UFS Lecture 16: Noise in Mode-Locked lasers

Pulse train from a mode-locked laser

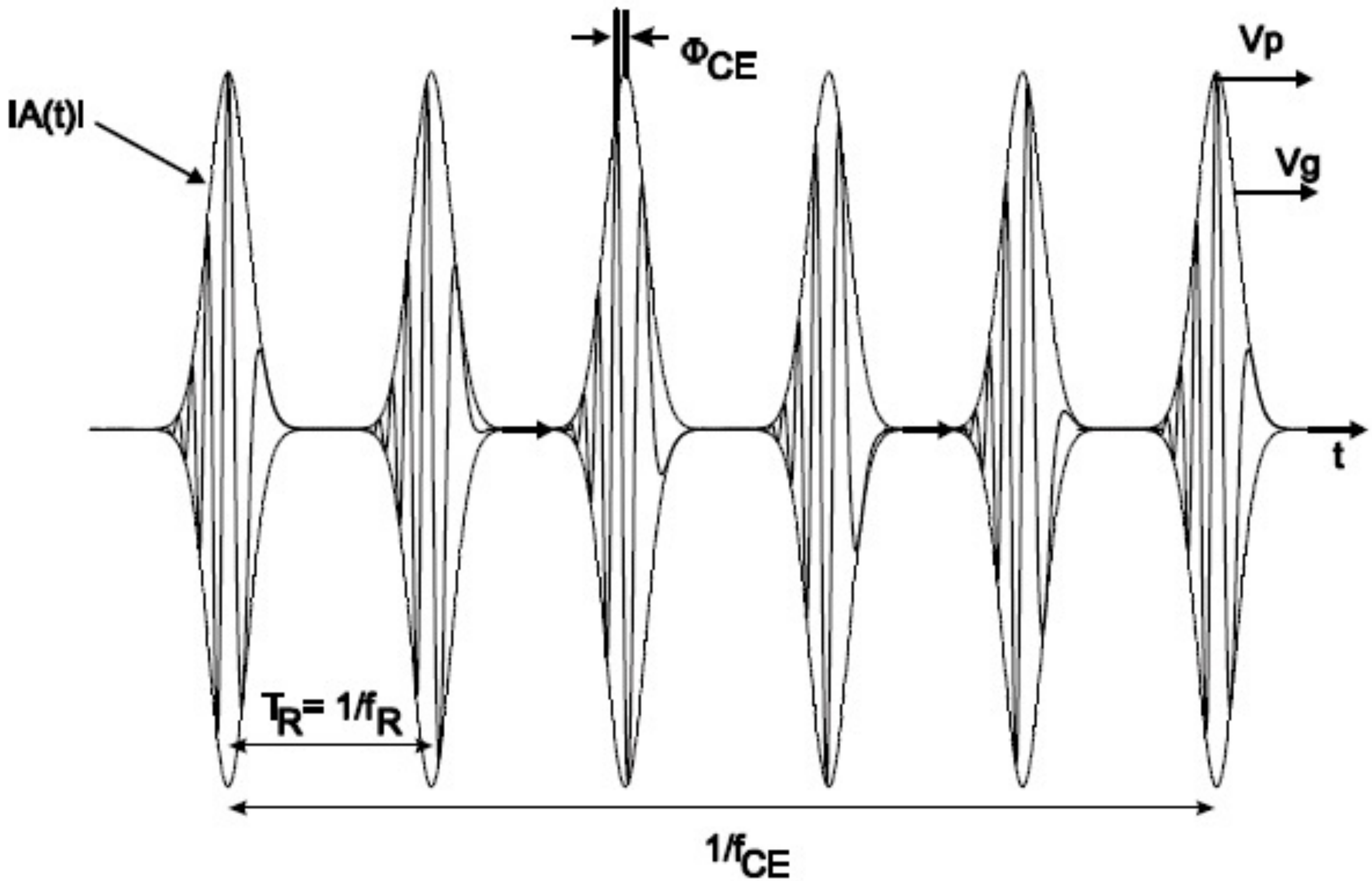


Figure 10.1: Pulse train from a mode-locked laser.

Optical spectrum of a mode-locked laser

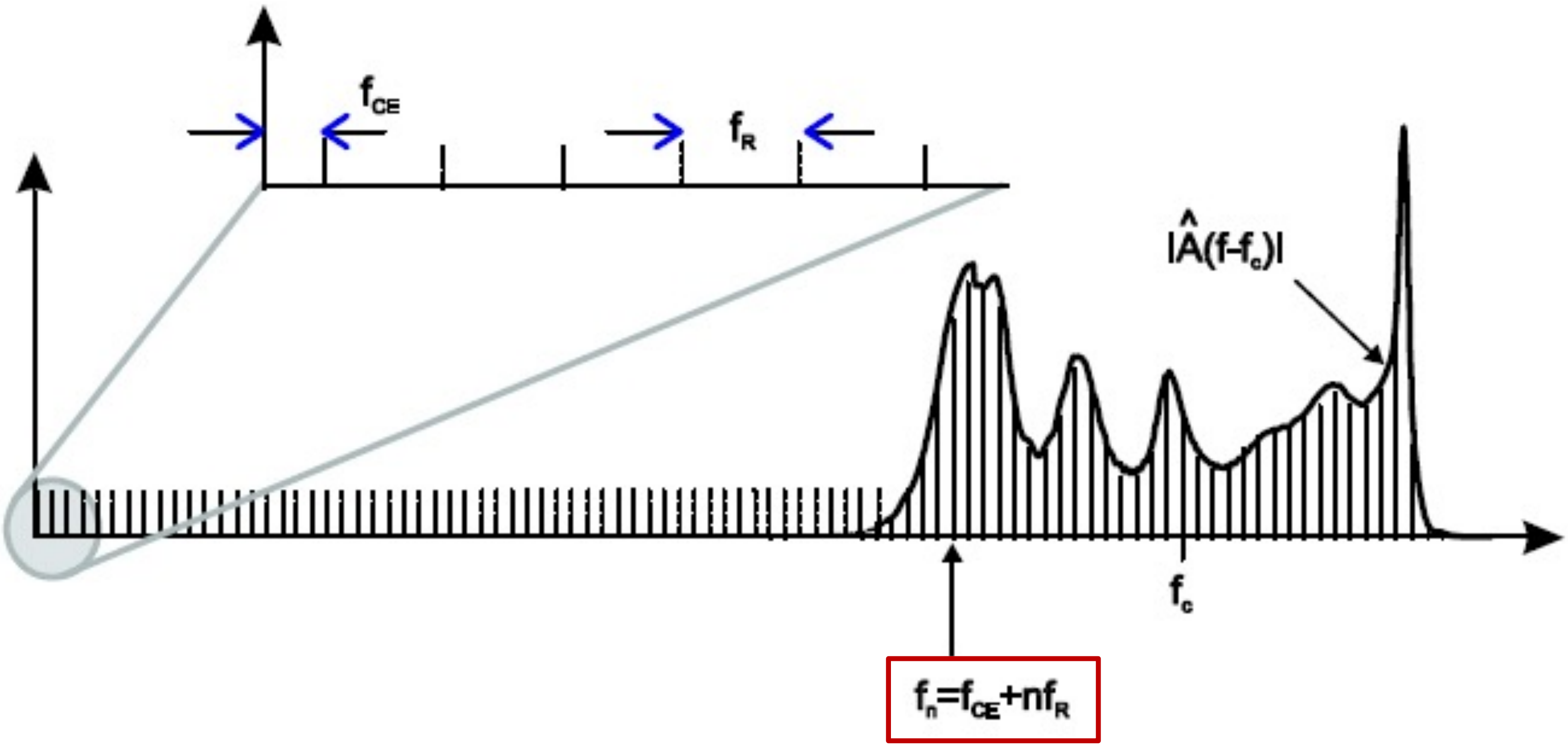


Figure 10.2: Optical mode comb of a mode-locked laser output.

