energy $E$ (pJ – mJ)
- Peak power $P_p$ (1 kW – 1 PW)
  $P_p \approx E/T$ (e.g., 1 nJ, 100 fs pulse leads to 10 kW peak power.)
- Repetition rate $f_R$ (10 MHz – 10 GHz)
- Average power $P$ (10 mW – 100 W)
  $P = E \times f_R$ (e.g., 1 nJ, 100 MHz rep-rate laser produces 100 mW average power.
- Center wavelength $\lambda_0$ (700 nm – 2000 nm)

# Measurement of pulse quantities using 'meters'




| Physical quantity | Measuring device |
|---|---|
| Average power | Power meter |
| Repetition rate | RF spectrum analyzer |
| Optical spectrum | Optical spectrum analyzer (OSA) |
| Optical pulse | Pulse meter (?) |

# What information do we need to fully determine an optical pulse?

A laser pulse has the time-domain electric field:

$$E(t) \sim \mathrm{Re}\,\{\mathbf{I(t)}^{1/2} \exp[j\omega_0 t - j\boldsymbol{\phi}\mathbf{(t)}\,]\,\}$$

Intensity

Phase

Equivalently, vs. frequency:

(neglecting the negative-frequency component)

$$\tilde{E}(\omega) \sim \mathbf{I(\omega-\omega_0)}^{1/2} \quad \exp[-j\boldsymbol{\varphi}\,\boldsymbol{(\omega-\omega_0)}]$$

Spectrum

Spectral Phase

<u>Can be measured by an optical spectrum analyzer.</u>

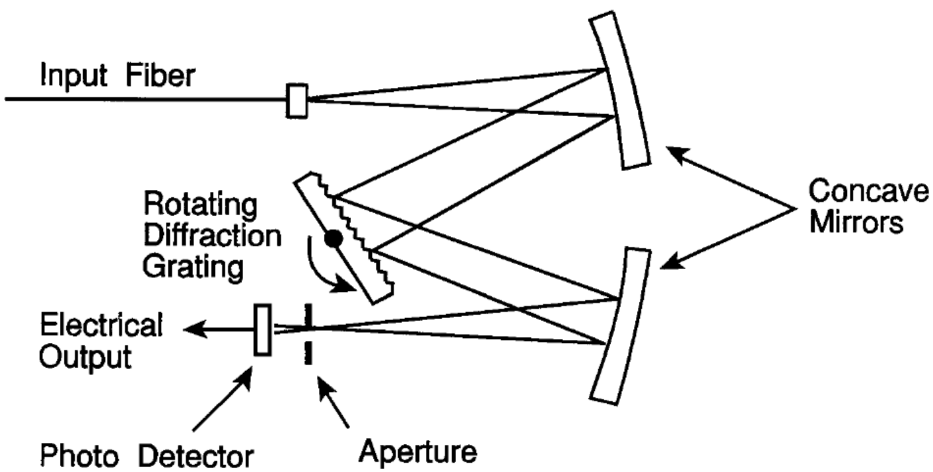
# Spectrum measurement by optical spectrum analyzer

$$E(\omega) = \mathrm{Re}\left\{ I(\omega-\omega_0)^{1/2} \ \exp\left[-j\varphi(\omega-\omega_0)\right] \right\}$$

Spectrum

Spectral Phase

Can be measured by an optical spectrum analyzer.



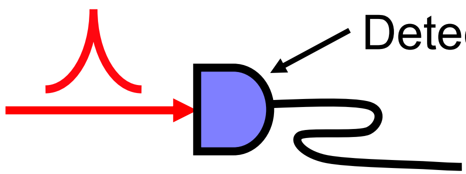


1. Spectral phase information is missing in the measurement.
2. Transform-limited pulse can be calculated from the measured spectrum.

# Measure pulse in time domain using photo-detectors

Photo-detectors are devices that emit electrons in response to photons.

Examples: Photo-diodes, Photo-multipliers



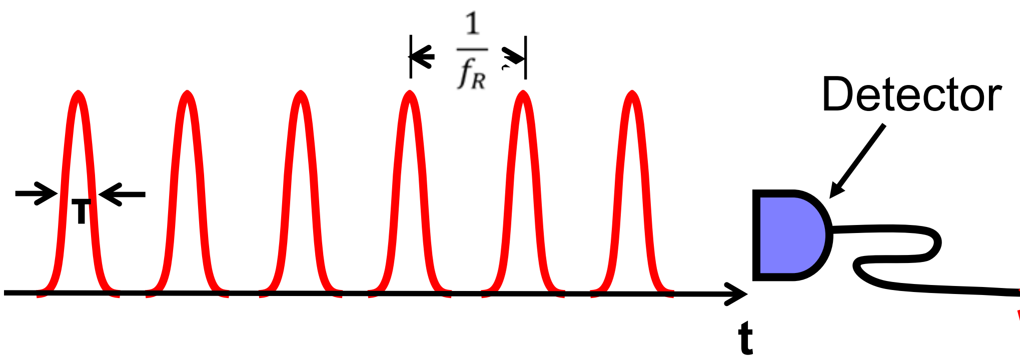

Detectors have very **slow** rise and fall times: ~ 1 nanosecond.

As far as we're concerned, detectors have **infinitely slow** responses.
They measure the time integral of the pulse intensity from $-\infty$ to $+\infty$:

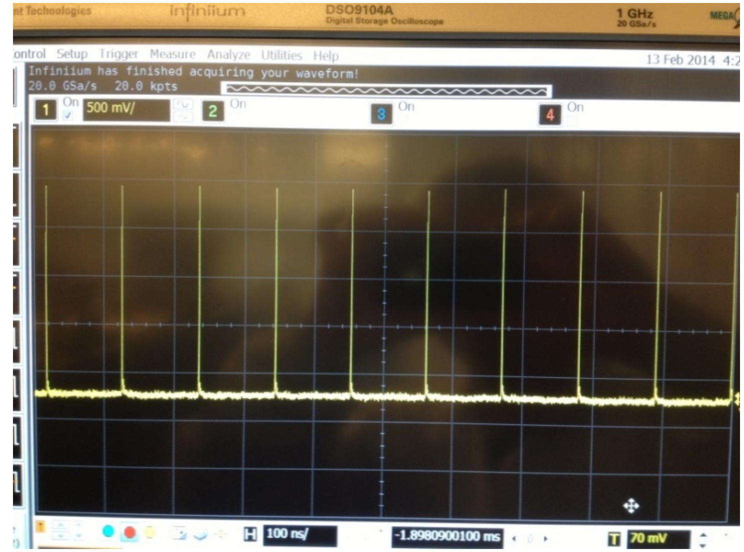
$$V_{detector} \propto \int_{-\infty}^{\infty} |E(t)|^2 \, dt$$

The detector output voltage is proportional to the pulse energy.
By themselves, detectors tell us little about a pulse.

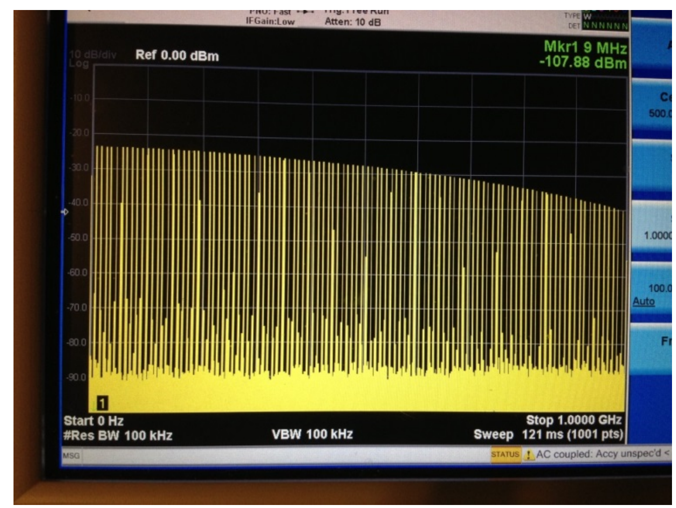
# But photo-detector can ‘see’ the pulse train



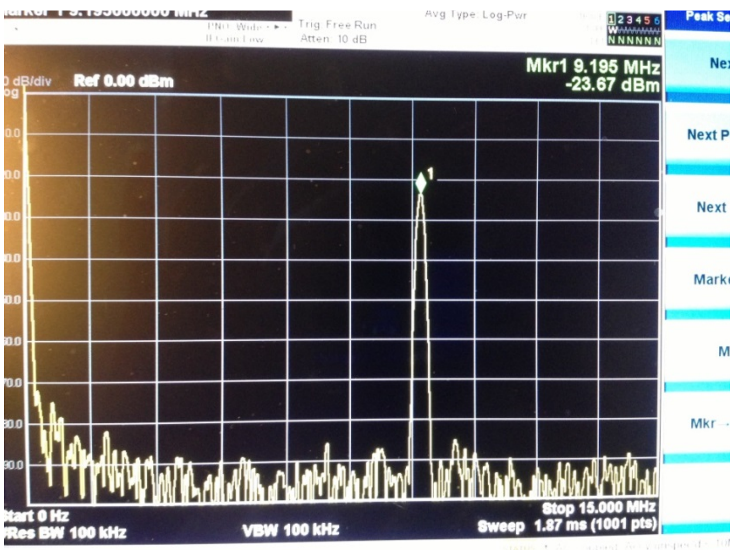

Optical pulse train under diagnosis




Pulse train measured by oscilloscope

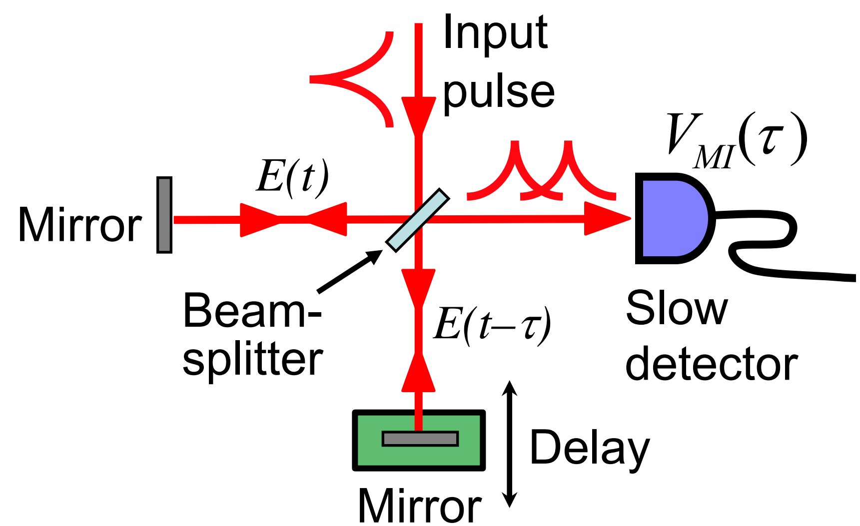



Pulse train measured by RF spectrum analyzer




Zoom in to see the RF spectrum of at the rep-rate frequency.

# Pulse measurement by field autocorrelation




$$V_{MI}(\tau) \propto \int_{-\infty}^{\infty} \left| E(t) - E(t-\tau) \right|^2 dt$$

$$= \int_{-\infty}^{\infty} \left| E(t) \right|^2 + \left| E(t-\tau) \right|^2 - 2\,\mathrm{Re}[E(t)E^*(t-\tau)]\; dt$$

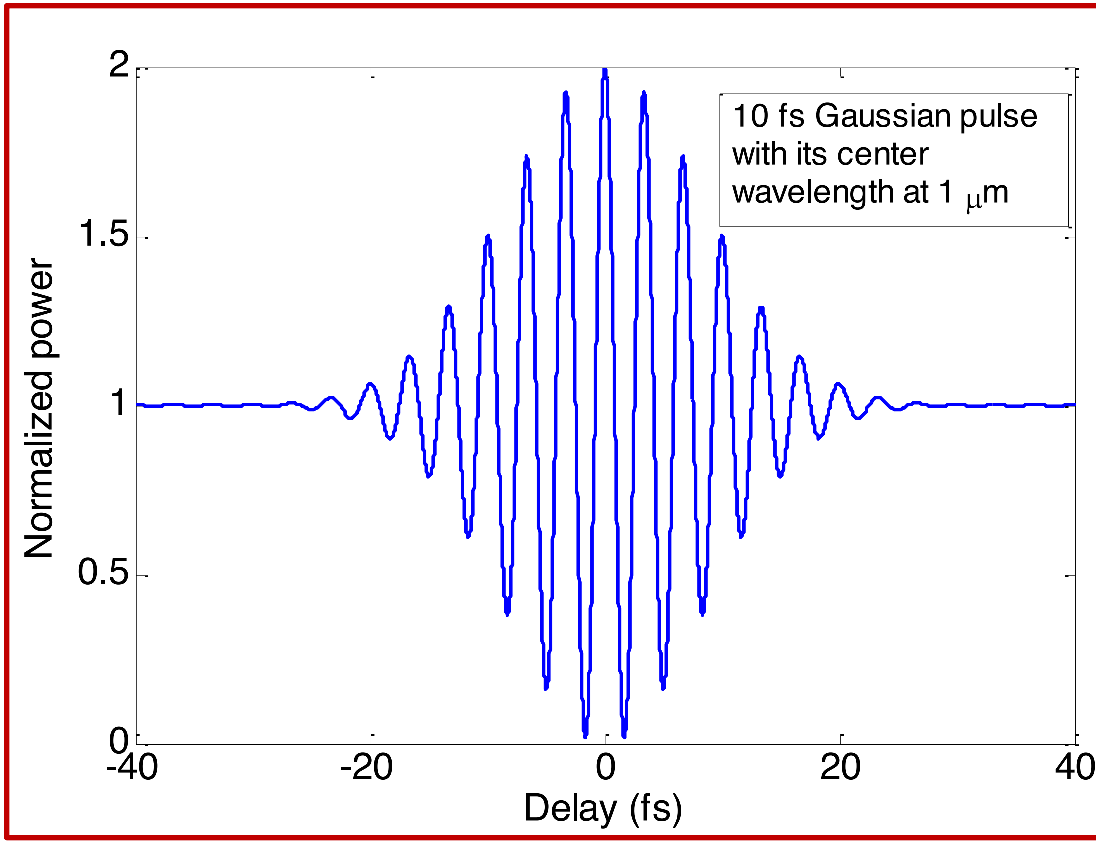
$$\Rightarrow \quad V_{MI}(\tau) \propto \underbrace{2\int_{-\infty}^{\infty} \left| E(t) \right|^2 dt}_{\propto \text{ Pulse energy}} - \underbrace{2\,\mathrm{Re}\int_{-\infty}^{\infty} E(t)E^*(t-\tau)\; dt}_{\text{Field autocorrelation}}$$

$$\mathrm{Re}\int_{-\infty}^{\infty} E(t)E^*(t-\tau)dt = \mathrm{Re}\,F^{-1}[E(\omega)E^*(\omega)] = \mathrm{Re}\,F^{-1}[I(\omega)]$$

<u>**Field autocorrelation measurement is equivalent to measuring the spectrum.**</u>

# Comments on field correlation measurement

The information obtained from measuring electric field correlation and measuring the optical power spectrum is identical.




The correlation time is roughly the inverse of the optical bandwidth.

Field correlation measurement gives no information about the spectral phase.

Field correlation measurement cannot distinguish a transform-limited pulse from a longer chirped pulse with the same bandwidth.

Coherent ultrashort pulse and continuous-wave incoherent light (i.e., noise) with the same optical spectra give the same result.

# How to measure both pulse intensity profile and the phase?

Result: Using only time-independent, linear filters, complete characterization of a pulse is **NOT** possible with a slow detector.

