UFS Lecture 14: Stochastic Process

1. Random Variables
2. Random Signals and Stationary Processes
3. Ergodic Processes and Wiener-Kintchin Theorem
4. Thermal Noise
5. Noise in linear systems
6. Ornstein Uhlenbeck Prozess: Amplitude Noise of Oscillators
7. Brownian Motion: Phase Noise of Oscillators

1. Random Variables

There are experiments that have a random outcome

Example I: Coin tossing. → Results is Head or Tail

Introduce Random Variable **X**, which can have two values

X = 0 (for Tail) or **X** = 1 (for Head)

There is a probability to find the value **X** = 0 called P(**X** = 0) and
a probability to find the value **X** = 1 called P(**X** = 1).

These probabilities are usually found by making repeated experiments and by noting how often the outcome **X** = 0, i.e. N_1 times or **X** = 1 is found, i.e. N_2 times in relation to the total numer of tosses made $N = N_1 + N_2$.

Then P(**X** = 0) = $\frac{N_1}{N}$ and P(**X** = 1) = $\frac{N_2}{N}$; for a symmetric coin those probabilities are 0.5.

We define a probability distribution: $p(x) = \begin{cases} 0.5, for\ x = 0 \\ 0.5, for\ x = 1 \end{cases}$ if we consider x to be a discrete variable.

Or we define $p(x)$ = $0.5\ \delta(x) + 0.5\ \delta(x-1)$,

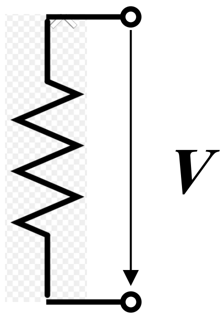
if we consider x to be a continuous variable.

Probability distributions are normalized:

for discrete variables: $\sum_i p(x_i) = 1$,

for continuous variables: $\int_{-\infty}^{+\infty} p(x)dx = 1$.

Example II: Voltage across and electronic resistor in thermal equilibrium:



$$p(v) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2}{2\sigma}}$$

We can perform experiments with a whole ensemble of equal resistors and compute expection values from the observations we make on all resistors.

Such an average is called ensemble average and denoted as:

$$\langle V \rangle = \int_{-\infty}^{+\infty} v\, p(v) dv = \int_{-\infty}^{+\infty} v \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2}{2\sigma}} dv = 0$$

Second order moment:

$$\langle V^2 \rangle = \int_{-\infty}^{+\infty} v^2\, p(v) dv = \sigma^2$$

n-th order moment:

$$\langle V^n \rangle = \int_{-\infty}^{+\infty} v^n\, p(v) dv$$

Variance and standard deviation

Variance: Var(V) = $\langle (V - \langle V \rangle)^2 \rangle$= $\langle V^2 \rangle - \langle V \rangle^2$

Standard deviation: sdev(V) = $\sqrt{Var(V)}$

For centered Gaussian distribution:

Var(V) = $\langle V^2 \rangle = \sigma^2$

sdev(V) = σ

Charateristic function of a random variable

$$C(s) = \mathcal{F}\{p(v)\} = \int_{-\infty}^{+\infty} p(v) e^{-jvs} dv \,,$$

is the generating function for all moments of a probability distribution

$$\frac{d^n}{ds^n} C(s) = \int_{-\infty}^{+\infty} (-jv)^n p(v) e^{-jvs} dv \quad \text{or}$$

. $$j^n \left. \frac{d^n}{ds^n} C(s) \right|_{s=0} = \int_{-\infty}^{+\infty} v^n p(v) dv = \langle V^n \rangle,$$ especially $C(s=0) = 1$

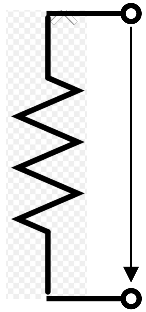
For example: Exponential distribution $p(x) = \begin{cases} \frac{1}{\sigma} e^{-\frac{x}{\sigma}}, for\ x \geq 0 \\ 0, for\ x < 0 \end{cases}$

$$C(s) = \int_{-\infty}^{+\infty} \frac{1}{\sigma} e^{-\frac{x}{\sigma} - jxs} dx = \frac{1}{\sigma\left(\frac{1}{\sigma} + js\right)} = \frac{1}{(1+js\sigma)}; \qquad j^n \frac{d^n}{ds^n} C(s) = \frac{n!\, \sigma^n}{(1+js\sigma)^{n+1}}$$

$$\langle V \rangle = \sigma; \quad \langle V^n \rangle = n!\, \sigma^n; \quad \langle V^2 \rangle = 2\, \sigma^2; \qquad sdev = \sigma$$

--> Fluctuations are as large as mean value.