Perturbation theory

$$T_R \frac{\partial}{\partial T} a = jD \frac{\partial^2}{\partial t^2} a - j\delta |a|^2 a + (g - l)a + D_f \frac{\partial^2}{\partial t^2} a + \gamma |a|^2 a + L_{\text{pert}}$$

The perturbations cause fluctuations in amplitude, phase, center frequency and timing of the soliton and generate background radiation, i.e. continuum

$$\begin{aligned} \Delta A(T,t) &= \Delta w(T) f_w(t) + \Delta\theta(T) f_\theta(t) + \Delta p(T) f_p(t) \\ &\quad + \Delta t(T) f_t(t) + a_c(T,t). \end{aligned} \tag{10.12}$$

The dynamics of the pulse parameters due to the perturbed NLSE can be projected out from the perturbation using the adjoint basis using the orthogonality relation

$$\mathrm{Re}\left\{ \int_{-\infty}^{+\infty} \bar{f}_i^*(t) f_j(t) dt \right\} = \delta_{i,j}.$$

Perturbation theory

$$\frac{\partial}{\partial T}\Delta w = -\frac{1}{\tau_w}\Delta w + \frac{1}{T_R}\mathrm{Re}\left\{\int_{-\infty}^{+\infty} \bar{f}_w^*(t) L_{\mathrm{pert}}(T,t)dt\right\} \quad (10.14)$$

$$\frac{\partial}{\partial T}\Delta\theta(T) = \frac{2\phi_o}{T_R}\frac{\Delta w}{w_o} + \frac{1}{T_R}\mathrm{Re}\left\{\int_{-\infty}^{+\infty} \bar{f}_\theta^*(t) L_{\mathrm{pert}}(T,t)dt\right\} \quad (10.15)$$

$$\frac{\partial}{\partial T}\Delta p(T) = -\frac{1}{\tau_p}\Delta p + \frac{1}{T_R}\mathrm{Re}\left\{\int_{-\infty}^{+\infty} \bar{f}_p^*(t) L_{\mathrm{pert}}(T,t)dt\right\} \quad (10.16)$$

$$\frac{\partial}{\partial T}\Delta t = \frac{-2|D|}{T_R}\Delta p + \frac{1}{T_R}\mathrm{Re}\left\{\int_{-\infty}^{+\infty} \bar{f}_t^*(t) L_{\mathrm{pert}}(T,t)dt\right\} \quad (10.17)$$

Physics behind:

(10.15)→ a change of soliton energy causes a cumulative change of phase since the contribution from the Kerr effect has changed.

(10.14) & (10.16)→ due to gain saturation, gain filtering, and saturable absorber action, the pulse energy and center frequency fluctuations are damped with decay constants

$$\frac{1}{\tau_w} = (2g_d - 2\gamma A_o^2) \qquad \frac{1}{\tau_p} = \frac{4}{3}\frac{g_s}{\Omega_g^2\tau^2}\frac{1}{T_R}.$$

(10.17)→ a change of carrier frequency causes a cumulative change of displacement due to a change in group velocity.

Noise as a perturbation

Many noise sources: Fluctuations of the pump power
Mirror vibrations
Air currents, air pressure fluctuations
Temperature fluctuations

Here, we consider only fundamental noise sources:
Spontaneous emission noise from amplifier

$$L_{\text{pert}} = \xi(t, T)$$

Modeled as Gaussian white noise sources with autocorrelation function:

$$\langle \xi(t', T') \xi^*(t, T) \rangle = \underline{T_R^2 P_n} \delta(t - t') \delta(T - T')$$

Power spectral density of noise source

Noise energy added to intracavity field within one roundtrip: $P_n \cdot T_R$

$$P_n = \Theta \frac{2 g_s}{T_R} \hbar \omega_c = \Theta \frac{\hbar \omega_c}{\tau_{ph}}$$

Excess noise factor of amplifier (non ideal amplifier)

Photon lifetime

Perturbations in amplitude, phase, carrier frequency and timing

$$\begin{aligned}
\frac{\partial}{\partial T}\Delta w &= -\frac{1}{\tau_w}\Delta w + S_w(T), \\
\frac{\partial}{\partial T}\Delta\theta(T) &= \frac{2\phi_o}{T_R}\frac{\Delta w}{w_o} + S_\theta(T), \\
\frac{\partial}{\partial T}\Delta p(T) &= -\frac{1}{\tau_p}\Delta p + S_p(T), \\
\frac{\partial}{\partial T}\Delta t &= \frac{-2|D|}{T_R}\Delta p + S_t(T),
\end{aligned}$$

Noise source that generates amplitude fluctuations

With:

$$\begin{aligned}
S_w(T) &= \frac{1}{T_R}\mathrm{Re}\left\{\int_{-\infty}^{+\infty} \bar{f}_w^*(t)\xi(T,t)dt\right\}, \\
S_\theta(T) &= \frac{1}{T_R}\mathrm{Re}\left\{\int_{-\infty}^{+\infty} \bar{f}_\theta^*(t)\xi(T,t)dt\right\}, \\
S_p(T) &= \frac{1}{T_R}\mathrm{Re}\left\{\int_{-\infty}^{+\infty} \bar{f}_p^*(t)\xi(T,t)dt\right\}, \\
S_t(T) &= \frac{1}{T_R}\mathrm{Re}\left\{\int_{-\infty}^{+\infty} \bar{f}_t^*(t)\xi(T,t)dt\right\}.
\end{aligned}$$

Correlation functions of reduced noise sources

$$\begin{aligned}
\langle S_w(T')S_w(T)\rangle &= \frac{P_n}{4w_0}\delta(T-T'), \\
\langle S_\theta(T')S_\theta(T)\rangle &= \frac{4}{3}\left(1+\frac{\pi^2}{12}\right)\frac{P_n}{w_o}\delta(T-T'), \\
\langle S_p(T')S_p(T)\rangle &= \frac{4}{3}\frac{P_n}{w_o}\delta(T-T'), \\
\langle S_t(T')S_t(T)\rangle &= \frac{\pi^2}{3}\frac{P_n}{w_o}\delta(T-T'), \\
\langle S_i(T')S_j(T)\rangle &= 0 \text{ for } i \neq j.
\end{aligned}$$

Noise sources are white and independent!

Define power spectra of amplitude, phase, frequency and timing fluctuations:
e.g. amplitude fluctuations:

$$|\Delta\hat{w}(\Omega)|^2 = \int_{-\infty}^{+\infty} \langle \Delta\hat{w}(T+\tau)\Delta\hat{w}(T)\rangle e^{-j\Omega\tau} d\tau, \text{ etc.}$$

Power spectral densities

$$\begin{aligned}
\left|\frac{\Delta\hat{w}(\Omega)}{w_o}\right|^2 &= \frac{4}{1/\tau_w^2+\Omega^2}\frac{P_n}{w_o}, \\
|\Delta\hat{\theta}(\Omega)|^2 &= \frac{1}{\Omega^2}\left[\frac{4}{3}\left(1+\frac{\pi^2}{12}\right)\frac{P_n}{w_o}+\frac{16}{(1/\tau_p^2+\Omega^2)}\frac{\phi_o^2}{T_R^2}\frac{P_n}{w_o}\right] \\
|\Delta\hat{p}(\Omega)\tau|^2 &= \frac{1}{1/\tau_p^2+\Omega^2}\frac{4}{3}\frac{P_n}{w_o}, \\
\left|\frac{\Delta\hat{t}(\Omega)}{\tau}\right|^2 &= \frac{1}{\Omega^2}\left[\frac{\pi^2}{3}\frac{P_n}{w_o}+\frac{1}{(1/\tau_p^2+\Omega^2)}\frac{4}{3}\frac{4|D|^2}{T_R^2\tau^4}\frac{P_n}{w_o}\right].
\end{aligned}$$