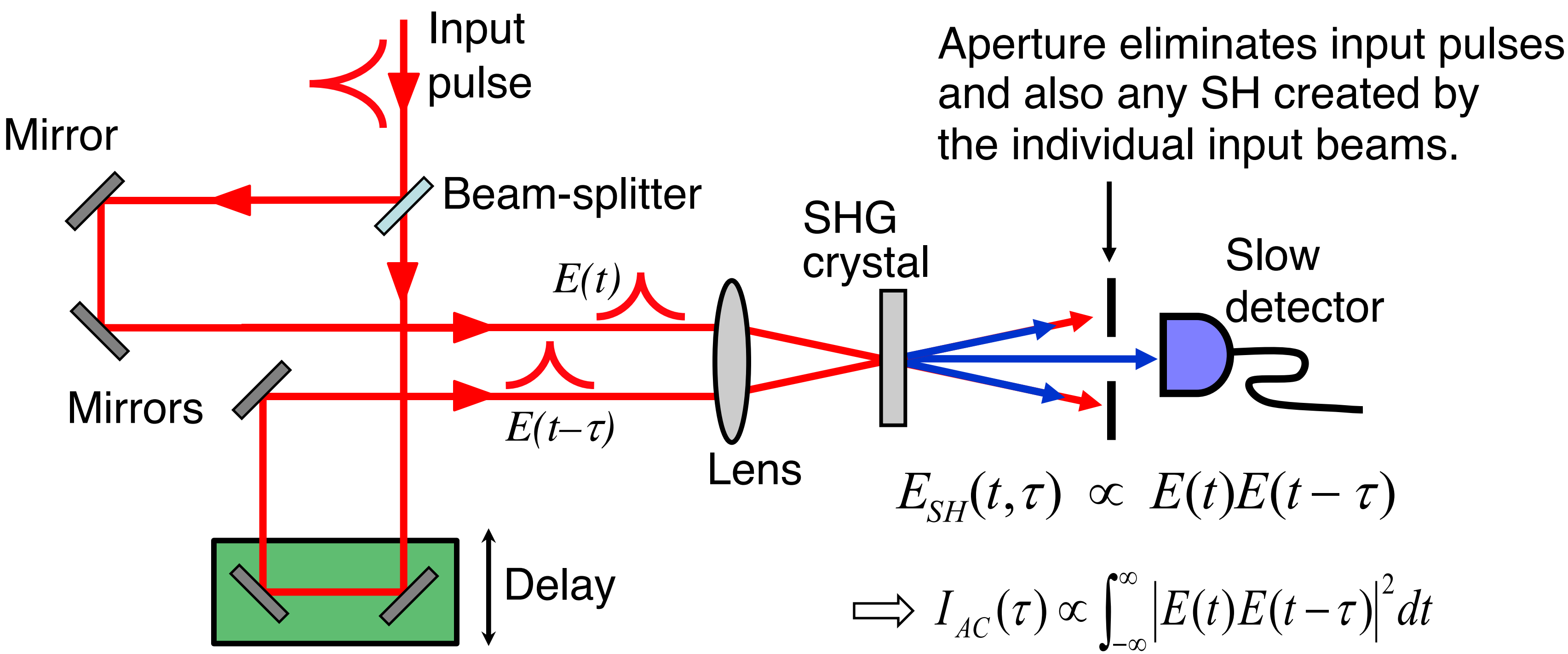
Translation: If you don't have a detector or modulator that is fast compared to the pulse width, you **CANNOT** measure the pulse intensity and phase with only linear measurements, such as a detector, interferometer, or a spectrometer.

V. Wong & I. A. Walmsley, Opt. Lett. **19**, 287-289 (1994)
I. A. Walmsley & V. Wong, J. Opt. Soc. Am B, **13**, 2453-2463 (1996)

**We need a shorter event, and we don't have one.**
**But we do have the pulse itself, which is a start.**
**And we can devise methods for the pulse to gate itself using optical nonlinearities.**

# Background-free intensity autocorrelation

Crossing beams in an second-harmonic generation (SHG) crystal, varying the delay between them, and measuring the second-harmonic (SH) pulse energy vs. delay yields the **Intensity Autocorrelation**:

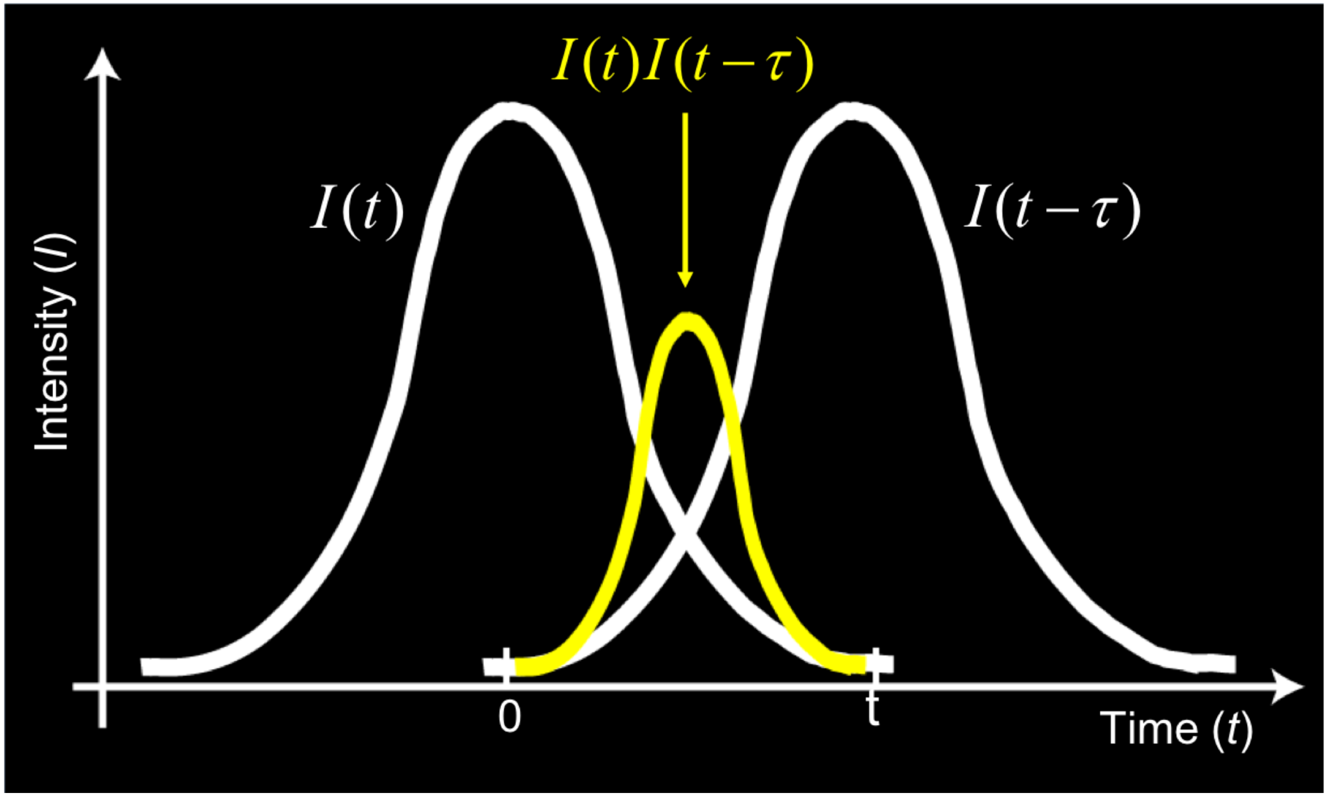



$$E_{SH}(t,\tau) \propto E(t)E(t-\tau)$$

$$\Rightarrow I_{AC}(\tau) \propto \int_{-\infty}^{\infty} \left|E(t)E(t-\tau)\right|^2 dt$$

The Intensity Autocorrelation: $$\boxed{I_{AC}(\tau) \propto \int_{-\infty}^{\infty} I(t)I(t-\tau)dt}$$

# Background-free intensity autocorrelation




- The intensity autocorrelation is only nonzero when the pulses overlap.
- The Intensity Autocorrelation is always symmetrical with respect to delay, thus cannot tell anything about the direction of time of a pulse

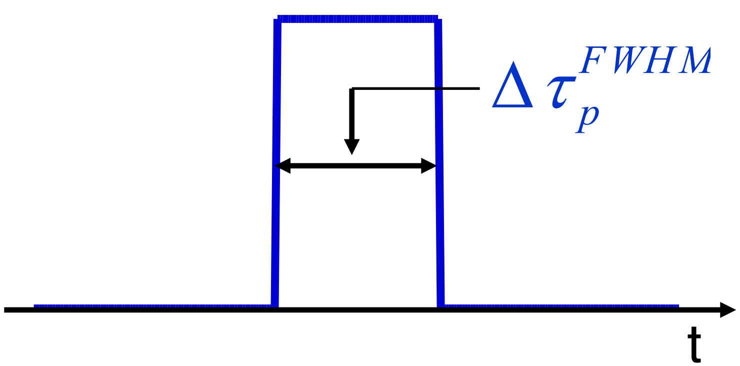
# Square pulse and its autocorrelation

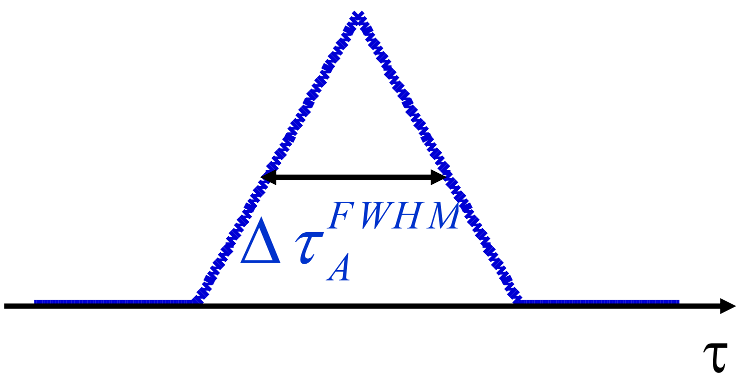
Pulse

$$I(t) = \begin{cases} 1; \ |t| \le \Delta\tau_p^{FWHM}/2 \\ 0; \ |t| > \Delta\tau_p^{FWHM}/2 \end{cases}$$

Autocorrelation

$$A^{(2)}(\tau) = \begin{cases} 1 - \left|\dfrac{\tau}{\Delta\tau_A^{FWHM}}\right|; \ |\tau| \le \Delta\tau_A^{FWHM} \\ 0; \ |\tau| > \Delta\tau_A^{FWHM} \end{cases}$$



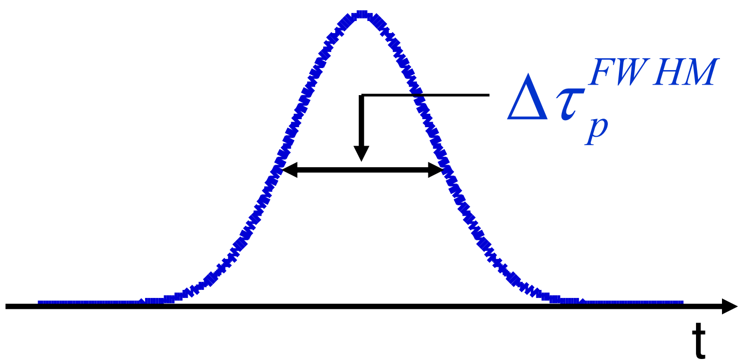




$$\Delta\tau_A^{FWHM} = \Delta\tau_p^{FWHM}$$

# Gaussian pulse and its autocorrelation

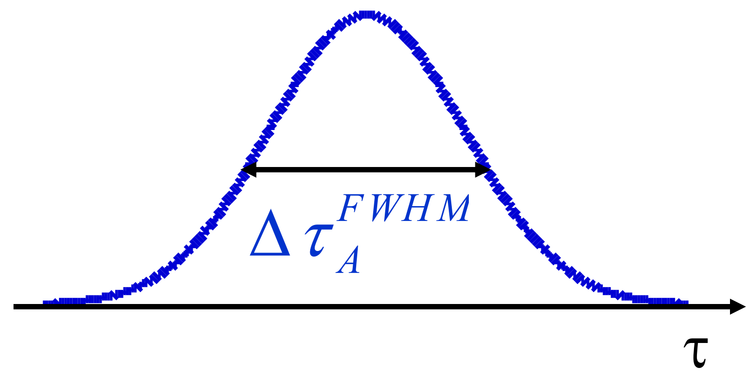
Pulse

$$I(t) = \exp\left[-\left(\frac{2\sqrt{\ln 2}\,t}{\Delta\tau_p^{FWHM}}\right)^2\right]$$




Autocorrelation

$$A^{(2)}(\tau) = \exp\left[-\left(\frac{2\sqrt{\ln 2}\,\tau}{\Delta\tau_A^{FWHM}}\right)^2\right]$$




$$\Delta\tau_A^{FWHM} = 1.41\,\Delta\tau_p^{FWHM}$$

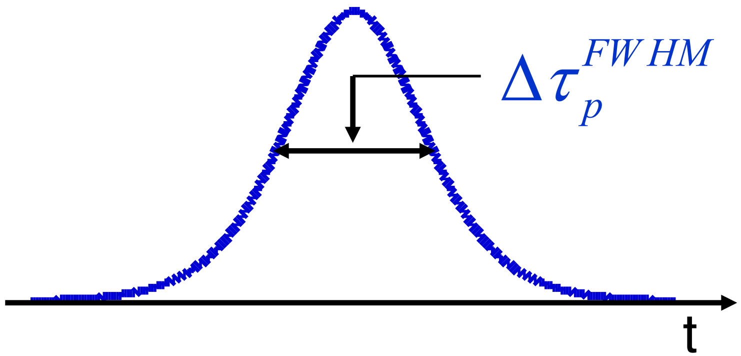
# Sech$^2$ pulse and its autocorrelation

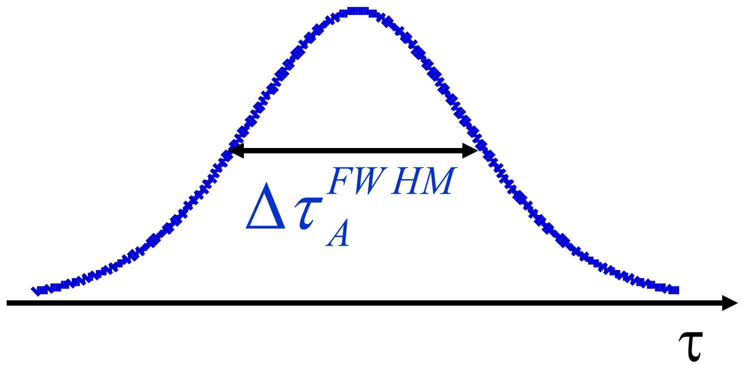
Pulse

$$I(t) = \operatorname{sech}^2\left[\frac{1.7627t}{\Delta t_p^{FWHM}}\right]$$

Autocorrelation

$$A^{(2)}(\tau) = \frac{3}{\sinh^2\left(\frac{2.7196\tau}{\Delta\tau_A^{FWHM}}\right)}\left[\frac{2.7196\tau}{\Delta\tau_A^{FWHM}}\coth\left(\frac{2.7196\tau}{\Delta\tau_A^{FWHM}}\right) - 1\right]$$




$$\Delta\tau_A^{FWHM} = 1.54\,\Delta\tau_p^{FWHM}$$

Theoretical models for passively mode-locked lasers often predict sech$^2$ pulse shapes.

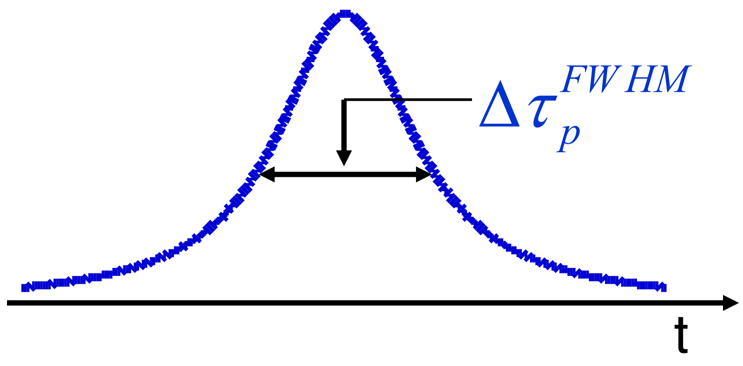
# Lorentzian Pulse and Its Autocorrelation

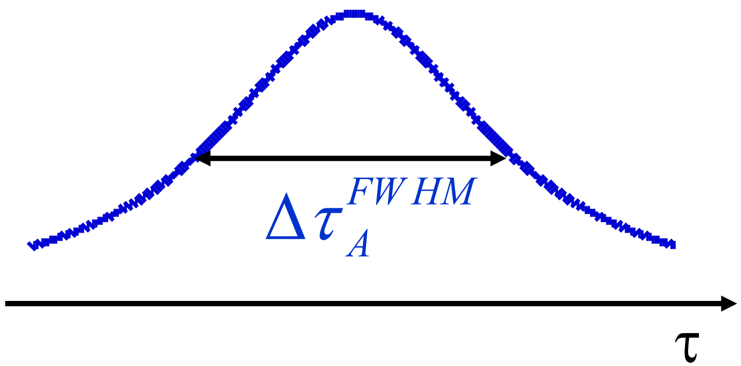
Pulse

Autocorrelation

$$I(t) = \frac{1}{1 + \left(2t / \Delta\tau_p^{FWHM}\right)^2}$$

$$A^{(2)}(\tau) = \frac{1}{1 + \left(2\tau / \Delta\tau_A^{FWHM}\right)^2}$$




$$\Delta\tau_A^{FWHM} = 2.0\,\Delta\tau_p^{FWHM}$$

# Properties of intensity autocorrelation

1) It is always symmetric, and assumes its maximum value at $\tau = 0$.

$$I_{AC}(\tau) \propto \int_{-\infty}^{\infty} I(t)I(t-\tau)dt \qquad I_{AC}(\tau) = I_{AC}(-\tau)$$

2) Width of the correlation peak gives information about the pulse width.

3) Pulse phase information is missing from the background-free Intensity autocorrelation.

4) Intensity autocorrelation trace is broader than the pulse itself. To get the pulse duration, it is necessary to assume a pulse shape, and to use the corresponding deconvolution factor.