Finite energy and center frequency fluctuations:

$$\boxed{\begin{aligned}
\left\langle\left(\frac{\Delta w}{w_o}\right)^2\right\rangle &= 2\frac{P_n\tau_w}{w_o} \\
\langle(\Delta p\tau)^2\rangle &= \frac{2}{3}\frac{P_n\tau_p}{w_o}
\end{aligned}}$$

Phase noise and timing jitter

Undergo a diffusive motion with variances:

$$\sigma_\theta^2(T) = \langle(\Delta\theta(T)-\Delta\theta(0))^2\rangle = \frac{4}{3}\left(1+\frac{\pi^2}{12}\right)\frac{P_n}{w_o}|T| + 16\frac{\phi_o^2}{T_R^2}\frac{P_n}{w_o}\tau_w^3\left(\exp\left[-\frac{|T|}{\tau_w}\right]-1+\frac{|T|}{\tau_w}\right)$$

$$\sigma_t^2(T) = \left\langle\left(\frac{\Delta t(T)-\Delta t(0)}{\tau}\right)^2\right\rangle = \frac{\pi^2}{3}\frac{P_n}{w_o}|T| + \frac{4}{3}\frac{4|D|^2}{T_R^2\tau^4}\frac{P_n}{w_o}\tau_p^3\left(\exp\left[-\frac{|T|}{\tau_p}\right]-1+\frac{|T|}{\tau_p}\right)$$

Causes fundamental linewidth of optical lines and the microwave photo current spectrum

Phase Noise

Phase difference: $\varphi = \Delta\theta(T) - \Delta\theta(0)$ Gaussian random variable!

with probability density: $p(\varphi) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{\varphi^2}{2\sigma}}$, with $\sigma = \langle \varphi^2 \rangle$ variance

Expectation value of phasor: *exp(jφ)*

$$\begin{aligned} \left\langle e^{j\varphi} \right\rangle &= \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{+\infty} e^{-\frac{\varphi^2}{2\sigma}} e^{j\varphi} d\varphi \\ &= e^{-\frac{1}{2}\sigma}. \end{aligned}$$

10.4.1 The Optical Spectrum

(Neglecting amplitude and frequency noise)

$$\begin{aligned} A(t) &= \sum_{m=-\infty}^{+\infty} A_0 \operatorname{sech}\left(\frac{t - mT_R - \Delta t(mT_R)}{\tau}\right) \\ & e^{j\Delta\phi_{CE} \cdot m} e^{j\omega_c t} e^{-j\Delta\theta(mT_R)} \end{aligned}$$

$$S_{AA}(\omega) = \lim_{T=2NT_R\to\infty} \frac{1}{T}\langle \hat{A}_T^*(\omega)\hat{A}_T(\omega)\rangle$$

with

$$\hat{A}_T(\omega) = \int_{-T}^{T} A(t)e^{-j\omega t}dt \;=\; \hat{a}_0(\omega-\omega_c)\sum_{m=-N}^{N} e^{jmT_R\left(\frac{\Delta\phi_{CE}}{T_R}-\omega\right)}$$

$$e^{-j[(\omega-\omega_c)\Delta t(mT_R)+\Delta\theta(mT_R)]}$$

Fourier transform of pulse

$$\hat{a}_0(\omega) = \int_{-\infty}^{\infty} A_0 \operatorname{sech}\left(\frac{t}{\tau}\right)\, e^{-j\omega t}dt = A_0\pi\tau \operatorname{sech}\left(\frac{\pi}{2}\omega\tau\right)$$

$$S_{AA}(\omega) \;=\; \lim_{N\to\infty}|\hat{a}_s(\omega-\omega_c)|^2\frac{1}{2NT_R}\sum_{m'=-N}^{N}\sum_{m=-N}^{M} e^{jT_R\left(\frac{\phi_{CE}}{T_R}-\omega\right)(m-m')}$$

$$\left\langle e^{+j[(\omega-\omega_c)(\Delta t(mT_R)-\Delta t(m'T_R))-(\theta(mT_R)-\theta(m'T_R))]}\right\rangle$$

$$\boxed{k = m - m'}$$

$$S_{AA}(\omega) \;=\; |\hat{a}_s(\omega-\omega_c)|^2\frac{1}{T_R}\sum_{k=-\infty}^{\infty} e^{jT_R\left(\frac{\Delta\phi_{CE}}{T_R}-\omega\right)k}$$

m=0!

c

$$\left\langle e^{+j[2\pi(\omega-\omega_{\cancel{0}})(\Delta t((m+k)T_R)-\Delta t(mT_R))]}\right\rangle \left\langle e^{-j(\Delta\theta((m+k)T_R)-\Delta\theta(mT_R))}\right\rangle$$

$$S_{AA}(\omega) = \frac{|\hat{a}_s(\omega-\omega_c)|^2}{T_R}\sum_{k=-\infty}^{\infty} e^{jT_R\left(\frac{\phi_{CE}}{T_R}-\omega\right)k} e^{-\frac{1}{2}\sigma_\theta(kT_R)}$$
$$e^{-\frac{1}{2}\left[((\omega-\omega_c)\tau)^2\sigma_t(kT_R)\right]}$$

Noise close to line center is determined by correlation function for large T:

$$\sigma_\theta(T) = \frac{4}{3}\left(1+\frac{\pi^2}{12}+16\frac{\tau_w^2}{T_R^2}\phi_o^2\right)\frac{P_n}{w_o}|T| = 2\Delta\omega_\phi|T|,$$
$$\sigma_t(T) = \frac{1}{3}\left(\pi^2+\frac{\tau_p^2}{T_R^2}\left(\frac{D}{\tau^2}\right)^2\right)\frac{P_n}{w_o}|T| = 2\Delta\omega_t|T|.$$

with:

$$\Delta\omega_\phi = \frac{2}{3}\left(1+\frac{\pi^2}{12}+16\frac{\tau_w^2}{T_R^2}\phi_o^2\right)\frac{P_n}{w_o},$$
$$\Delta\omega_t = \frac{1}{6}\left(\pi^2+\frac{\tau_p^2}{T_R^2}\left(\frac{D}{\tau^2}\right)^2\right)\frac{P_n}{w_o}.$$

Poisson formula

$$\sum_{k=-\infty}^{+\infty} h[k] e^{-jkx} = \sum_{n=-\infty}^{+\infty} G(x + 2n\pi)$$

$$G(x) = \int_{-\infty}^{+\infty} h[k] e^{-jkx} dk,$$

$$\boxed{S_{AA}(\omega) = \frac{|\hat{a}_0(\omega - \omega_c)|^2}{T_R^2} \sum_{n=-\infty}^{+\infty} \frac{2\Delta\omega_n}{(\omega - \omega_n)^2 + \Delta\omega_n^2}}$$

Lorentzian lines at mode comb positions:

$\omega_n = n\omega_R + \dfrac{\Delta\phi_{CE}}{T_R},$ **with HWHM** $\Delta\omega_n = \Delta\omega_\phi + \underline{[\tau(\omega_n - \omega_c)]^2}\Delta\omega_t$

negligible at line center

$$\Delta\omega_\phi = \frac{2}{3}\left(1+\frac{\pi^2}{12}+16\frac{\tau_w^2}{T_R^2}\phi_o^2\right)\frac{\Theta 2g_s}{N_0 T_R}$$
$$= \frac{2}{3}\left(1+\frac{\pi^2}{12}+16\frac{\tau_w^2}{T_R^2}\phi_o^2\right)\frac{\Theta}{N_0 \tau_{ph}}$$