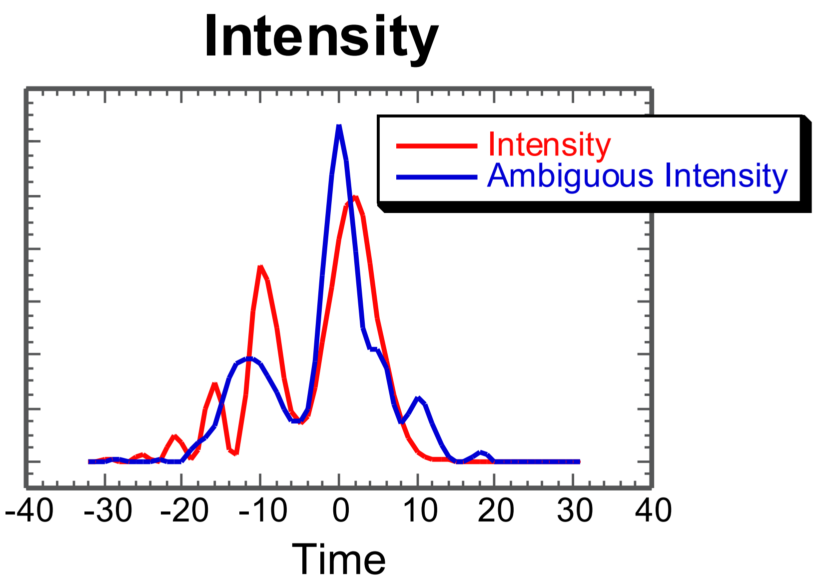
4) For short pulses, very thin crystals must be used to guarantee enough phase-matching bandwidth. This reduces the efficiency and hence the sensitivity of the device.

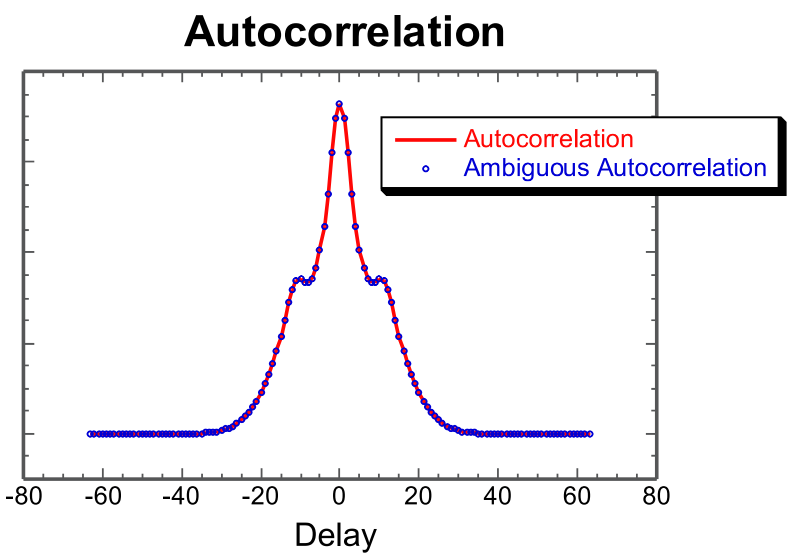
5) Conversion efficiency must be kept low, or distortions due to “depletion” of input light fields will occur.

6) The Intensity autocorrelation is **not** sufficient to determine the intensity profile.

# Autocorrelations of more complex intensities

Autocorrelations nearly always have considerably less structure than the corresponding intensity.

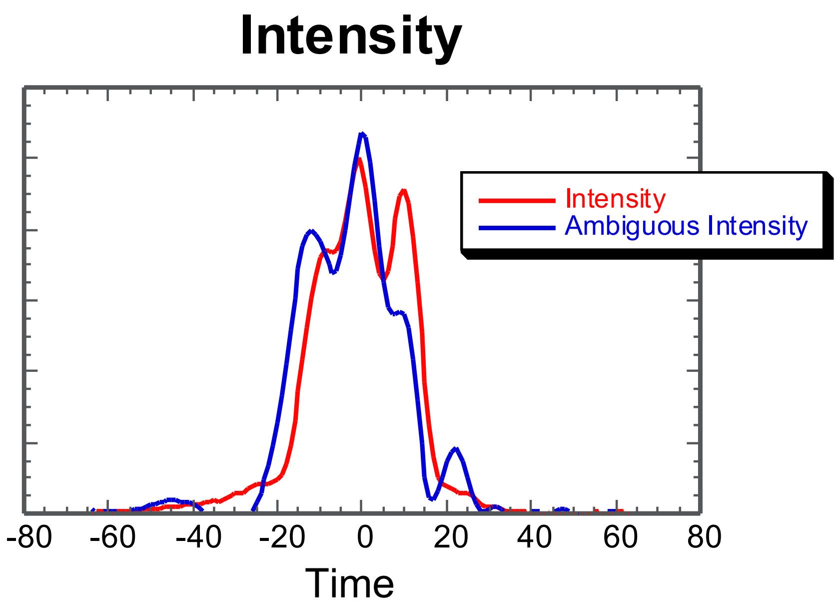


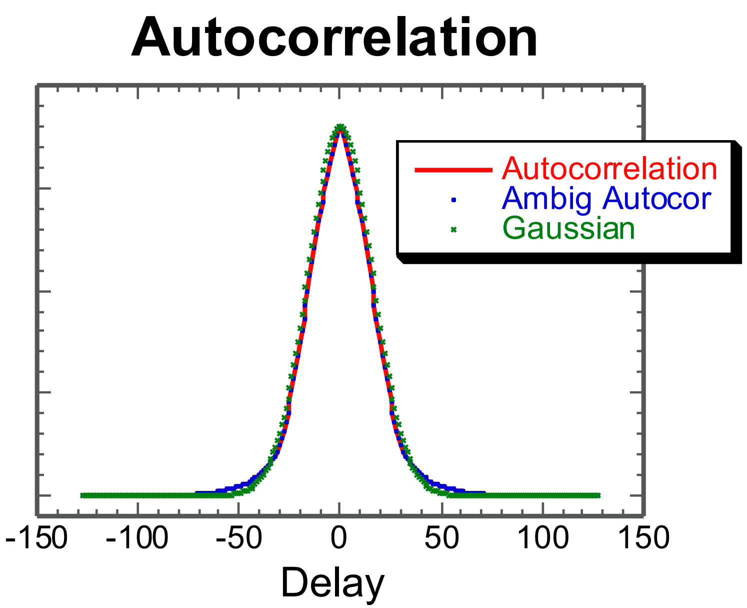




An autocorrelation typically corresponds to more than one intensity. Thus the autocorrelation does not uniquely determine the intensity.

# Even nice autocorrelations have ambiguities

These complex intensities have nearly Gaussian autocorrelations.

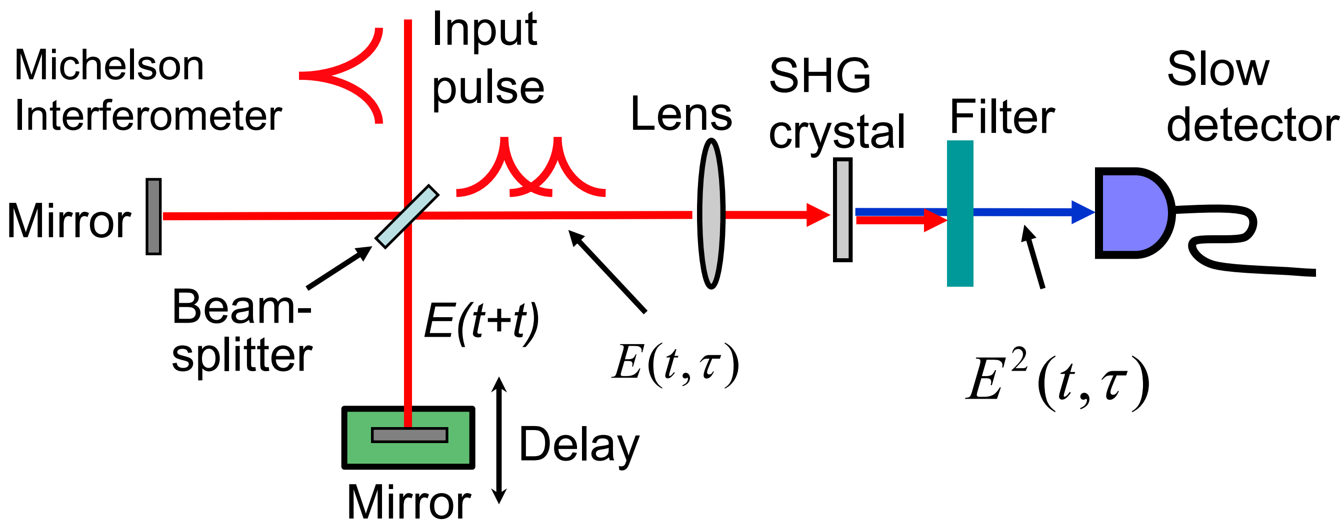



Conclusions drawn from an autocorrelation are unreliable.

# Interferometric autocorrelation (IAC)

What if we use a **collinear beam geometry**, and allow the autocorrelator signal light to interfere with the SHG from each individual beam?




Developed by J-C Diels

Diels and Rudolph, Ultrashort Laser Pulse Phenomena, Academic Press, 1996.

$$IA^{(2)}(\tau) \equiv \int_{-\infty}^{\infty} \left| [E(t) - E(t-\tau)]^2 \right|^2 dt$$

$$IA^{(2)}(\tau) \equiv \int_{-\infty}^{\infty} \left| \underbrace{E^2(t) + E^2(t-\tau)}_{\text{New terms}} - 2\underbrace{E(t)E(t-\tau)}_{\text{Autocorrelation term}} \right|^2 dt$$

# Some simple math

$$IA^{(2)}(\tau) \equiv \int_{-\infty}^{\infty} \left[ E^2(t) + E^2(t-\tau) - 2E(t)E(t-\tau) \right] \left[ E^{*2}(t) + E^{*2}(t-\tau) - 2E^*(t)E^*(t-\tau) \right] dt$$

$$\begin{aligned}
IA^{(2)}(\tau) = \int_{-\infty}^{\infty} \Big\{ & \left| E^2(t) \right|^2 + E^2(t) E^{*2}(t-\tau) - 2E^2(t) E^*(t) E^*(t-\tau) + \\
& E^2(t-\tau) E^{*2}(t) + \left| E^2(t-\tau) \right|^2 - 2E^2(t-\tau) E^*(t) E^*(t-\tau) + \\
& -2E(t)E(t-\tau)E^{*2}(t) - 2E(t)E(t-\tau)E^{*2}(t-\tau) + 4\left| E(t) \right|^2 \left| E(t-\tau) \right|^2 \Big\} \, dt
\end{aligned}$$

$$\begin{aligned}
= \int_{-\infty}^{\infty} \Big\{ & I^2(t) + E^2(t) E^{*2}(t-\tau) - 2I(t)E(t)E^*(t-\tau) + \\
& E^2(t-\tau) E^{*2}(t) + I^2(t-\tau) - 2I(t-\tau)E^*(t)E(t-\tau) + \\
& -2I(t)E(t-\tau)E^*(t) - 2I(t-\tau)E(t)E^*(t-\tau) + 4I(t)I(t-\tau) \Big\} \, dt
\end{aligned}$$

Where: $I(t) \equiv \left| E(t) \right|^2$

# Some simple math

From the math we can extract 4 terms:

$$= \int_{-\infty}^{\infty} I^2(t) + I^2(t-\tau)\, dt \quad = I_{back}$$ Background

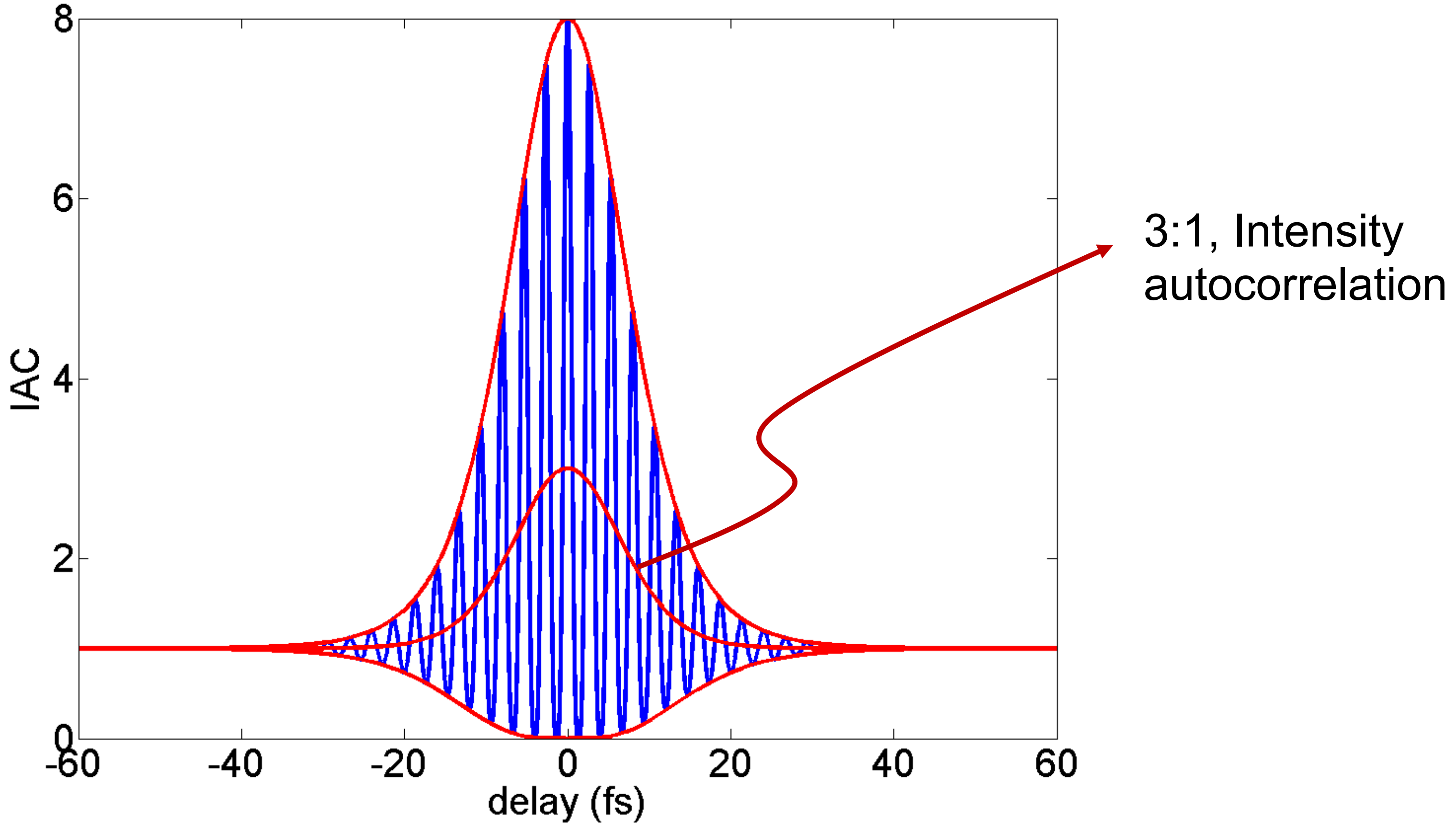
$$+\ 4\int_{-\infty}^{\infty} I(t) I(t-\tau)\, dt \quad = I_{int}$$ Intensity autocorrelation

$$-\ 2\int_{-\infty}^{\infty} \left[ I(t) + I(t-\tau) \right] E(t) E^*(t-\tau)\, dt \ +\ c.c = I_{\omega}$$ Interferogram of E(t), oscillating at $\omega$

$$+\ \int_{-\infty}^{\infty} E^2(t) E^{2*}(t-\tau)\, dt \ +\ c.c. = I_{2\omega}$$ Interferogram of the SH oscillating at $2\omega$

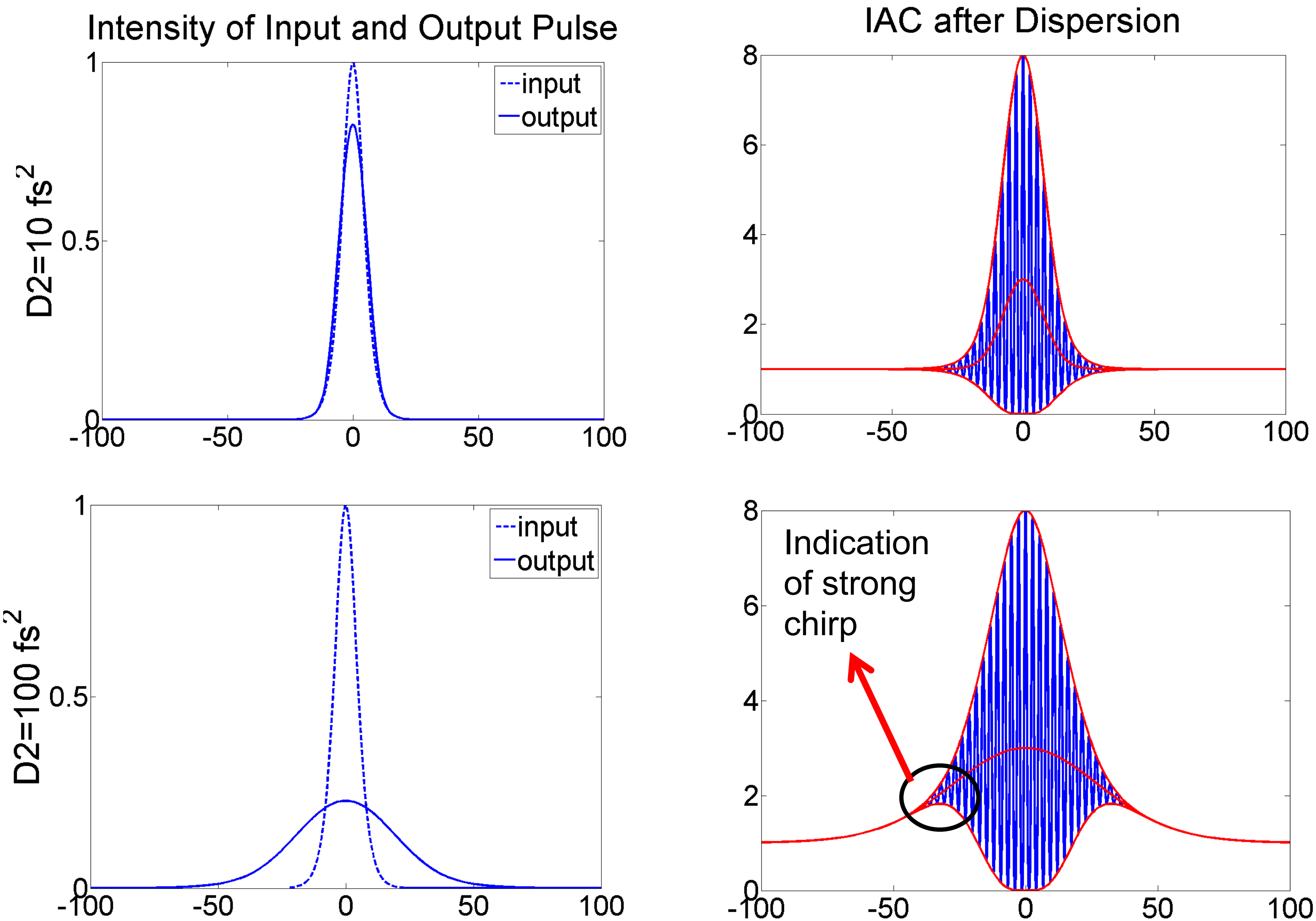
$$IA^{(2)}(\tau = 0) = 8 \qquad IA^{(2)}(\tau \rightarrow \infty) = 1$$

# IAC of 10 fs Sech-shaped pulse

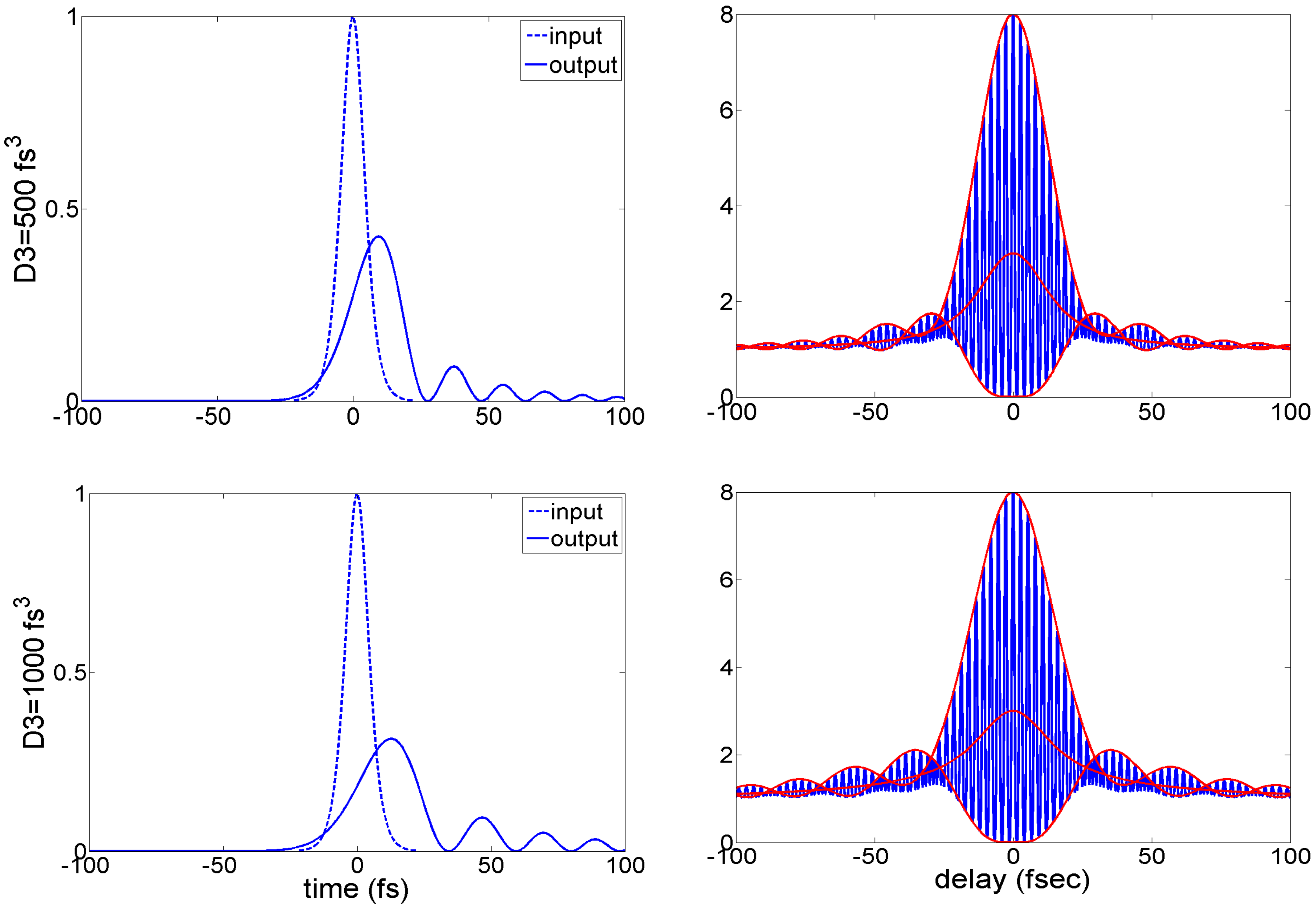



The interferometric autocorrelation simply combines several measures of the pulse into one (admittedly complex) trace. Conveniently, however, they occur with different oscillation frequencies: $0$, $\omega$, and $2\omega$.

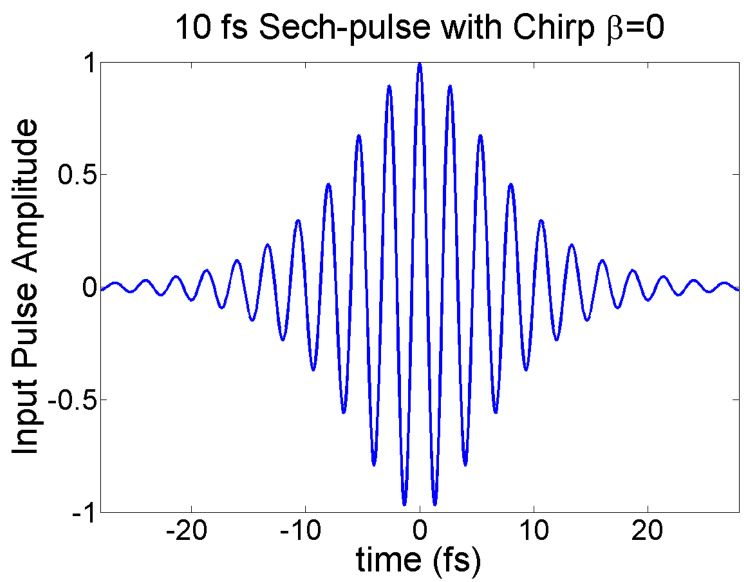
# Effects of second-order dispersion

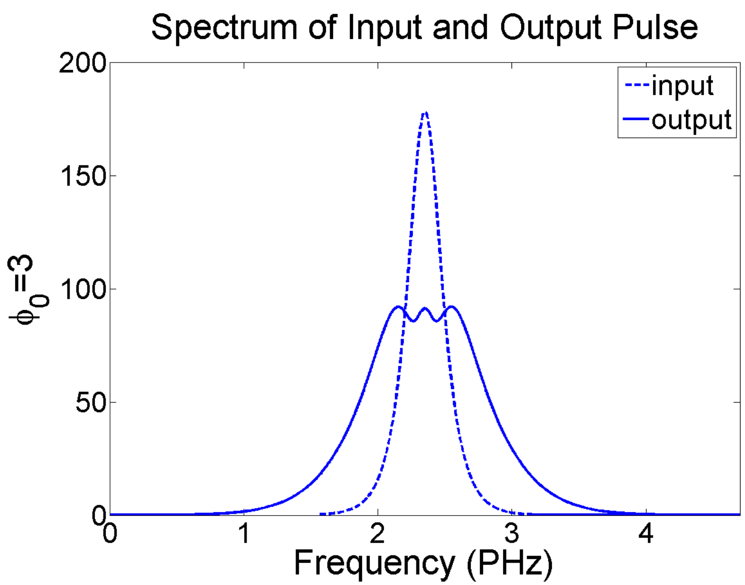


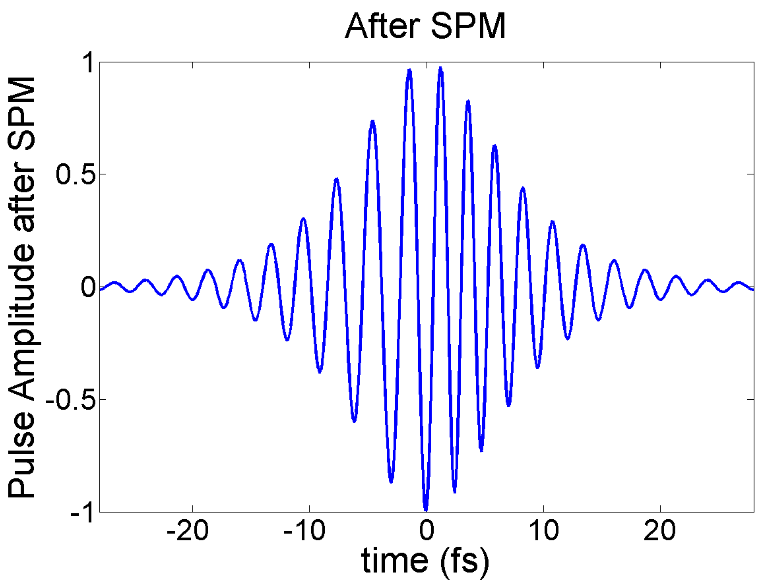
# Effects of third-order dispersion

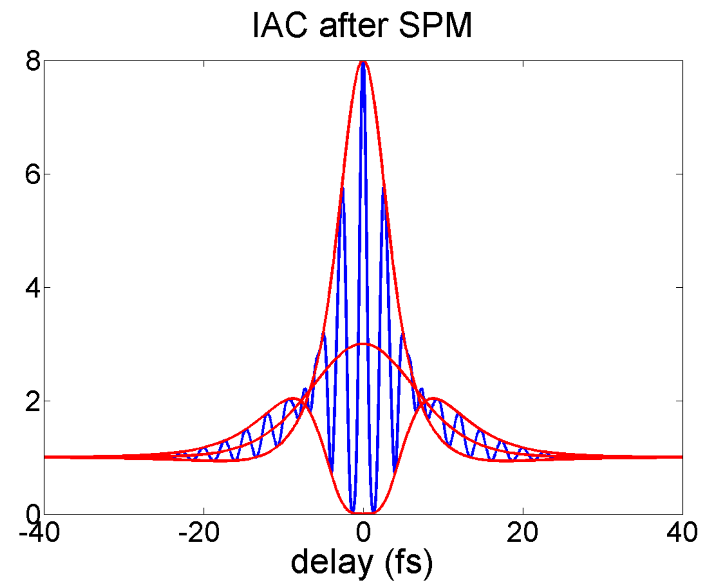


# Effects of self-phase modulation



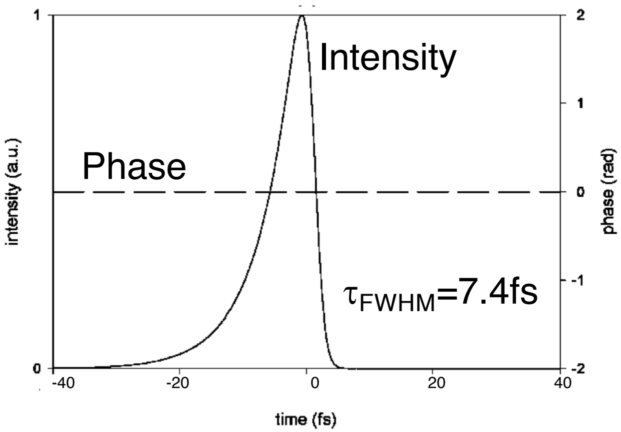



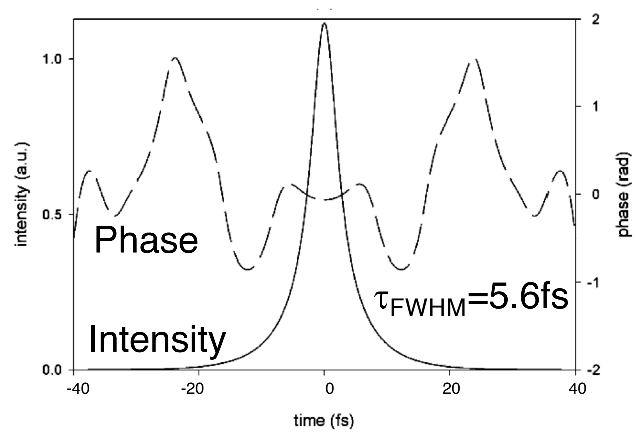



# Pulses with similar IAC

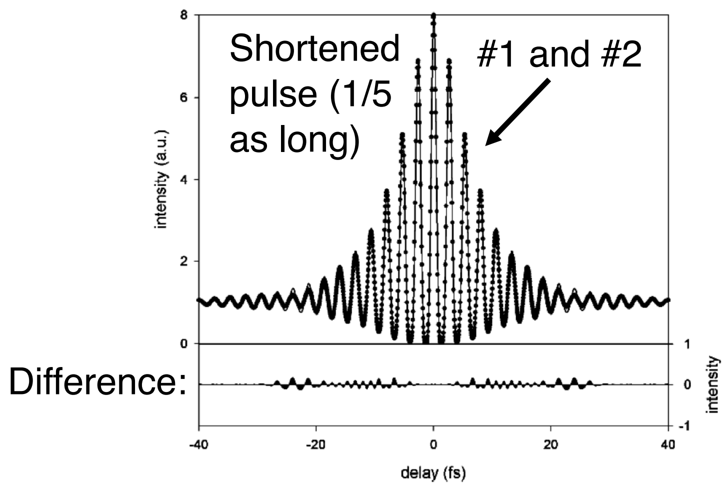
Pulse #1




Pulse #2




Interferometric Autocorrelations for Shorter Pulses #1 and #2




Chung and Weiner, IEEE JSTQE, 2001.

**The interferometric autocorrelation contains the full information of the pulse, however pulse retrieval is at times sensitive to noise.**

# Properties of IAC

1) It is always symmetric and the peak-to-background ratio should be 8.

2) This device is difficult to align; there are five very sensitive degrees of freedom in aligning two collinear pulses, but alignment shows up in result.

3) Dispersion in each arm must be the same, so it is necessary to insert a compensator plate in one arm.

4) Using optical spectrum and background-free intensity autocorrelator can determine the presence or absence of strong chirp. The interferometric autocorrelation serves as a clear visual indication of moderate to large chirp.

5) It is difficult to distinguish between different pulse shapes and, especially, different phases from interferometric autocorrelations (maybe).

6) Like the intensity autocorrelation, it must be curve-fit to an assumed pulse shape and so should only be used for rough estimates (wrong).

## How to measure both pulse intensity profile and the phase?

1) A pulse can be represented by two arrays of data with length N, one for the amplitude/intensity and the other for the phase. Totally we have 2N degrees of freedom (corresponding to the real and imaginary parts for the electric field).

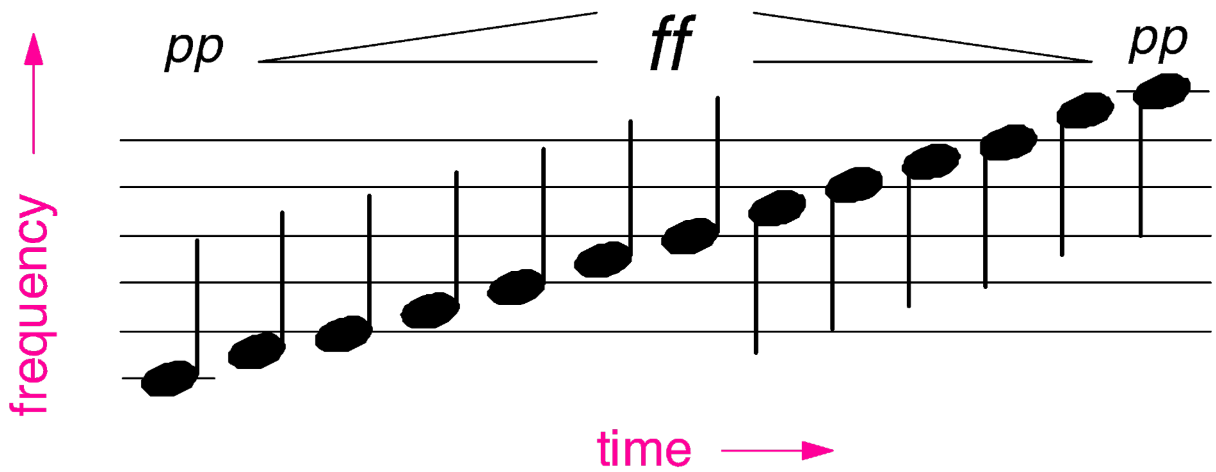
2) Intensity autocorrelator provides only one array of data with length N. Optical spectrum measurement can provide another array of data with length N. However some information, especially about phase, is missing from both measurements.