For cw-laser: Shawlow – Townes linewidth: $\Delta f_\phi = \frac{\Theta}{2\pi N_0 \tau_{ph}}$

Typical numbers: 50 nJ @1μm $N_0 = \frac{w_o}{\hbar\omega_c}$ = 2.5x10^{11} f_R = 100 MHz

$2\,l$ = 0.1 Θ= 2

→ $\Delta f_\phi = 8$ μHz x 100-10000

Expected optical linewidth:

→ $\Delta f_\phi \sim 1$ mHz - 1 Hz

10.4.2 The Microwave Spectrum

$$I(t) = \eta \frac{e}{\hbar\omega_c} |A(T,t)|^2 = \eta \frac{e}{\hbar\omega_c \tau} \times \sum_{m=-\infty}^{+\infty} \frac{w_0}{2} \mathrm{sech}^2 \left(\frac{t - mT_R - \Delta t(mT_R)}{\tau} \right)$$

$$\hat{I}_T(\omega) = \eta \frac{e w_0}{\hbar\omega_0 \tau} |a_0|^2 (\omega) \sum_{m=-N}^{+N} e^{-j\omega(mT_R + \Delta t(mT_R))}$$

Using the Poisson formula again results in

$$S_I(\omega) = \frac{(\eta e N_0)^2}{T_R^2} \left| |a_0|^2 (\omega) \right|^2 \sum_{n=-\infty}^{+\infty} \frac{2\Delta\omega_{I,n}}{(\omega - n\omega_R)^2 + \Delta\omega_{I,n}^2}$$

with the linewidth $\Delta\omega_{I,n}$ of the n-th harmonic

$$\Delta\omega_{I,n} = \left(2\pi n \frac{\tau}{T_R} \right)^2 \Delta\omega_t = \left(\frac{2\pi n}{M} \right)^2 \Delta\omega_t .$$

10-fs laser: $M=10^6$

10.4.3 Example: Er-fiber laser

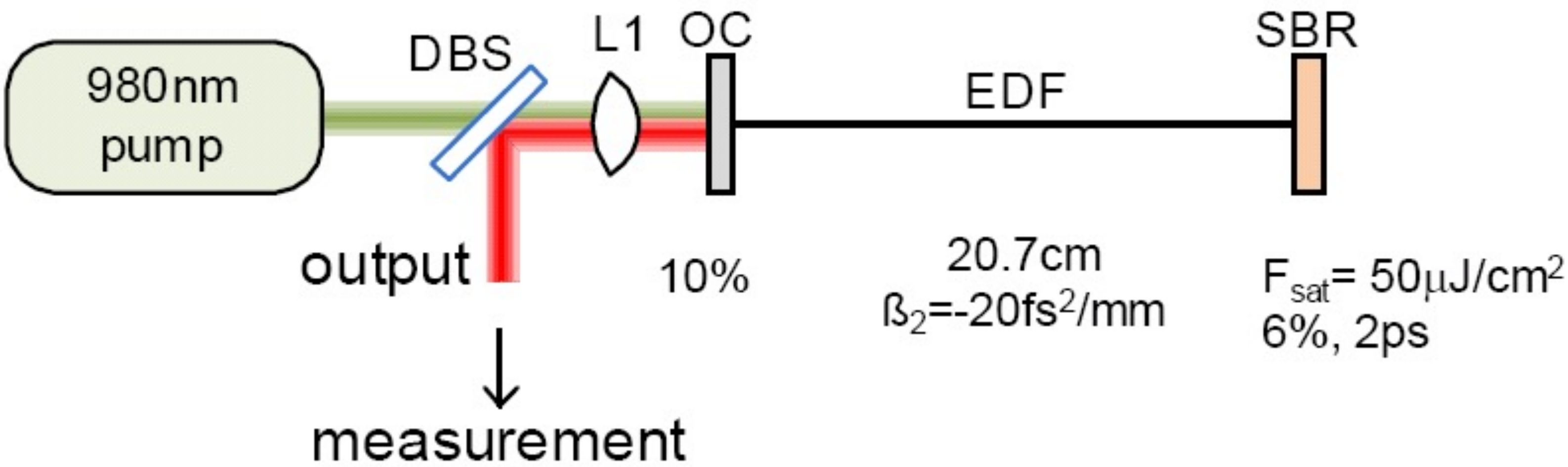


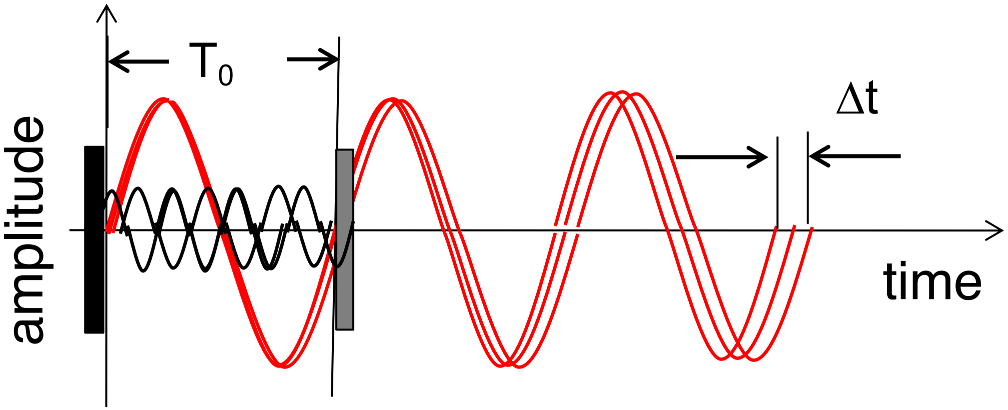
Figure 10.8: Schematic of soliton fiber laser mode-locked with a semiconductor saturable Bragg reflector (SBR)

Gain Half-Width Half Maximum	$\Omega_g = 2\pi \cdot \frac{0.3\mu\text{m/fs}}{(1.\mu\text{m})^2} 0.01\mu\text{m} = 19\text{THz}$
Saturated gain	$g_s = 0.13$
Pulse width	$\tau_{FWHM} = 180\text{fs},\ \tau = \tau_{FWHM}/1.76 = 100fs$
Pulse repetition time	$T_R = 2ns$
Decay time for center freq. fluctuations	$\frac{1}{\tau_p} = \frac{4}{3}\frac{g_s}{\Omega_g^2 \tau^2 T_R} = \frac{0.05}{T_R}$
Intracavity power	$P = 120\text{mW}$
Intra cavity pulse energy / photon number	$w_o = 240\text{pJ}, \quad N_0 = 0.2 \cdot 10^{10}$
Noise power spectral density	$P_n = \Theta \frac{2g_s}{T_R} \hbar\omega_o$
Amplifier excess noise factor	$\Theta = 2$
ASE noise	$\frac{P_n}{w_o} = \Theta \frac{2g_s}{T_R N_0} = 0.13Hz$
Dispersion	$-8240fs^2$
Frequency-to-timing conv.	$\frac{4}{\pi^2}\frac{4\|D\|^2}{\tau^4}\frac{\tau_p^2}{T_R^2} = \left(\frac{2}{\pi} \cdot 1.6 \cdot 20\right)^2 = (20)^2$
Timing jitter density	$\left\| \frac{\Delta\hat{t}(\Omega)}{\tau} \right\|^2 = \frac{1}{\Omega^2}\frac{\pi^2}{3}\frac{P_n}{w_o}\left(1 + \frac{4}{\pi^2}\frac{4\|D\|^2}{\tau^4}\frac{1}{(T_R^2/\tau_p^2 + T_R^2\Omega^2)}\right)$
Timing jitter $[f_{\min}, f_{\max}]$ for $f_{\min} << 1/\tau_p$, $f_{\min} = 10kHz$,	$\Delta t = \tau \sqrt{\frac{1}{12 \cdot f_{\min}}\frac{P_n}{w_o}\left(1 + \frac{4}{\pi^2}\frac{4\|D\|^2}{\tau^4}\frac{\tau_p^2}{T_R^2}\right)} = 2fs$