3) Need to have more data, providing enough redundancy to recover the both the amplitude and phase.

**How about measuring the spectrum of the autocorrelation pulse at each delay? NxN data points**

# Frequency-Resolved Optical Gating (FROG): SHG-FROG

Frequency vs Time → SPECTROGRAM
A spectrogram can be seen as a musical score!




*Frequency-Resolved Optical Gating: The Measurement of Ultrashort Laser Pulses, Rick Trebino*

# Frequency-Resolved Optical Gating (FROG): SHG-FROG

If $E(t)$ is the waveform of interest, its spectrogram is:

$$\Sigma_E(\omega,\tau) \equiv \left| \int_{-\infty}^{\infty} E(t)\, g(t-\tau) \exp(-i\omega t)\, dt \right|^2$$

where $g(t\text{-}\tau)$ is a variable-delay gate function and *t* is the delay

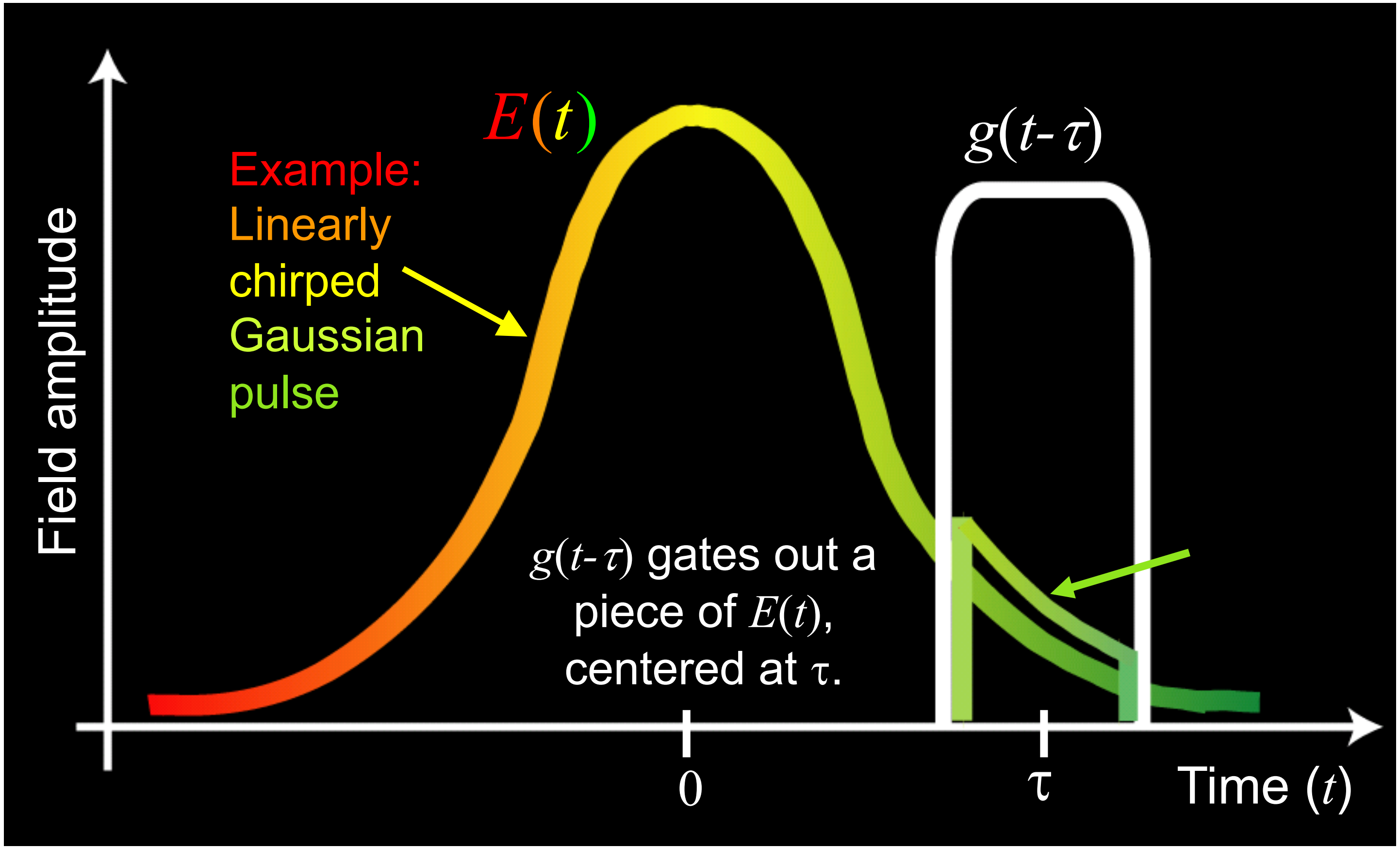
Without $g(t\text{-}\tau)$, $\Sigma_E(\omega,\tau)$ would simply be the spectrum

The spectrogram is a function of $\omega$ and $\tau$

It is the set of spectra of all temporal slices of $E(t)$

# Frequency-Resolved Optical Gating (FROG): SHG-FROG

We must compute the spectrum of the product: $E_{sig}(t,\tau) = E(t)\, g(t-\tau)$

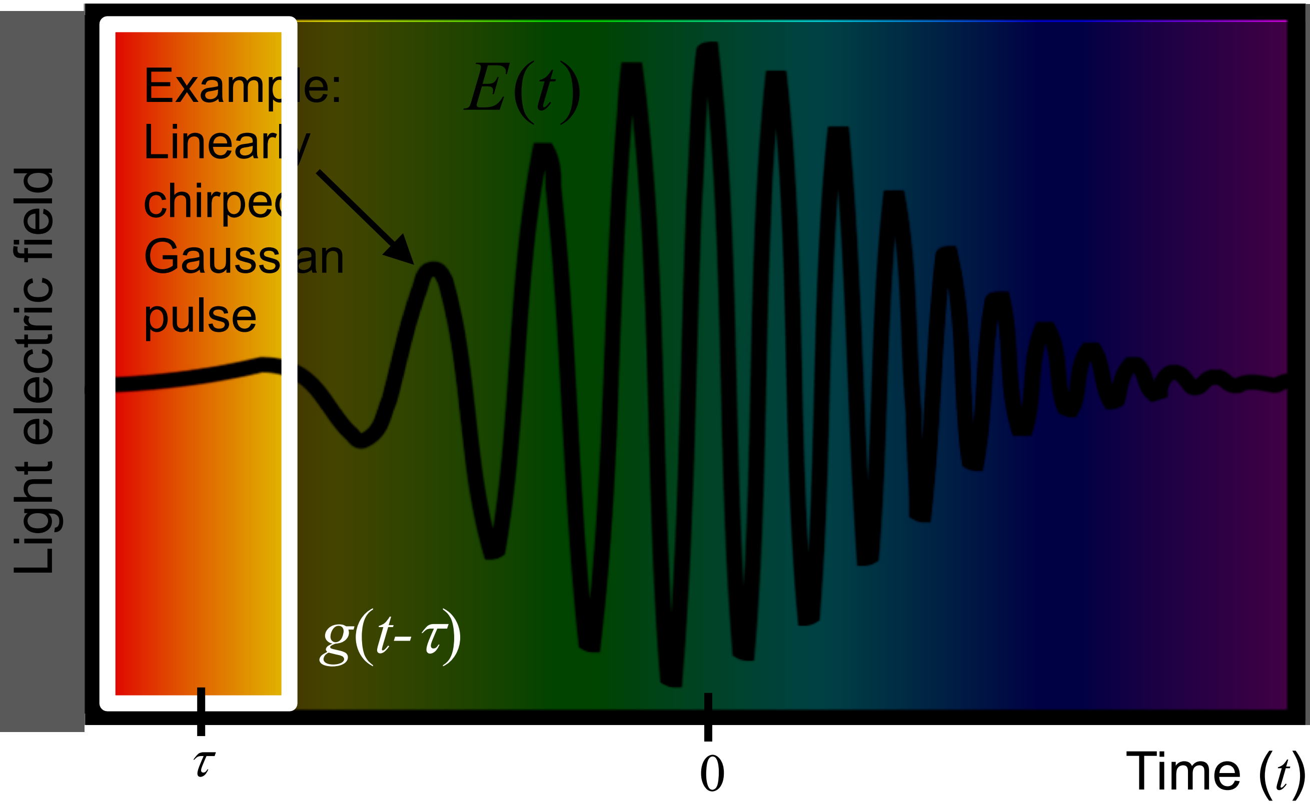



The spectrogram contains the color and intensity of $E(t)$ at each time $t$

# Frequency-Resolved Optical Gating (FROG): SHG-FROG

**The spectrogram**

$$\Sigma_E(\omega,\tau) \equiv \left| \int_{-\infty}^{\infty} E(t)\, g(t-\tau) \exp(-i\omega t)\, dt \right|^2$$

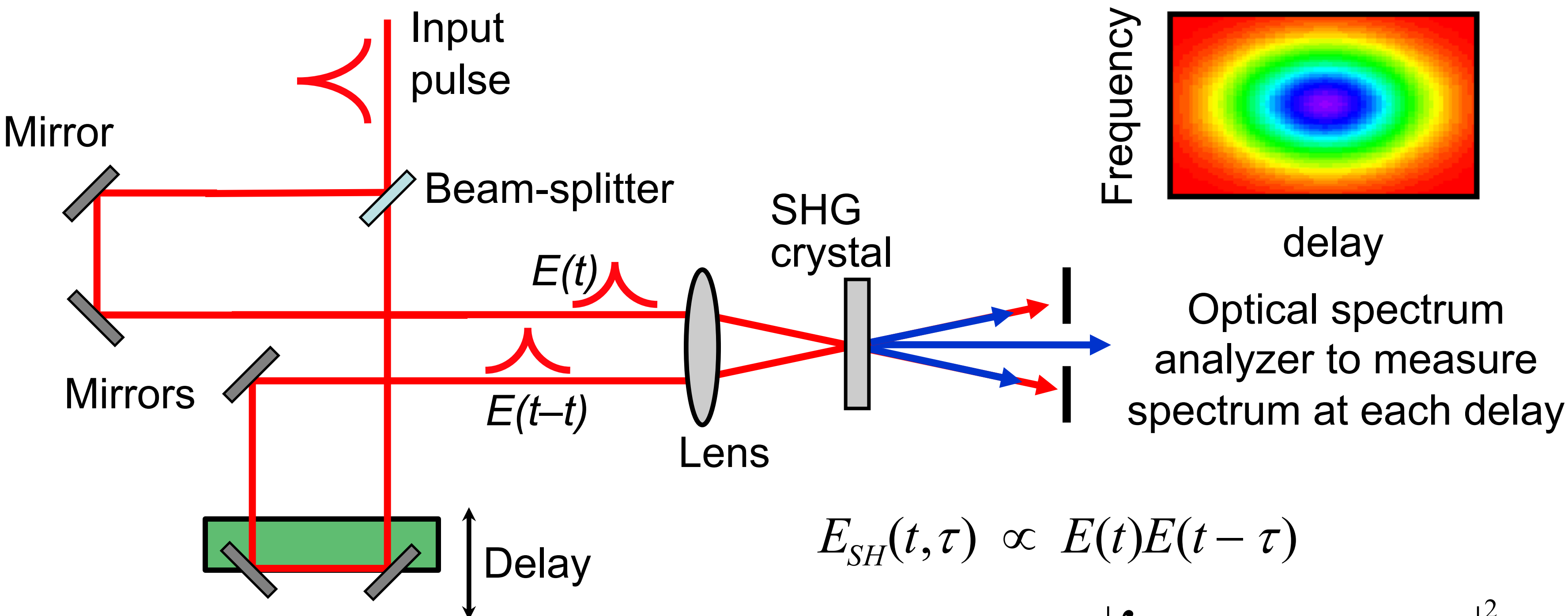



$g(t\text{-}t)$ gates out a portion of $E(t)$, centered at $t$.

The spectrogram contains the color and intensity of $E(t)$ at each time $t$

# Frequency-Resolved Optical Gating (FROG): SHG-FROG

Background-free intensity autocorrelator + optical spectrum analyzer

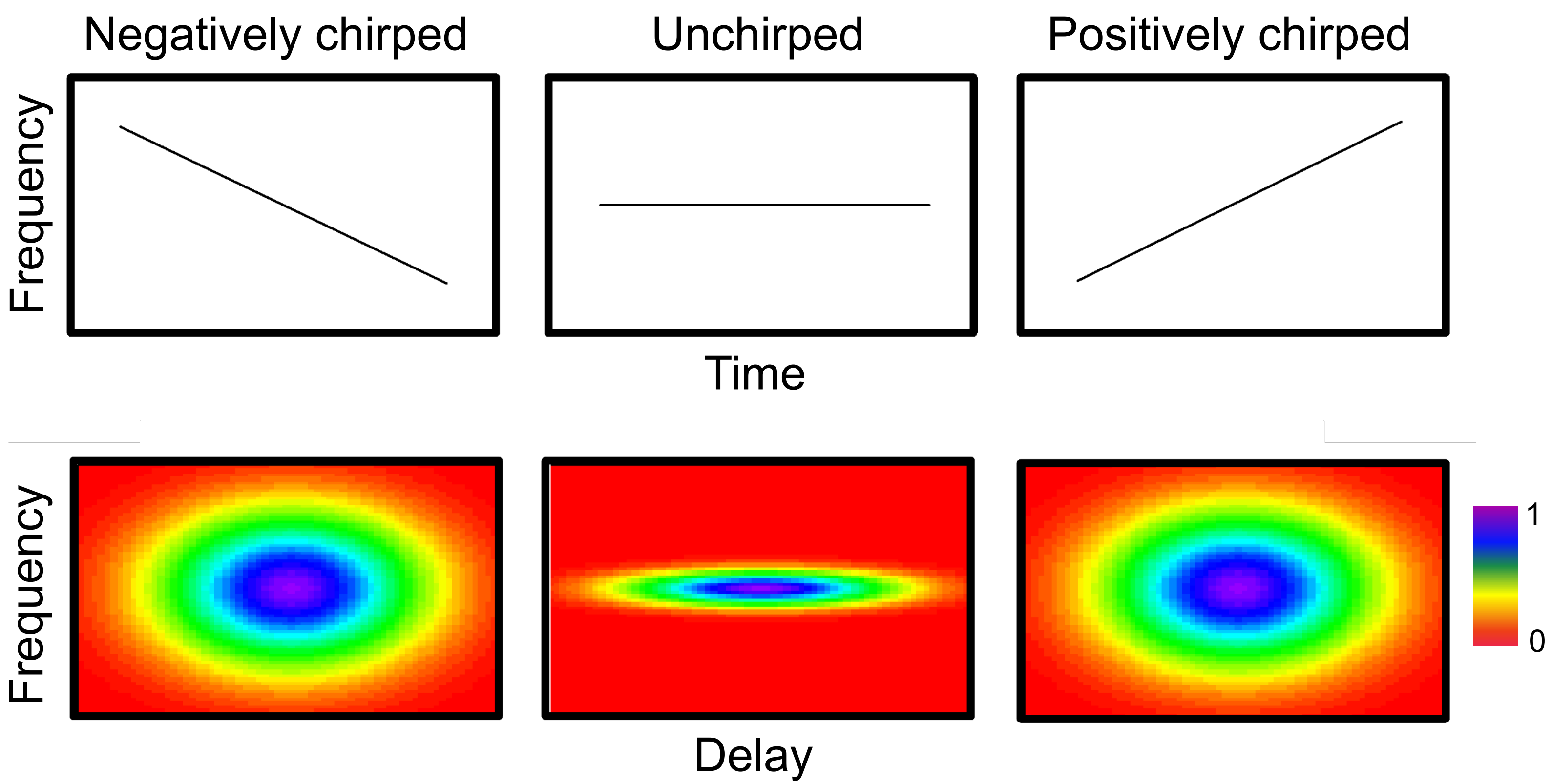



$$E_{SH}(t,\tau) \propto E(t)E(t-\tau)$$

$$\Rightarrow \quad I_{FROG}(\omega,\tau) \propto \left| \int E(t)E(t-\tau)e^{j\omega t}dt \right|^2$$

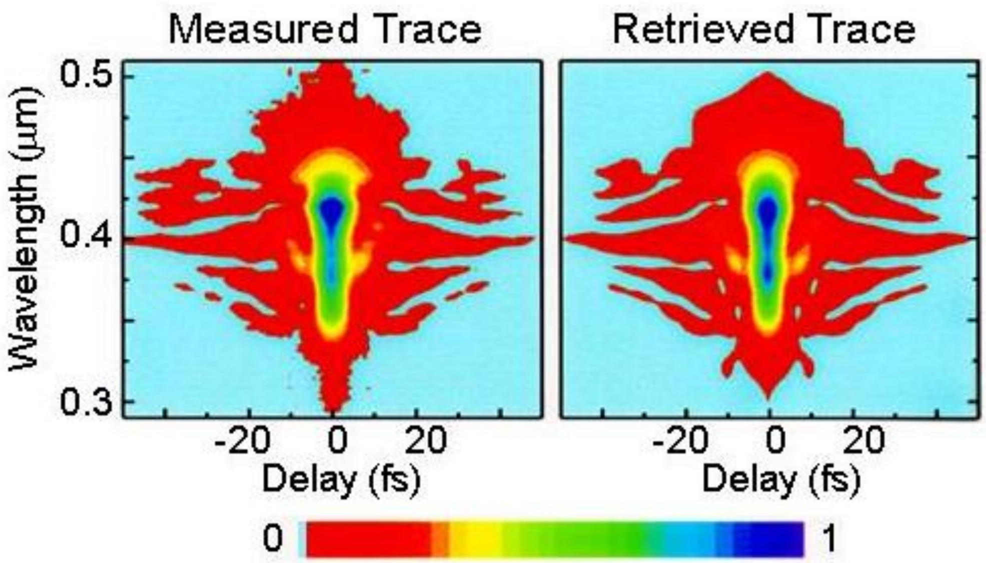
Now we have N X N data points. Iterative algorithm can retrieve both the amplitude and phase of the measured optical pulse.

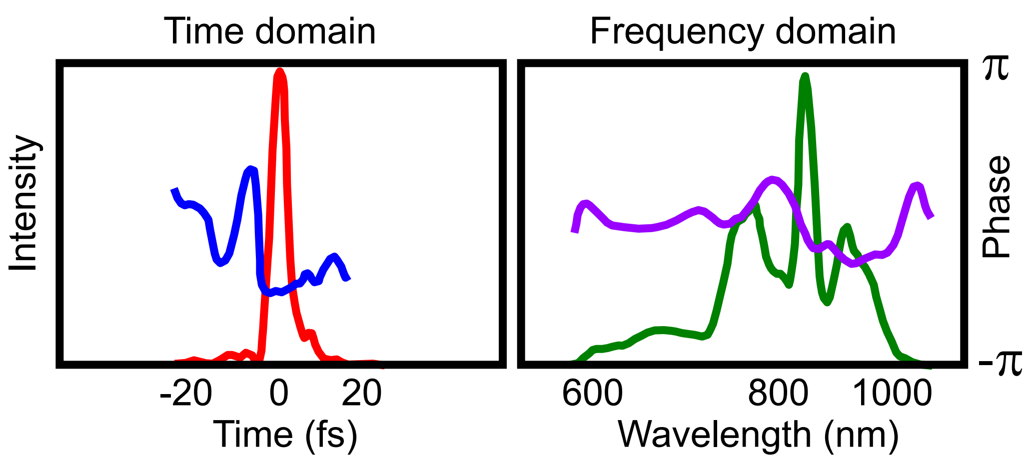
# SHG FROG traces are symmetrical with respect to delay




SHG FROG has an ambiguity in the direction of time, but it can be removed.

# SHG FROG measurements of a 4.5-fs pulse




Agreement between the experimental and reconstructed FROG traces provides a nice check on the measurement.

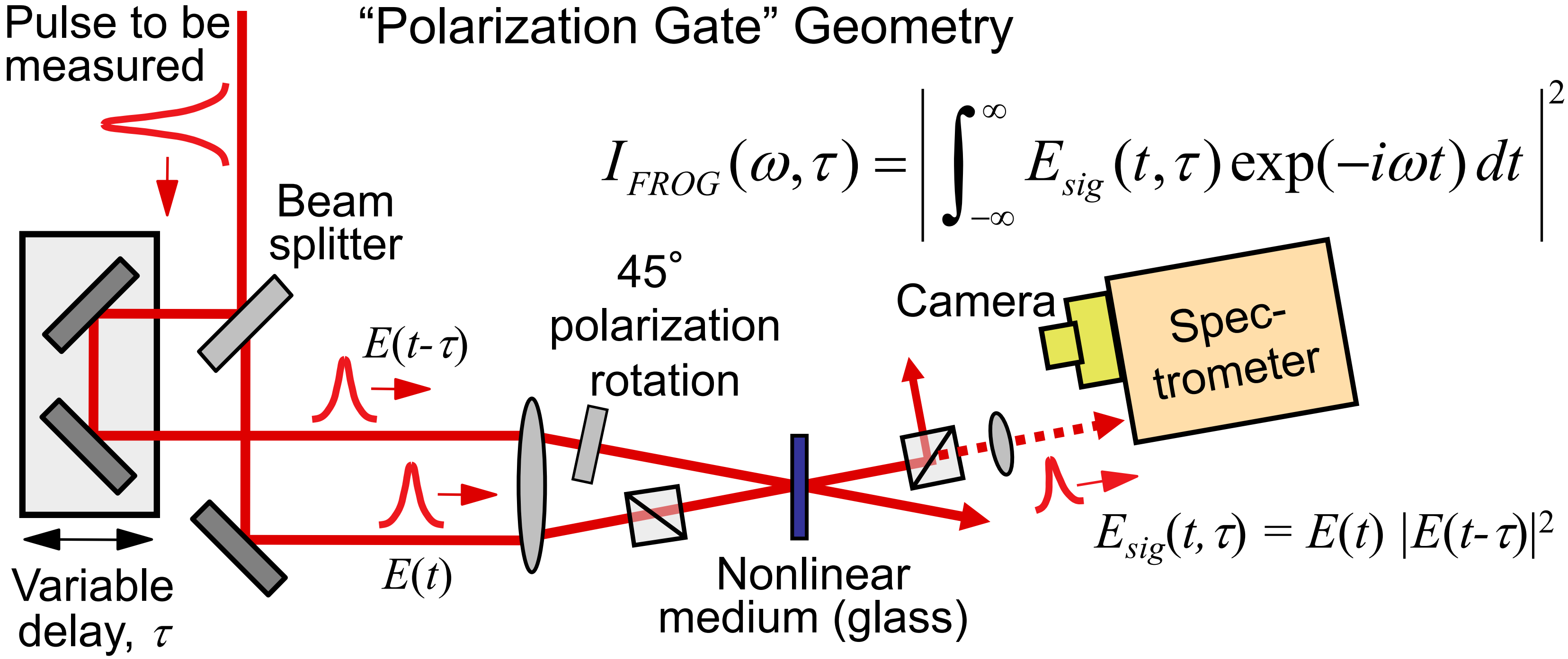
Baltuska, Pshenichnikov, and Weirsma, J. Quant. Electron., 35, 459 (1999).

# Polarization gating FROG (PG-FROG)

FROG involves gating the pulse with a variably delayed replica of itself in an instantaneous nonlinear-optical medium and then spectrally resolving the gated pulse vs. delay.

"Polarization Gate" Geometry

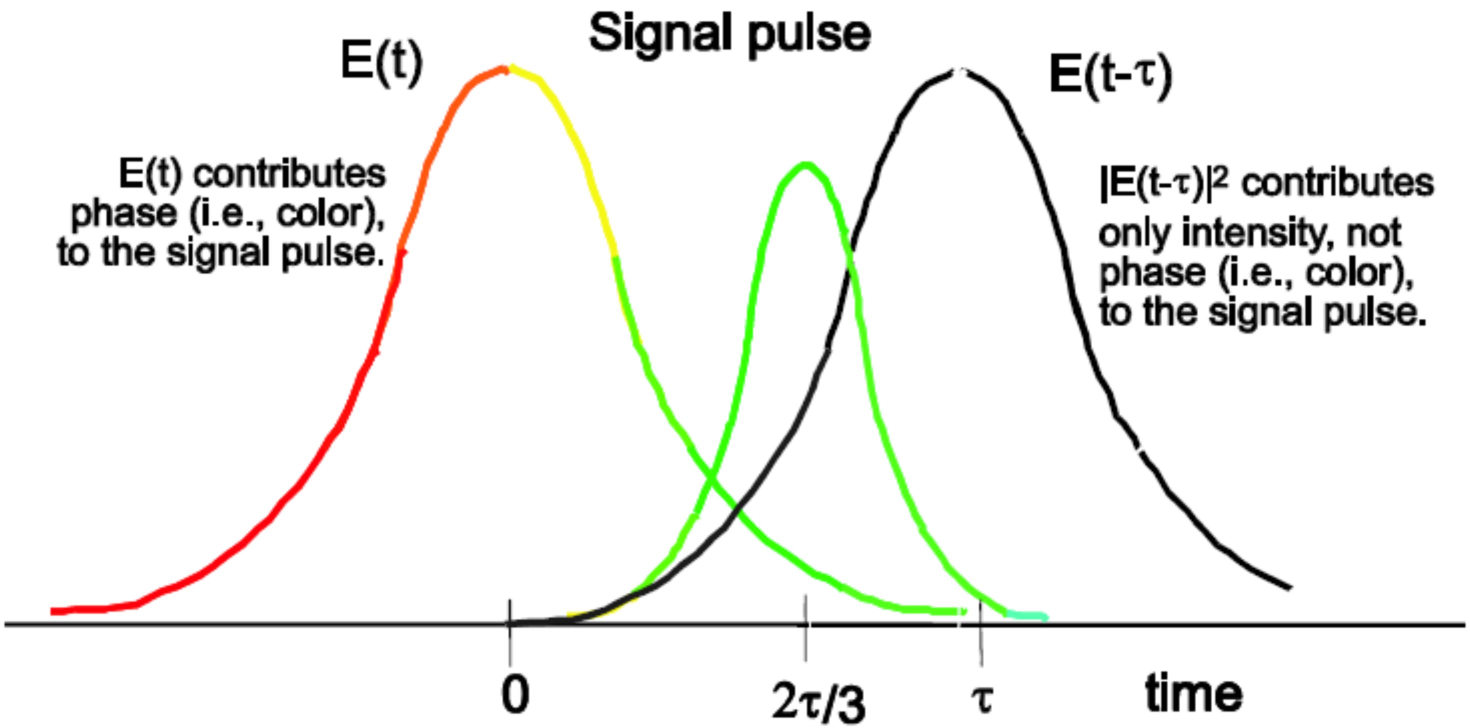
$$I_{FROG}(\omega,\tau) = \left| \int_{-\infty}^{\infty} E_{sig}(t,\tau)\exp(-i\omega t)\,dt \right|^2$$




Use any ultrafast nonlinearity: Second-harmonic generation, etc.

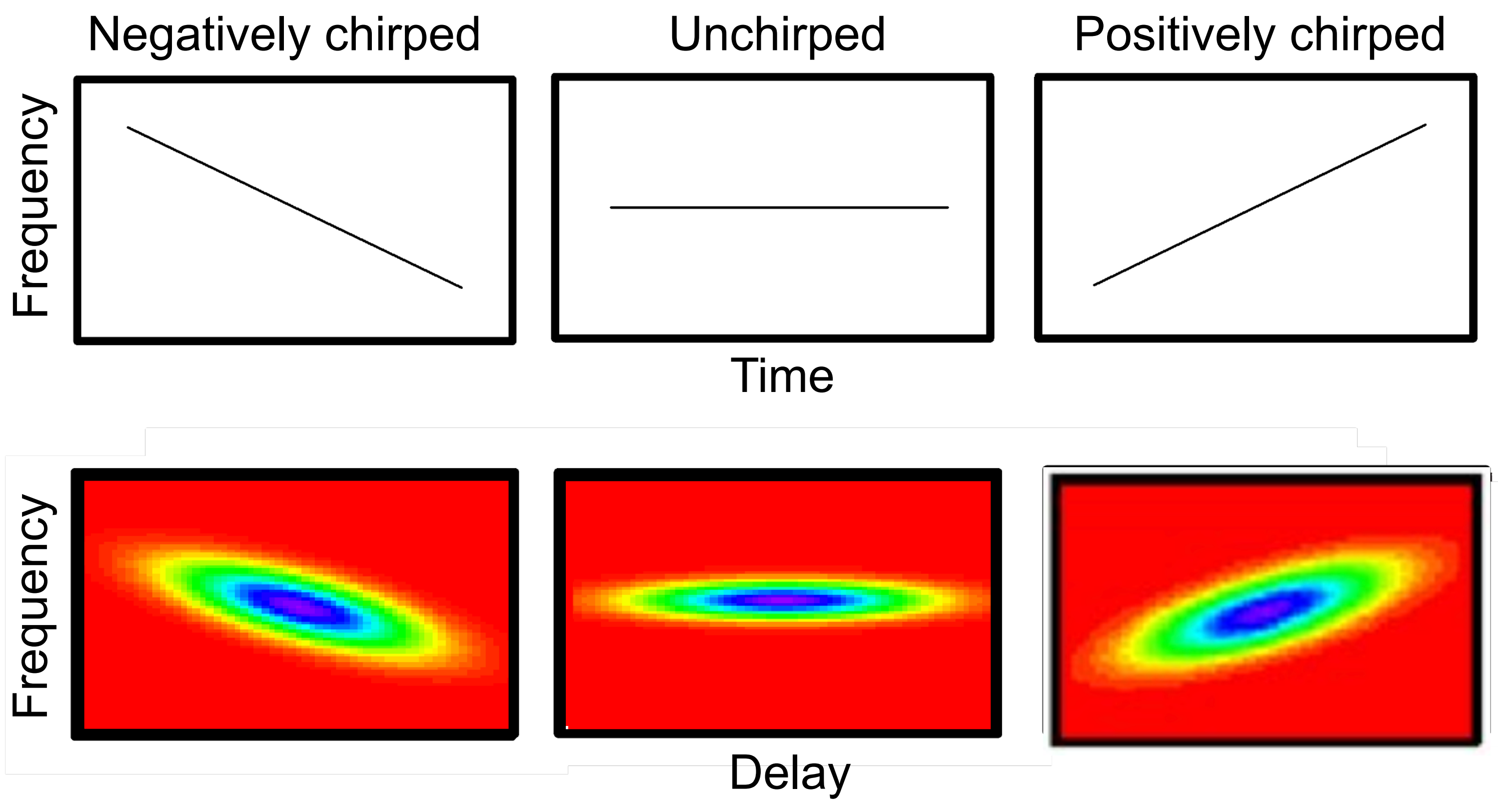
# Polarization gating FROG (PG-FROG)

$$E_{sig}(t,\tau) \propto E(t)\,|E(t-\tau)|^2$$




The gating is more complex for complex pulses, but it still works.
And it also works for other nonlinear-optical processes.

# PG-FROG traces for linearly chirped pulses

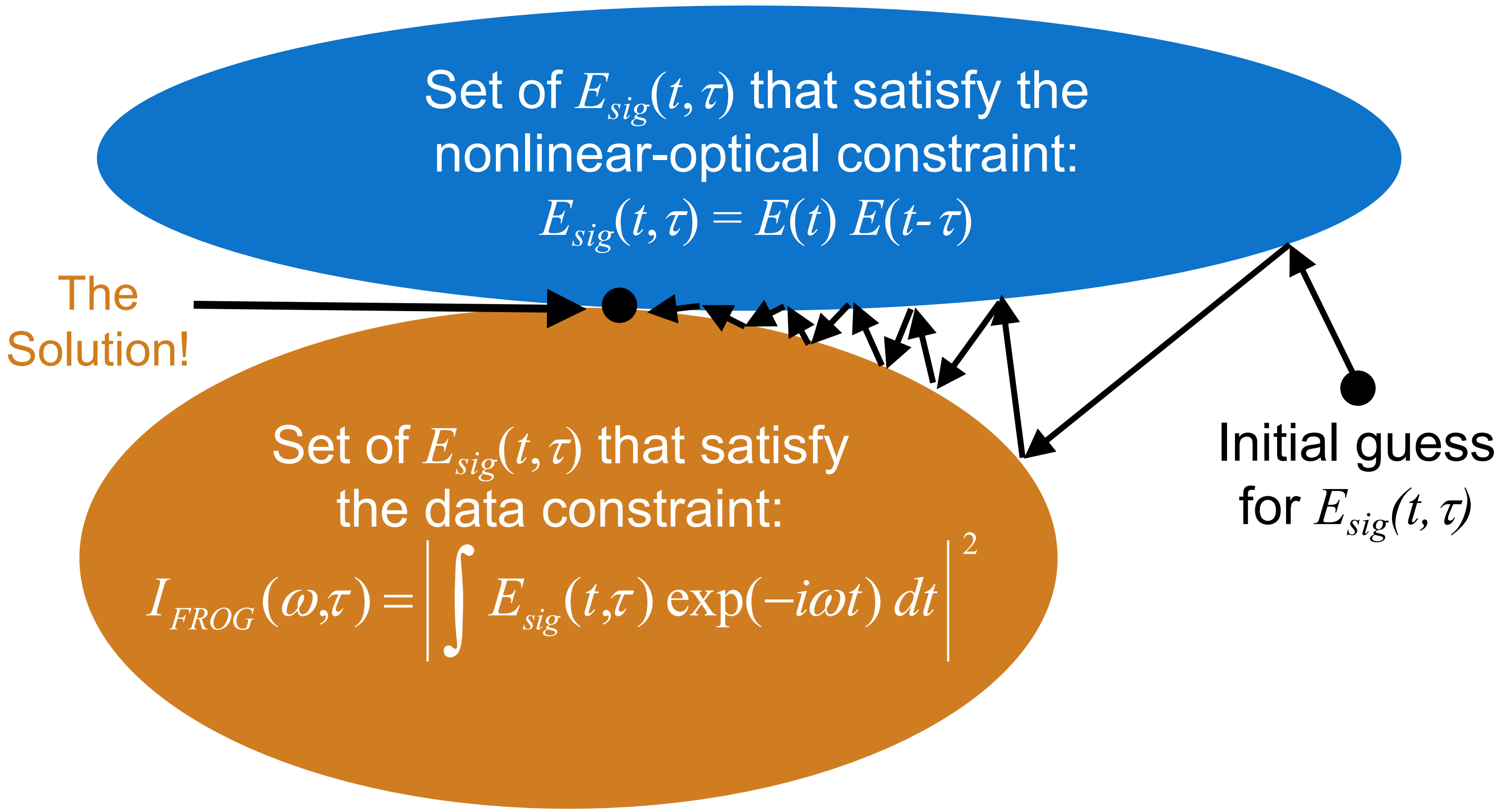



Like a musical score, the FROG trace visually reveals the pulse frequency vs. time—for simple and complex pulses.