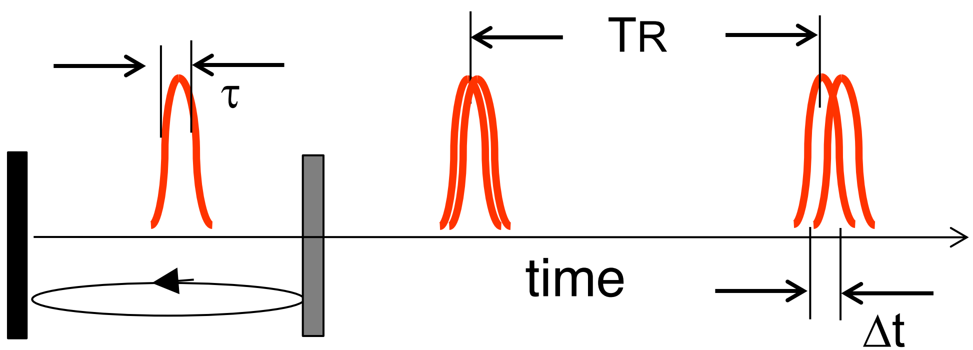
Table 10.1: Parameters for the soliton laser of Figure 10.8.

Timing Jitter of Femtosecond Lasers

Electronic Oscillator



Femtosecond Laser



Optical Cavity

Dissipation-Fluctuation Theorem

$$\frac{d}{dt} < \Delta t_{RF}^2 > \approx T_0^2 \cdot \frac{1}{W_{mode}} \cdot \frac{kT}{\tau_{cav}}$$

period ~100ps

cavity lifetime

10^{-6}

pulse width ~100fs

$$\frac{d}{dt} < \Delta t_{ML}^2 > \approx \tau^2 \cdot \frac{1}{W_{pulse}} \cdot \frac{\hbar\omega_c}{\tau_{cav}}$$

10^{-4}

$$\hbar\omega_c \sim 50kT$$

kT = thermal energy

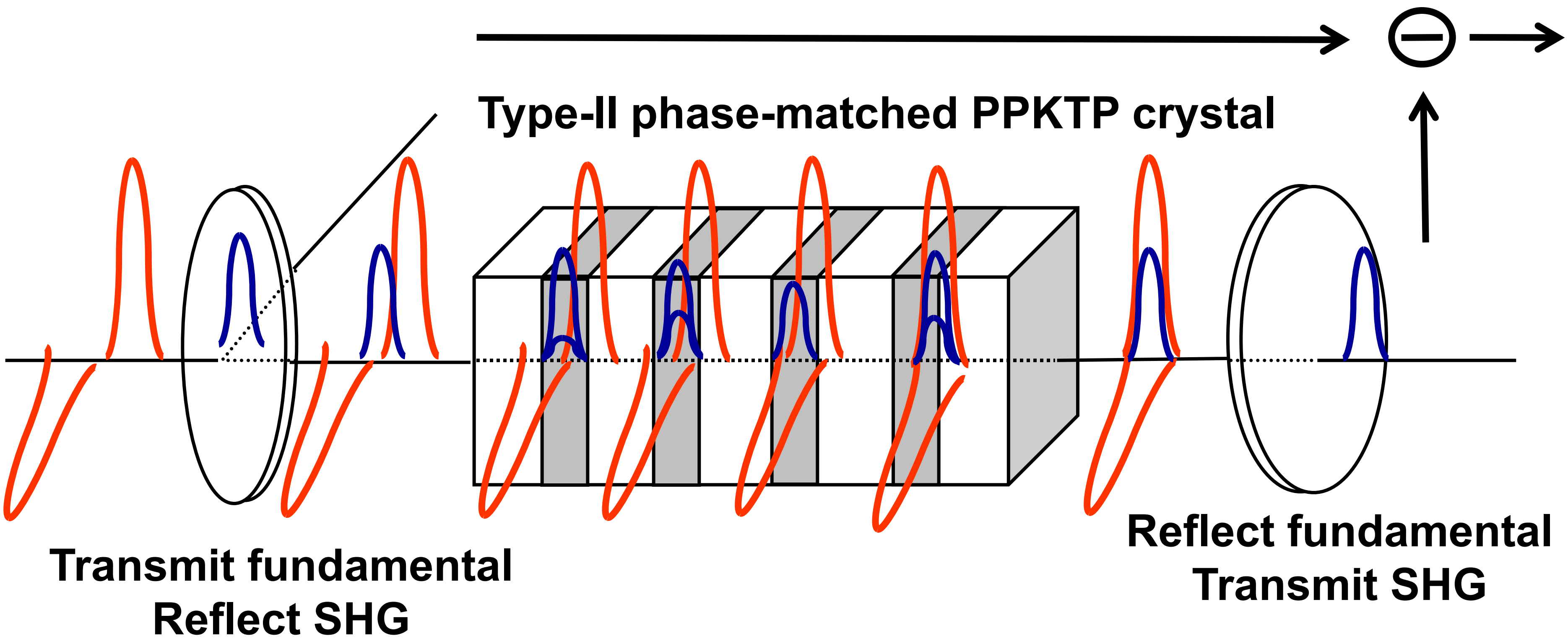
$\hbar\omega_c$ = photon energy

H. A. Haus and A. Mecozzi, IEEE JQE *29, 983* (1993).

J. Kim and F. X. Kärtner, Laser & Phot. Rev., 1–25 (2009).

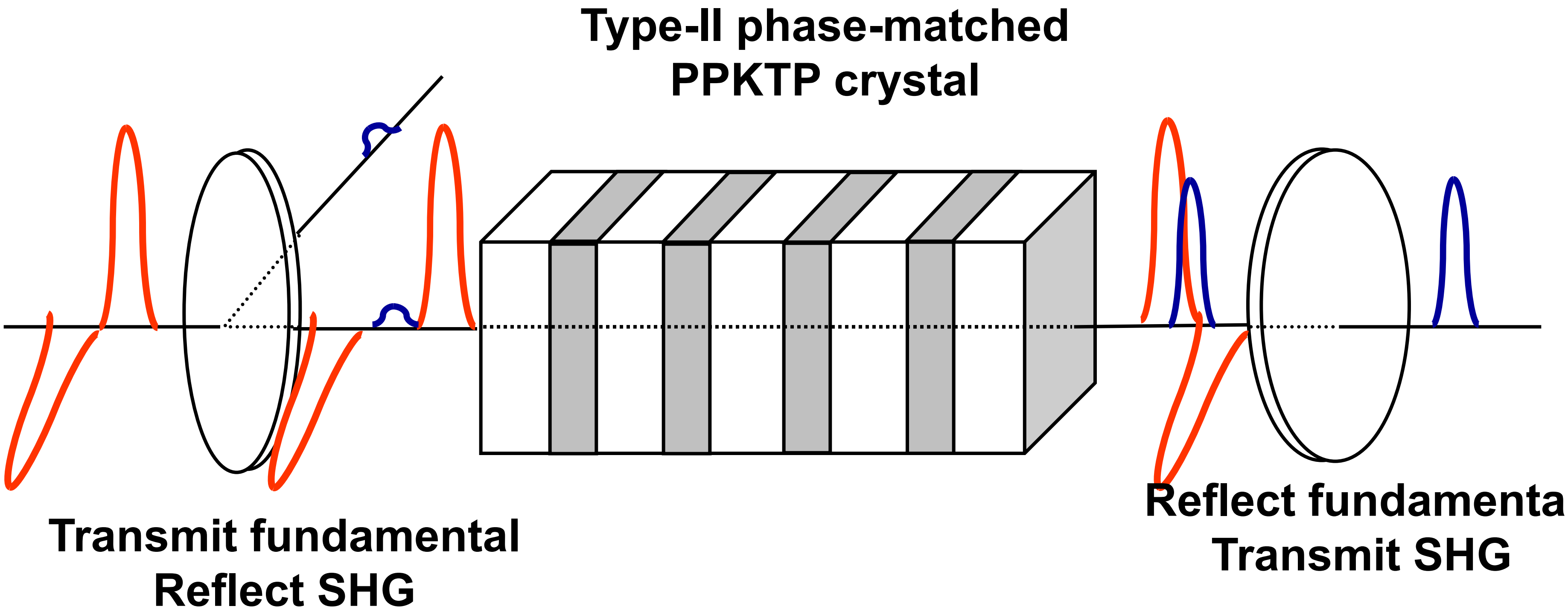
Single-crystal balanced cross-correlator

T. Schibli et al, OL 28, 947 (2003)



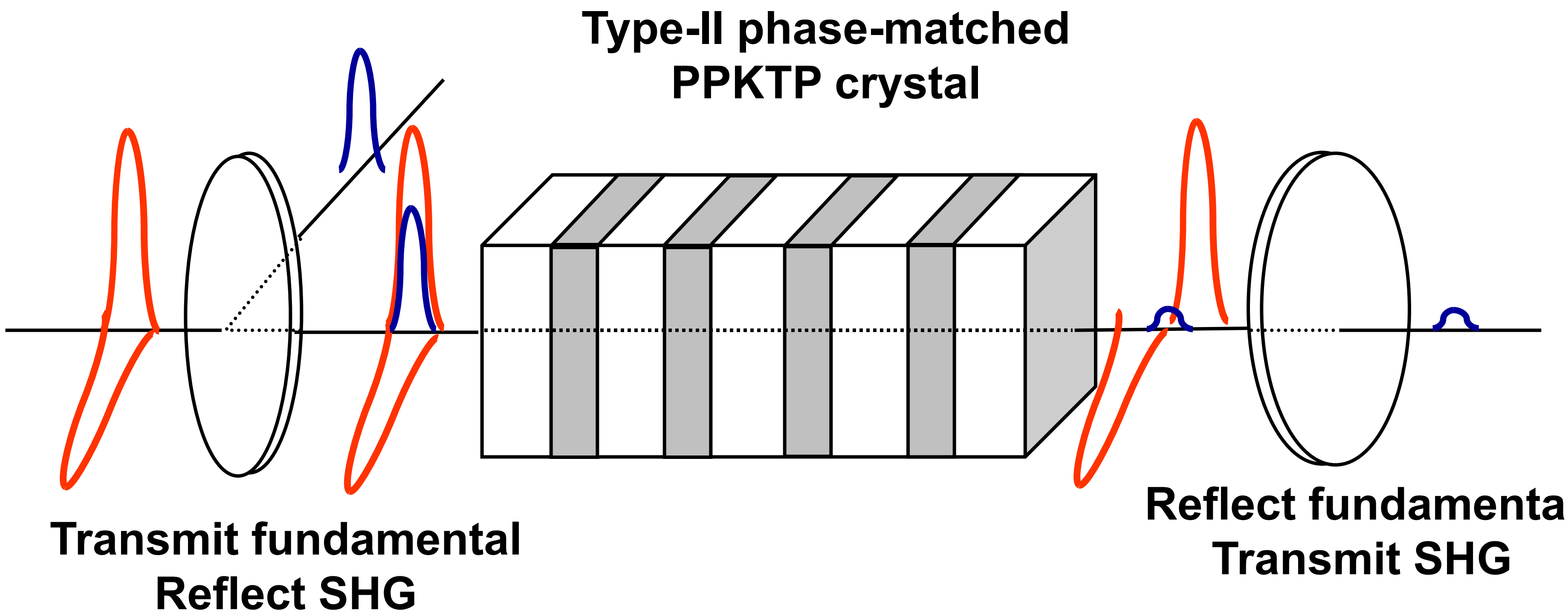
J. Kim et al., Opt. Lett. 32, 1044 (2007)

Single-crystal balanced cross-correlator



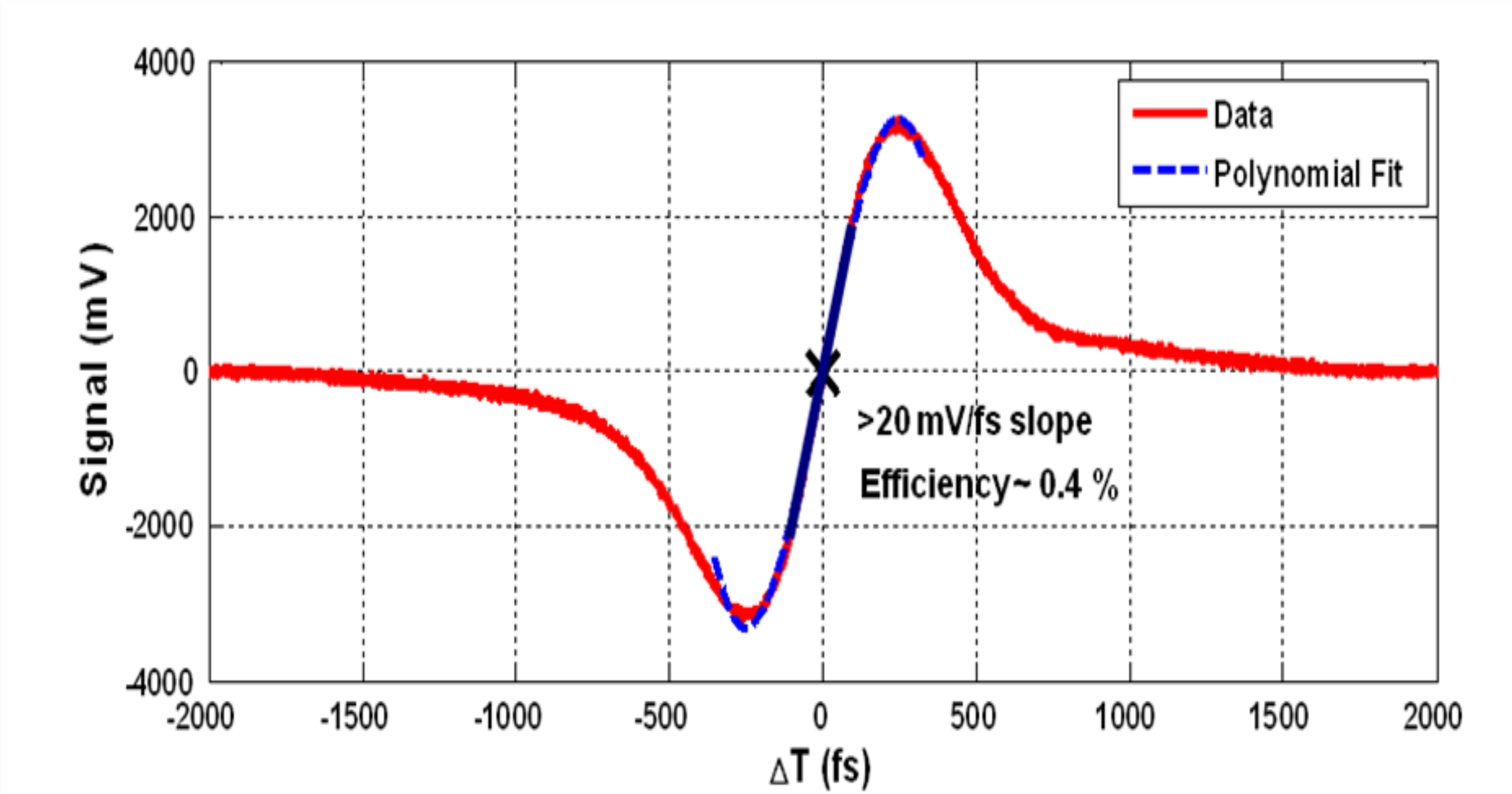
J. Kim et al., Opt. Lett. 32, 1044 (2007)