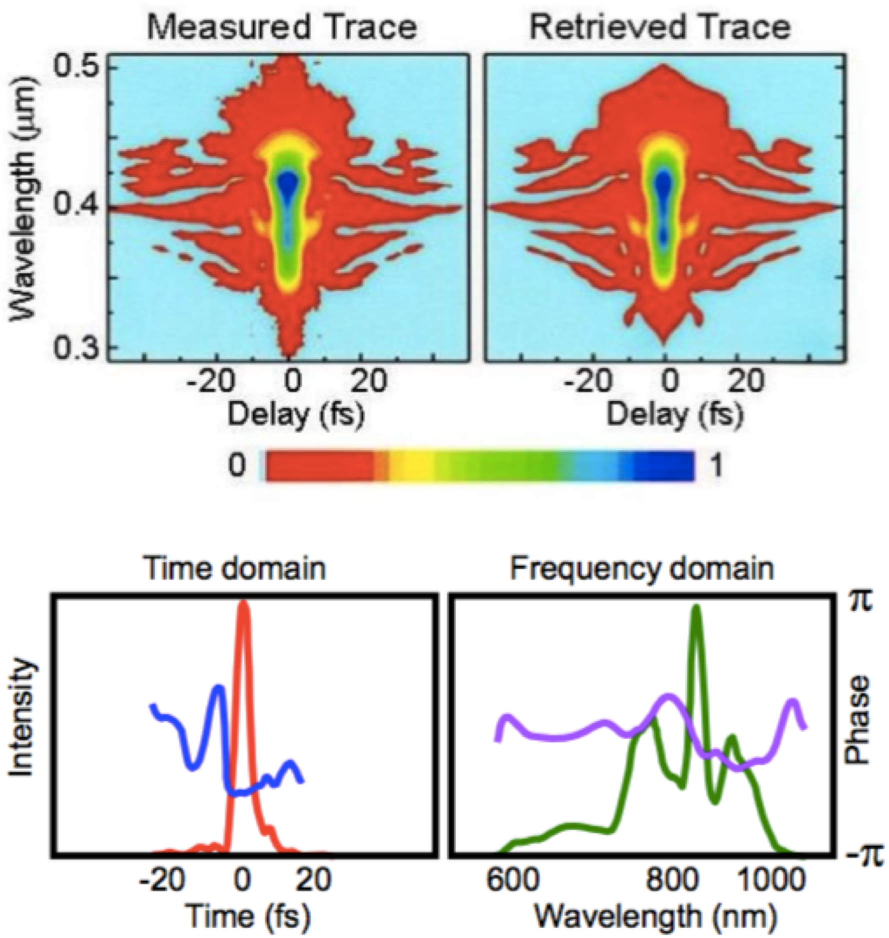
# FROG: pulse reconstruction

## Generalized projections algorithm

*E(t)* can be fully retrieved from the measured spectrogram by applying iterative reconstruction algorithms



# FROG: short pulse




Agreement between the experimental and reconstructed FROG traces provides a nice check on the measurement.

Baltuska, Pshenichnikov, and Weirsma, J. Quant. Electron., 35, 459 (1999).

# What information do we need to fully determine an optical pulse?

A laser pulse has the time-domain electric field:

$$E(t) \sim \mathrm{Re}\,\{\underbrace{I(t)}_{\text{Intensity}}{}^{1/2} \exp\,[\,j\omega_0 t - j\underbrace{\phi(t)}_{\text{Phase}}\,]\,\}$$

Equivalently, vs. frequency:

$$\tilde{E}(\omega) \sim \underbrace{I(\omega-\omega_0)}_{\text{Spectrum}}{}^{1/2} \;\; \exp\,[\,-j\underbrace{\varphi(\omega-\omega_0)}_{\text{Spectral Phase}}\,]$$

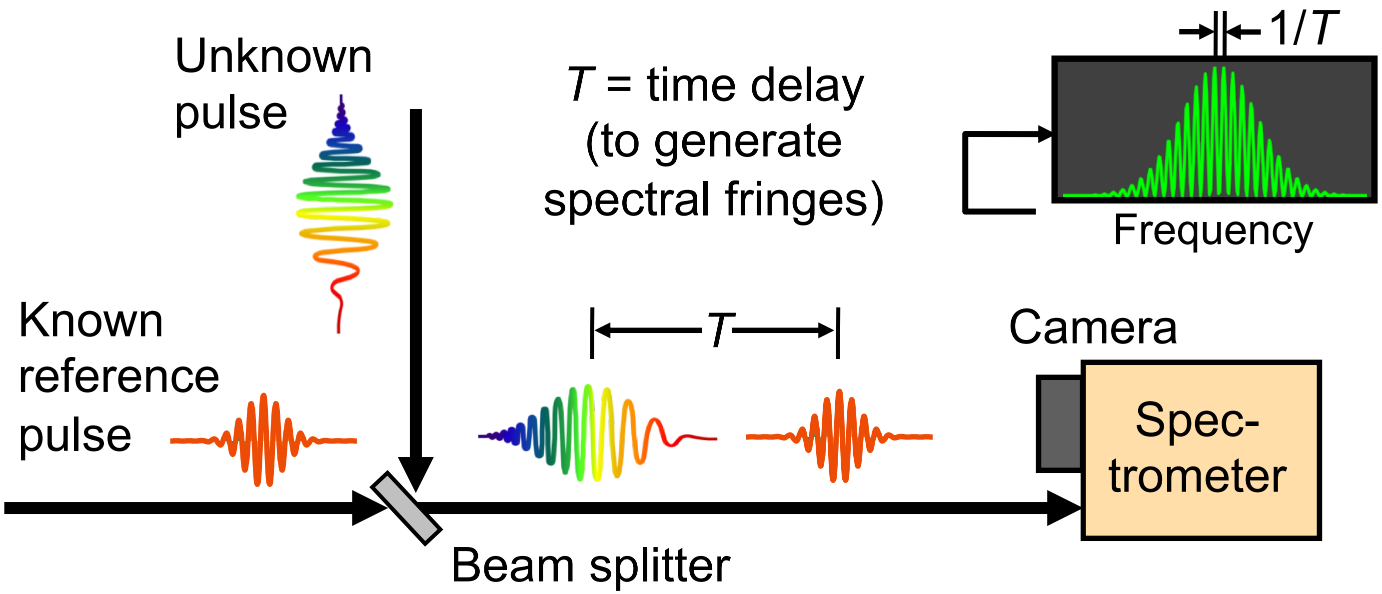
(neglecting the negative-frequency component)

Can be measured by an optical spectrum analyzer.

# Spectral interferometry

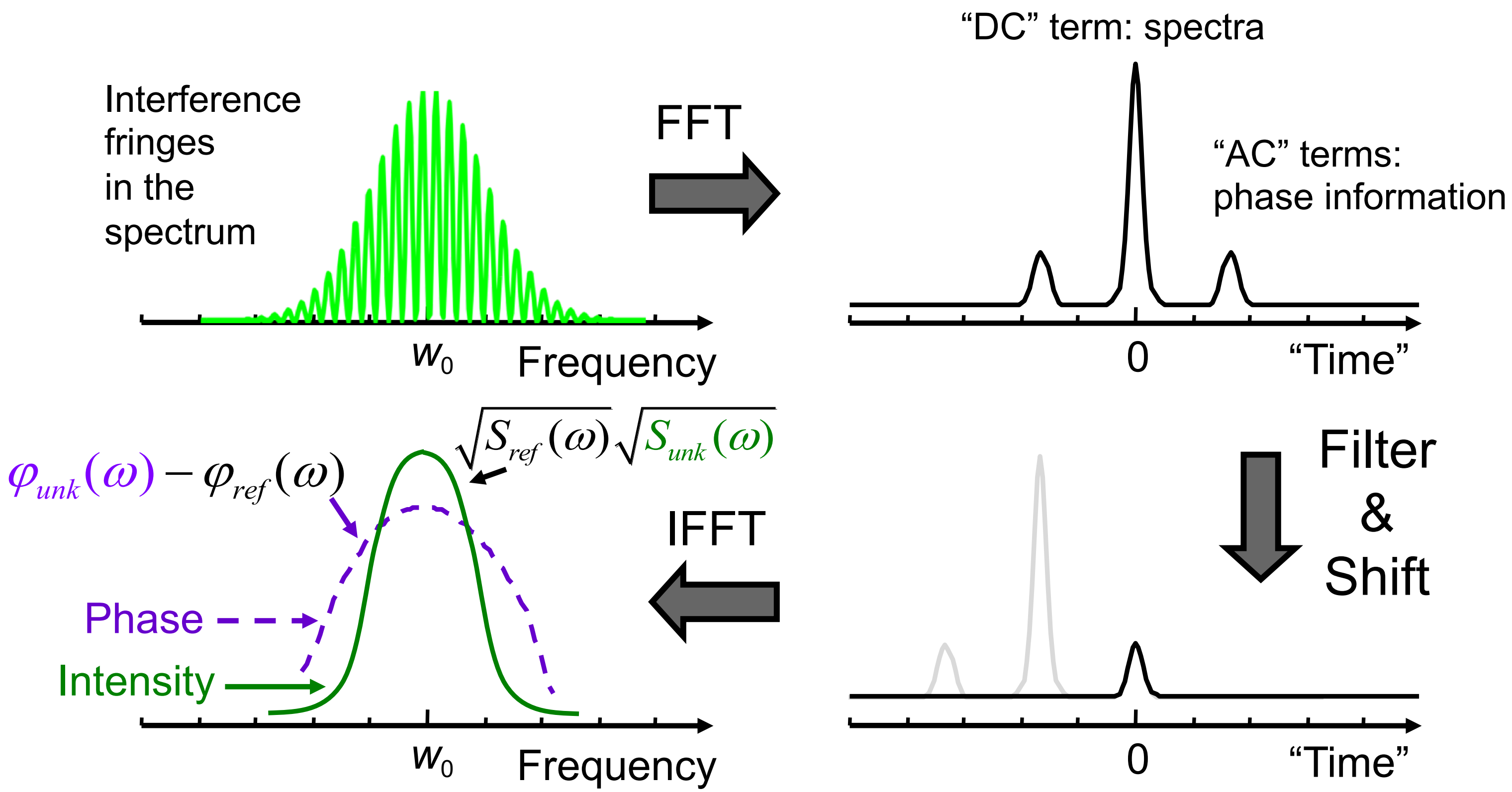
Measure the spectrum of the sum of a known and unknown pulse

Retrieve the unknown pulse from the spectral fringes




$$S_{SI}(\omega) = S_{ref}(\omega) + S_{unk}(\omega) + 2\sqrt{S_{ref}(\omega)}\sqrt{S_{unk}(\omega)}\cos[\varphi_{unk}(\omega) - \varphi_{ref}(\omega) + \omega T]$$

# Spectral interferometry




This retrieval algorithm is quick, direct, and reliable

A reference pulse is usually not available!

# Spectral Phase Interferometry for Direct Electric-Field Reconstruction (SPIDER)

If we perform spectral interferometry between a pulse and itself, the spectral phase cancels out. Perfect sinusoidal fringes always occur:

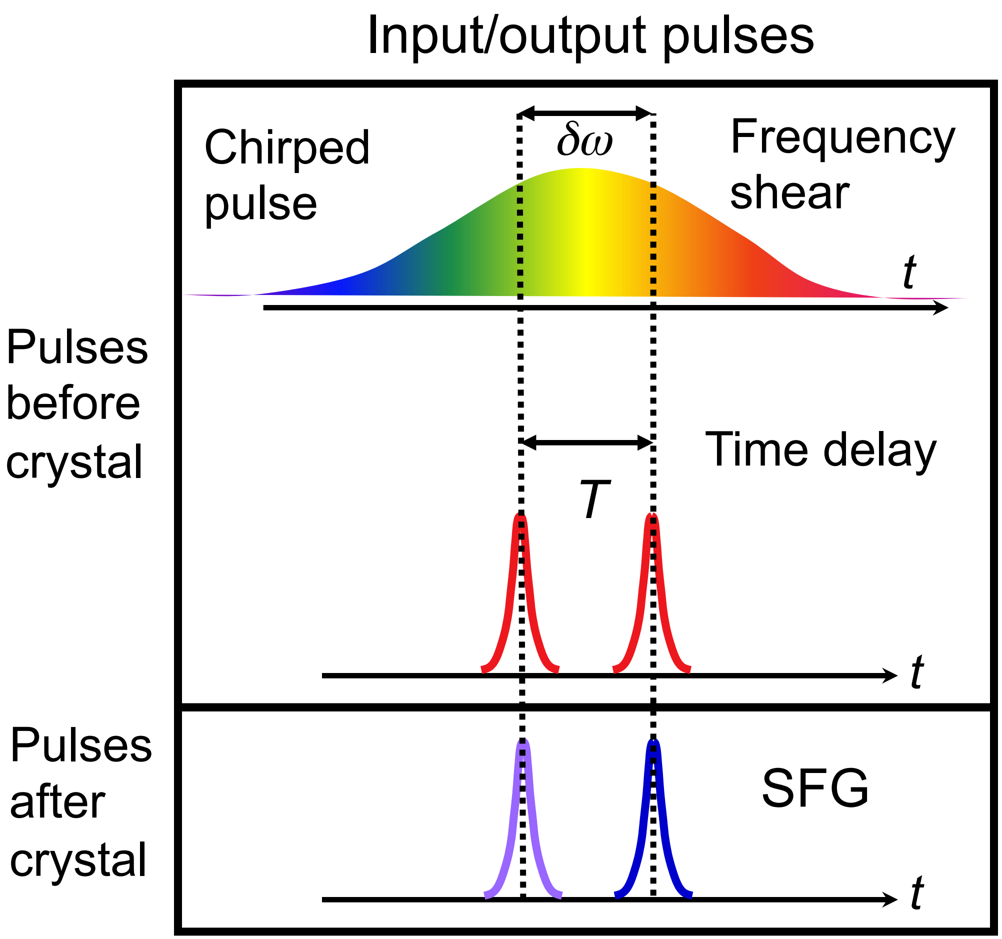
$$S_{SI}(\omega) = S_{unk}(\omega) + S_{unk}(\omega) + 2\sqrt{S_{unk}(\omega)}\sqrt{S_{unk}(\omega)}\cos[\cancel{\varphi_{unk}}(\omega) - \cancel{\varphi_{unk}}(\omega) + \omega T]$$

$$S_{SI}(\omega) = S(\omega) + S(\omega+\delta\omega) + 2\sqrt{S(\omega)}\sqrt{S(\omega+\delta\omega)}\cos[\varphi(\omega+\delta\omega) - \varphi(\omega) + \omega T]$$

$$\phi_{SPIDER} = \varphi(\omega+\delta\omega) - \varphi(\omega) + \omega T = \overbrace{\boxed{\delta\omega}}^{\text{frequency shear}} \overbrace{\boxed{\frac{d\varphi}{d\omega}}}^{\text{group delay vs. }\omega} + \overbrace{\boxed{\omega T}}^{\text{Time delay}}$$

This measures the derivative of the spectral phase (the group delay)