Single-crystal balanced cross-correlator



J. Kim et al., Opt. Lett. 32, 1044 (2007)

Single-crystal balanced cross-correlator

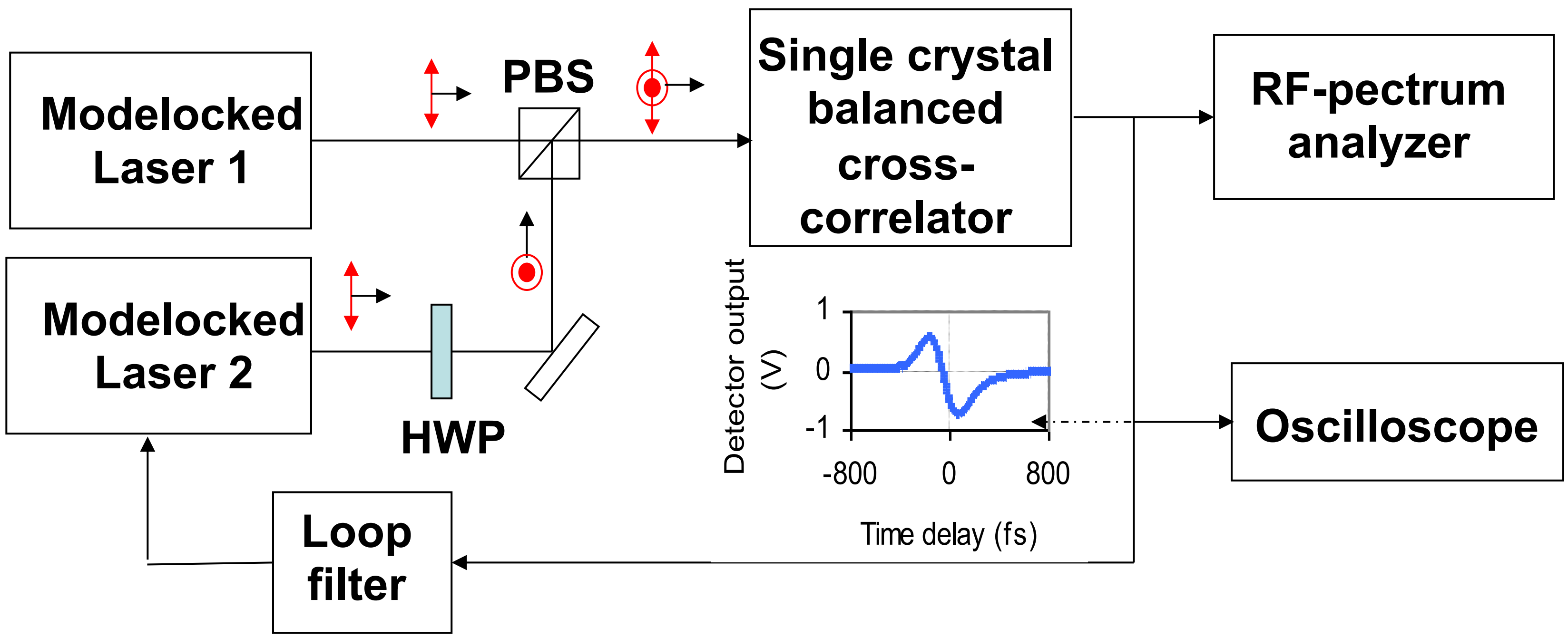


80 pJ, 200 fs
1550nm input pulses
at 200 MHz rep. rate

In comparison:
Typical microwave mixer
Slope ~1 μV/fs @ 10 GHz
Greatly reduced thermal drifts!

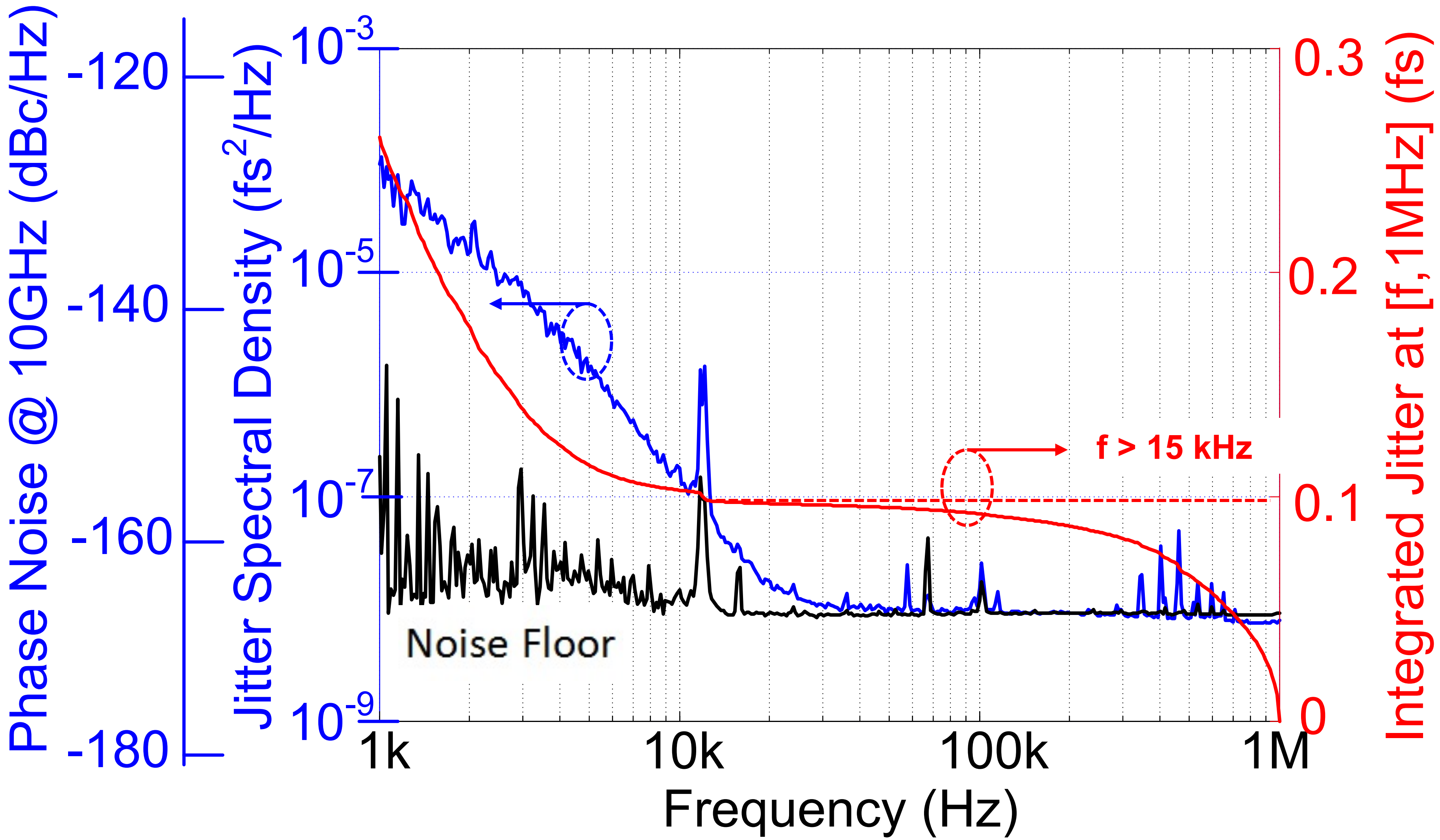
Timing jitter of lasers

Phase detector method → Timing Detector method



J. Kim, et al. , Opt. Lett. 32, 3519 (2007).

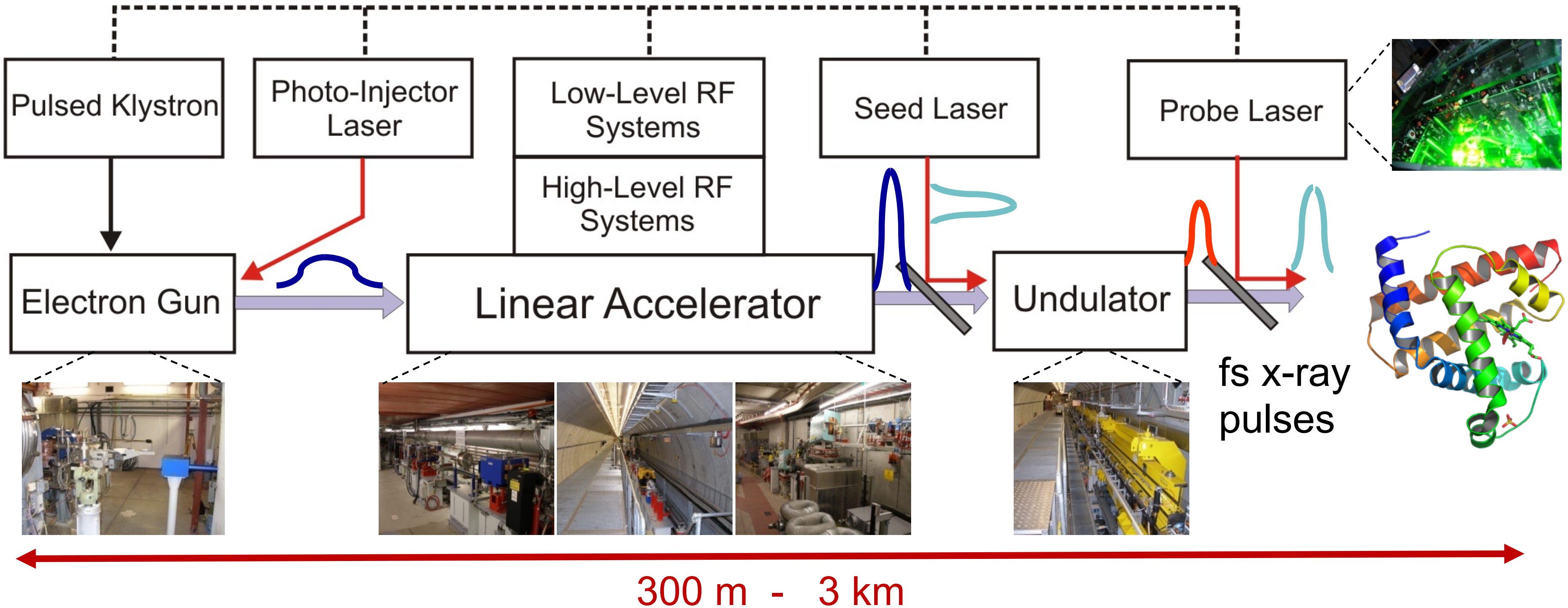
Timing jitter of OneFive:Origami Laser



K. Safak et al., Int. J. Structural Dynamics 2:(4) 041715 (2015).

Application: Timing of XFELs

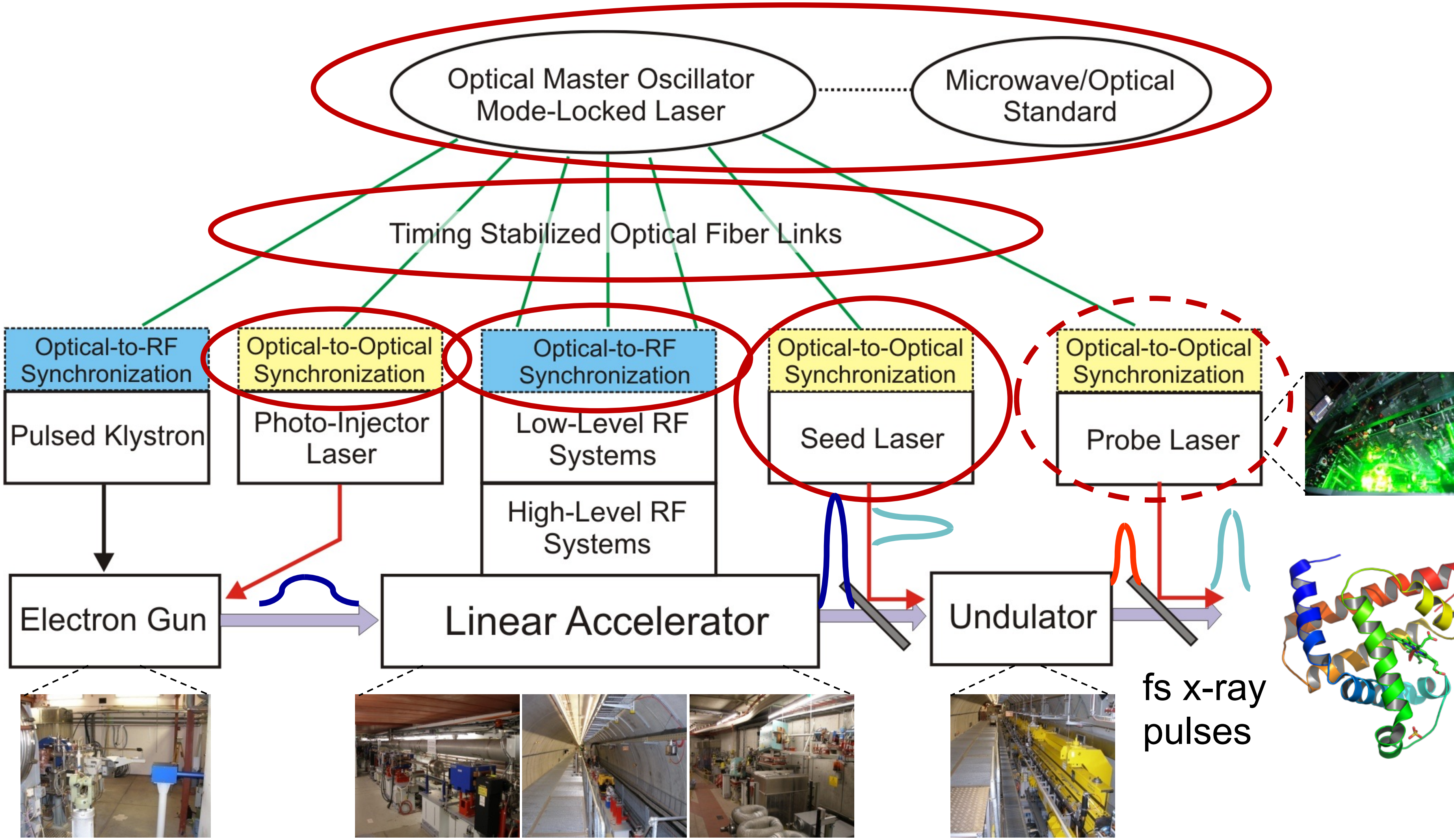
Seeded X-Ray FELs



Long-term sub-10 fs synchronization over entire facility desired.

Upcoming Attosecond FELs → sub-fs synchronization

Timing Distribution and Synchronization



J. Kim et al, FEL 2004.