# Spectral Phase Interferometry for Direct Electric-Field Reconstruction (SPIDER)

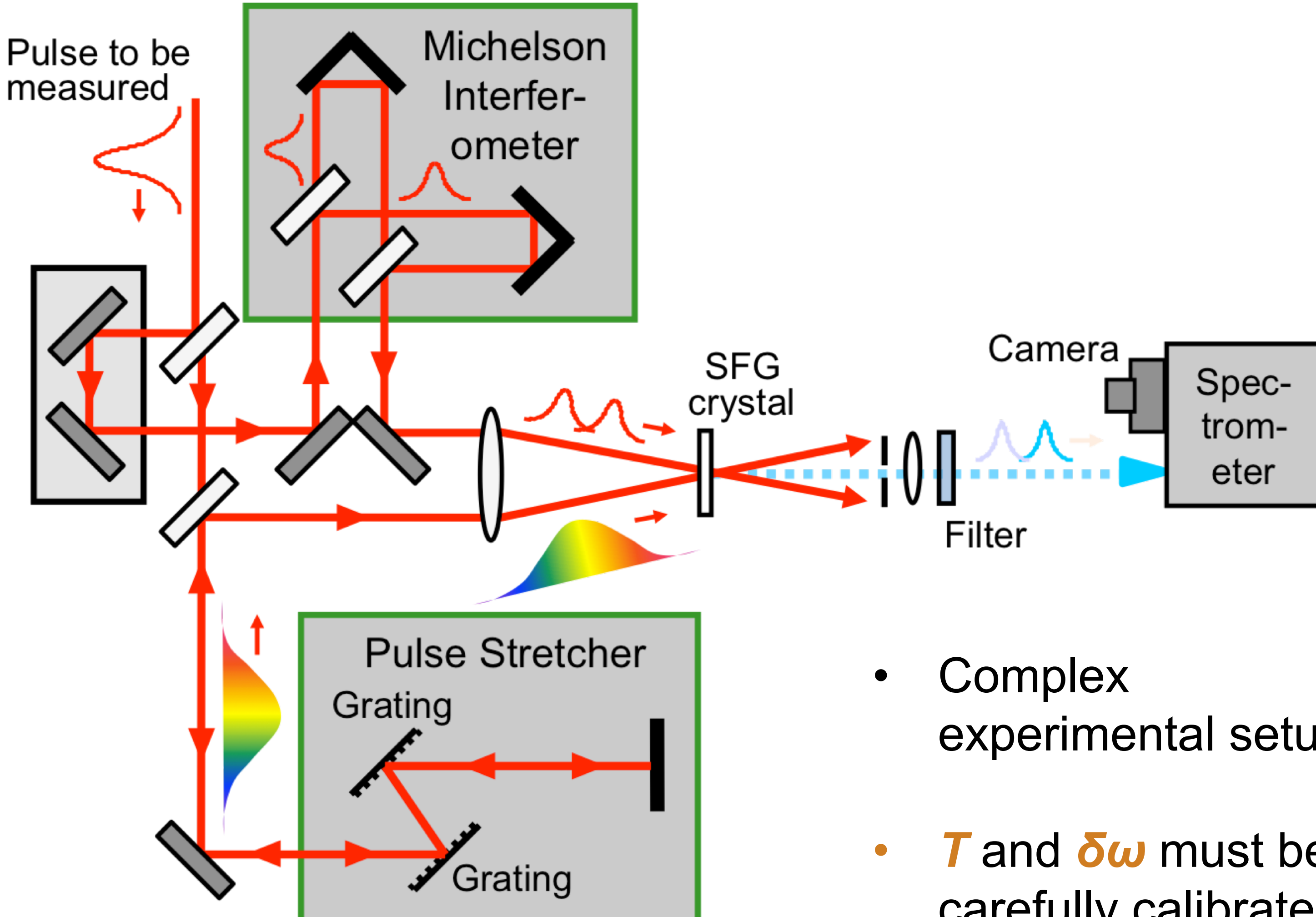



1) Make a very chirped pulse

2) Create two replicas of the pulse

3) Frequency shift the 2 replicas by SFG with the broadband pulse and perform SI

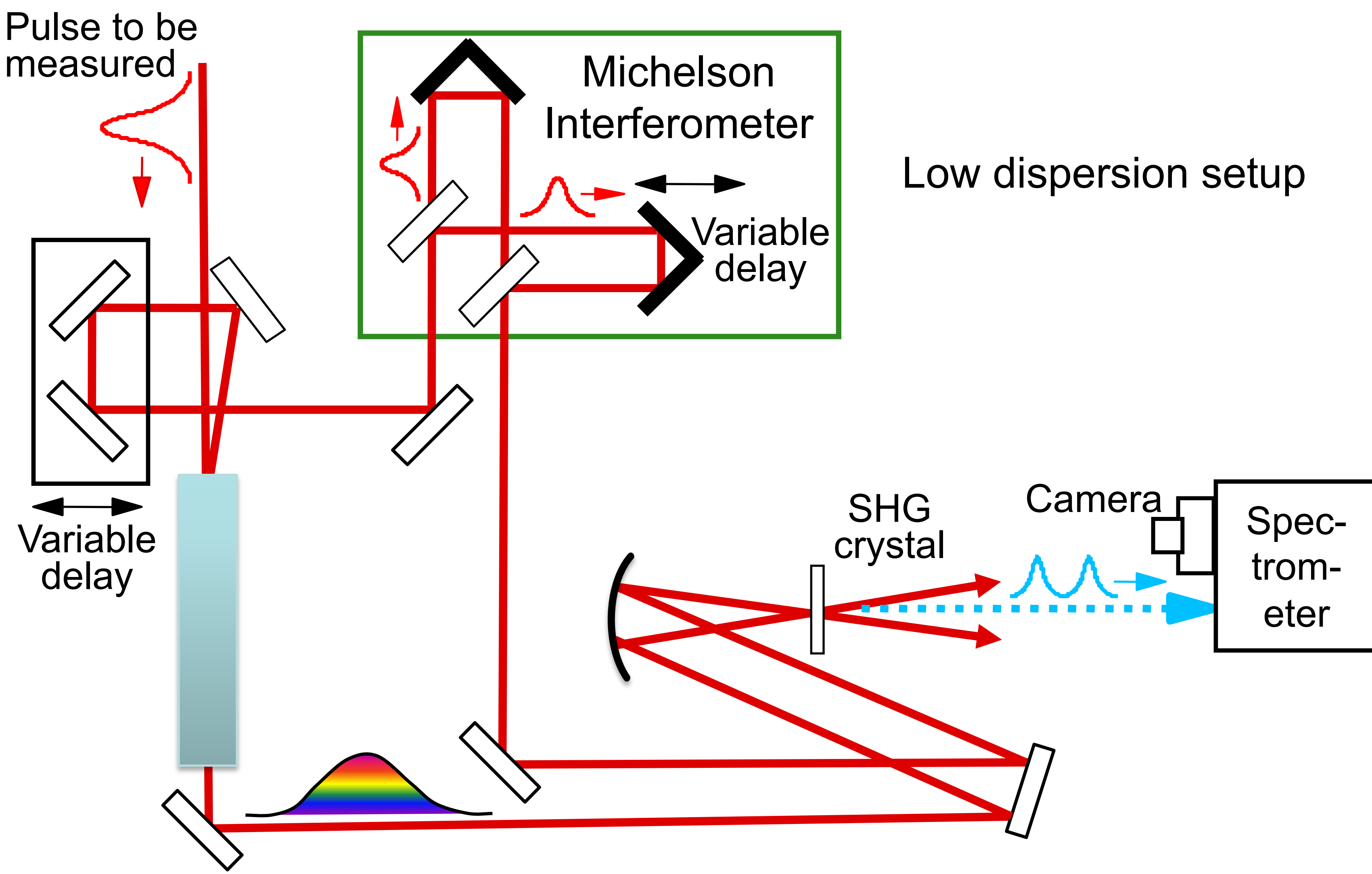
# Spectral Phase Interferometry for Direct Electric-Field Reconstruction (SPIDER)



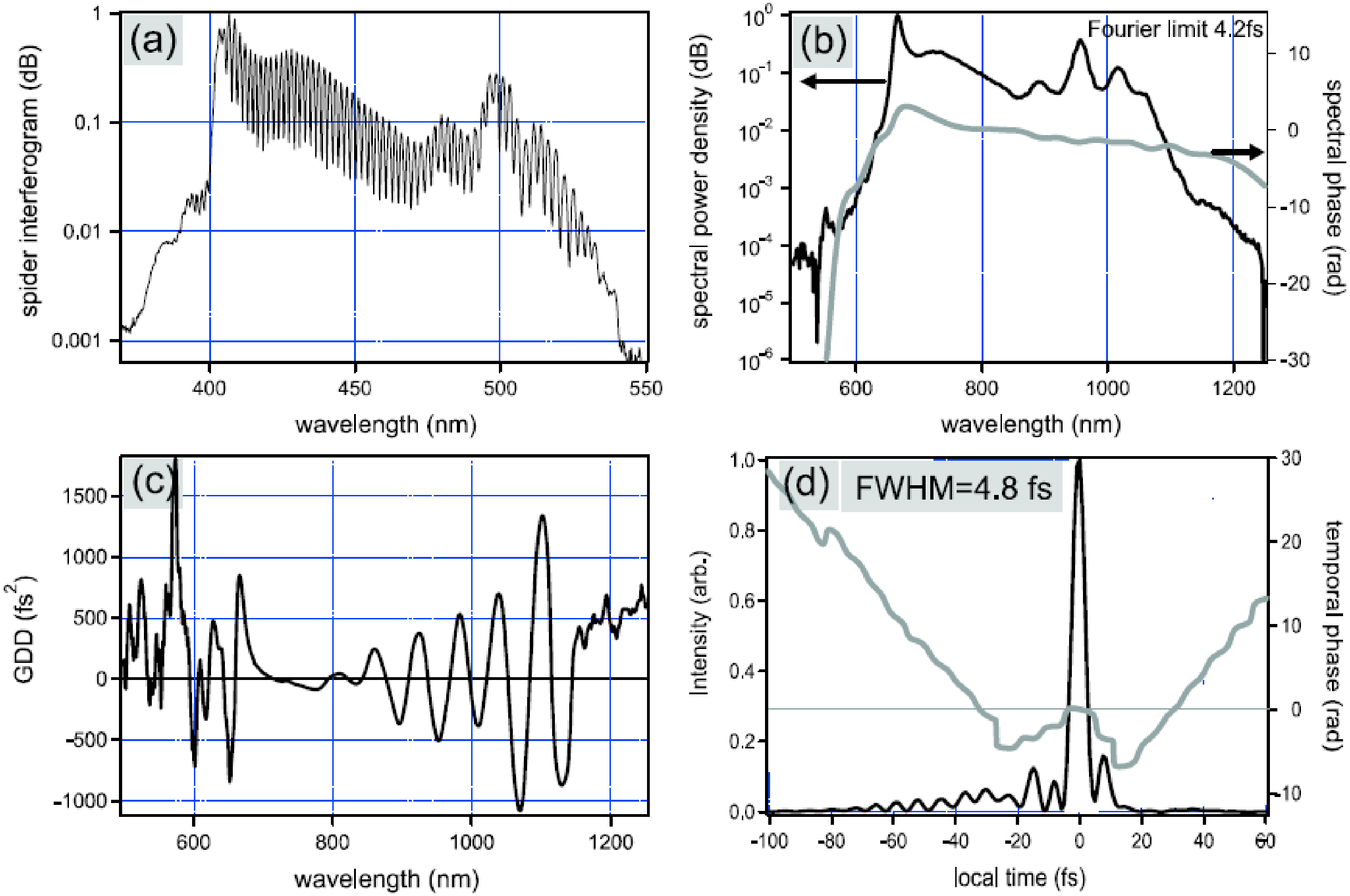

- Complex experimental setup
- *T* and *δω* must be carefully calibrated!

# Spectral Phase Interferometry for Direct Electric-Field Reconstruction (SPIDER)

## ZAP (zero additional phase)-SPIDER

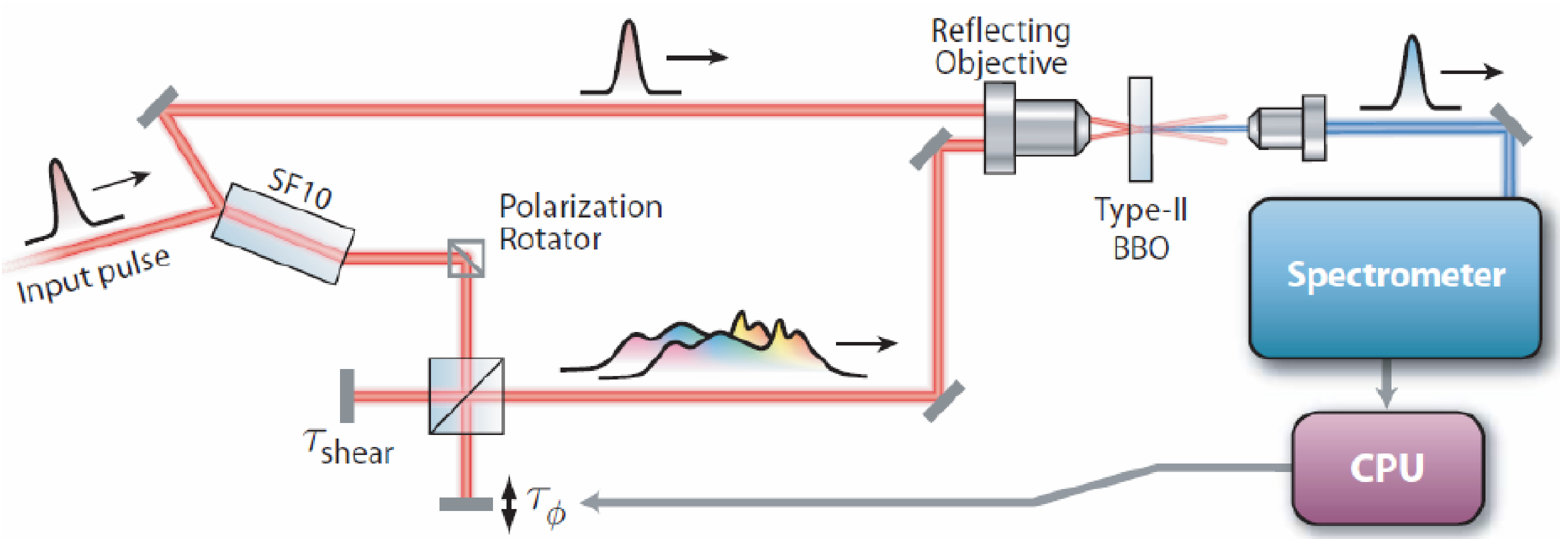


# Measurement Results



# 2DSI (Two Dimensional Spectral Shearing Interferometer)

The technique does not suffer from the calibration sensitivities of SPIDER nor the bandwidth limitations of FROG or interferometric autocorrelation (IAC).

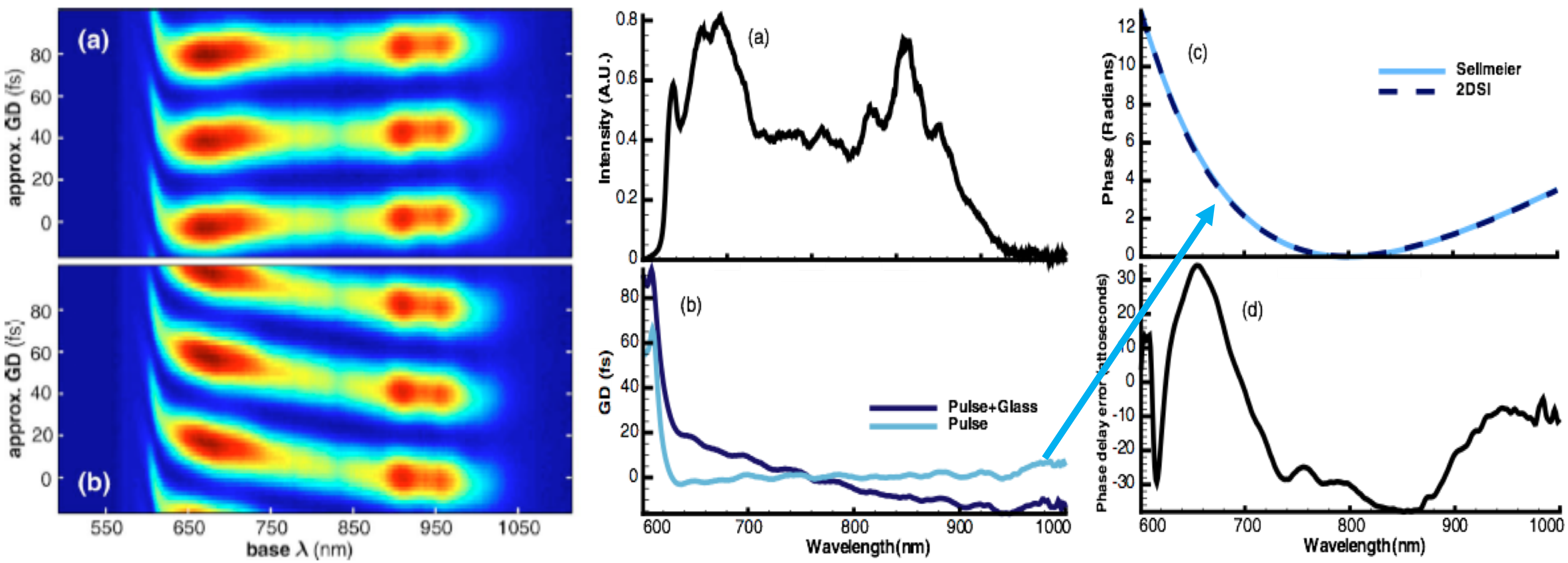



$$I(\omega,\tau_{cw}) = |A(\omega)|^2 + |A(\omega-\Omega)|^2 + 2|A(\omega)A(\omega-\Omega)|$$

$$\times \cos[\omega_{cw}\tau_{cw} + \underbrace{\phi(\omega) - \phi(\omega-\Omega)}_{\tau_g(\omega)\Omega + O[\Omega^2]}],$$

# 2DSI analysis

Relative fringe phase is what matters, so the delay scan does not need to be calibrated




**Phase delay error within ±35 attoseconds over 400 nm.**

$$I(\omega,\tau_{\rm cw}) = |A(\omega)|^2 + |A(\omega-\Omega)|^2 + 2|A(\omega)A(\omega-\Omega)|$$
$$\times \cos[\omega_{\rm cw}\tau_{\rm cw} + \underbrace{\phi(\omega) - \phi(\omega-\Omega)}_{\tau_g(\omega)\Omega + O[\Omega^2]}],$$

# Absolute accuracy