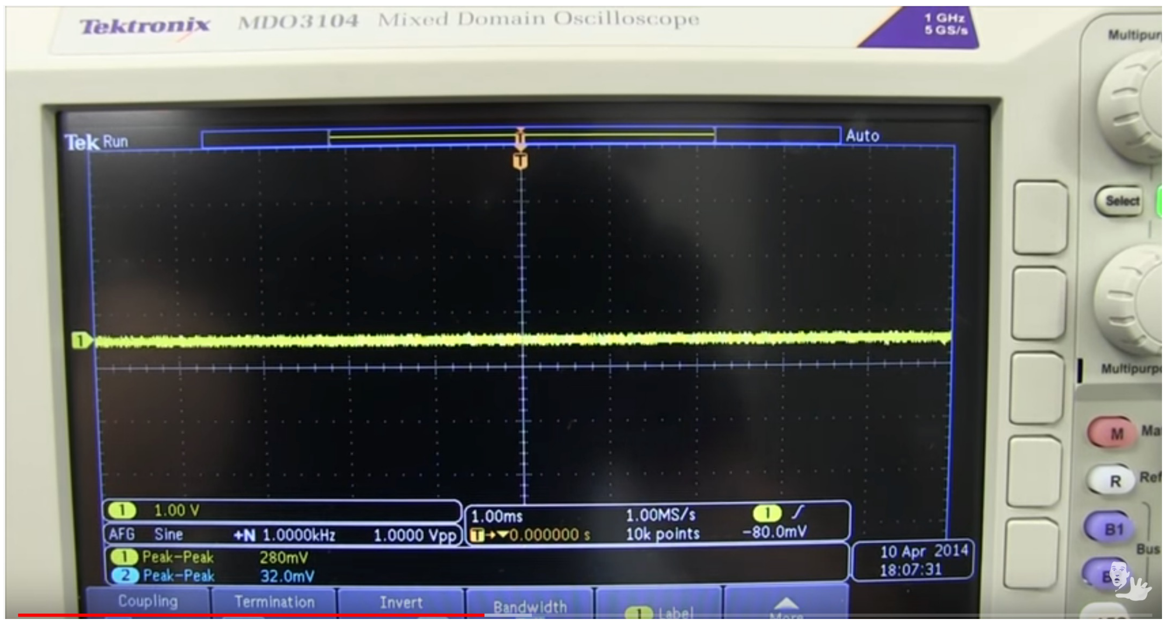
2. Random Signals and Stationary Process



V(t)

Is a random signal, i.e. a random variable at each point in time.

We can watch it on an oscilloscope!



But now we can build: Ensemble averages, i.e. expectation values

$$\langle V(t)^n \rangle \quad or\ correlation\ functions \quad \langle V(t)V(t+\tau)\rangle.$$

Stationary Processes:

$$\langle V(t)V(t+\tau)\rangle = \langle V(0)V(\tau)\rangle = CVV(\tau).$$ symmetric in τ

This average does not depend on the time we measure it!

Strictly speaking one can distinguish between n-th order stationary processes.

3. Ergotic Processes and Wiener-Kintchin Theorem

Instead of ensemble averages we can also build time averages!

Often (we assume always if nothing else is specified) it does not matter whether one performs a time average of a certain random variable of the system or we build an ensemble average over many identical systems, such systems are called ergodic. Depending on the system you may need a rather long time average until the system samples all of it‘s phase space.

For ergotic systems we have for example for the auto correlation function of variable V:

$$\boxed{C_{VV}(\tau) = \langle V(t)V(t+\tau)\rangle = \lim_{T\to\infty} \overline{V(t)V(t+\tau)}}$$

Wiener-Khintchin Theorem:

$$\langle V(t)V(t+\tau)\rangle = \langle V(0)V(\tau)\rangle = CVV(\tau).$$

We are interested in the power spectral density of stationary random signals $s(t)$:

and define time limited random signals and their Fourier Transforms

$$s_T(t) = \begin{cases} s(t), for\ |t| \le T \\ 0, for|t| > T \end{cases}, \qquad S_T(f) = \int_{-\infty}^{+\infty} s_T(t)\, e^{-j2\pi f t} dt$$

and

$$s_T(t) = \int_{-\infty}^{+\infty} S_T(f)\, e^{j2\pi f t} df$$

Then the auto-correlation function can be computed by

$$c_{ss}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{+\infty} s_T(t)\, s_T(t+\tau)\, dt =$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} S_T(f)\, e^{j2\pi f t} df \int_{-\infty}^{+\infty} S_T(f')\, e^{j2\pi f'(t+\tau)} df'\, dt =$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{+\infty} S_T(f)\, S_T(f') df \int_{-\infty}^{+\infty} \delta(f+f')\, e^{j2\pi f'\tau} df' =$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{+\infty} S_T^*(f)\, S_T(f) e^{j2\pi f\tau} df = \int_{-\infty}^{+\infty} \lim_{T\to\infty} \frac{1}{2T} < S_T^*(f) S_T(f) > e^{j2\pi f\tau} df$$

Correlation Spectrum:

We define the autocorrelation spectrum or power spectral density of the signal s(t):

$$C_{ss}(f) = \lim_{T\to\infty} \frac{1}{2T} < S_T^*(f) S_T(f) >$$

with

$$c_{ss}(\tau) = \int_{-\infty}^{+\infty} C_{ss}(f)\ e^{j2\pi f\tau} df$$

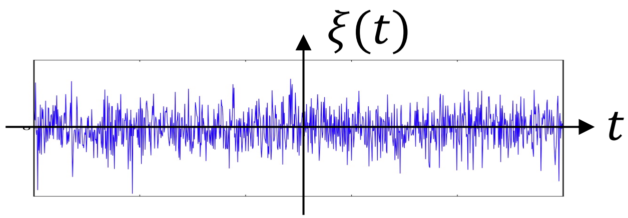
and

$$C_{ss}(f) = \int_{-\infty}^{+\infty} c_{ss}(\tau) e^{-j2\pi f\tau} d\tau$$

Wiener-Khintchin Theorem:

For ergodic processes the Fourier transform of the autocorrelation function is the autocorrelation spectrum which is equal to the power spectral density of the signal.

Example III: White Noise $\xi(t)$



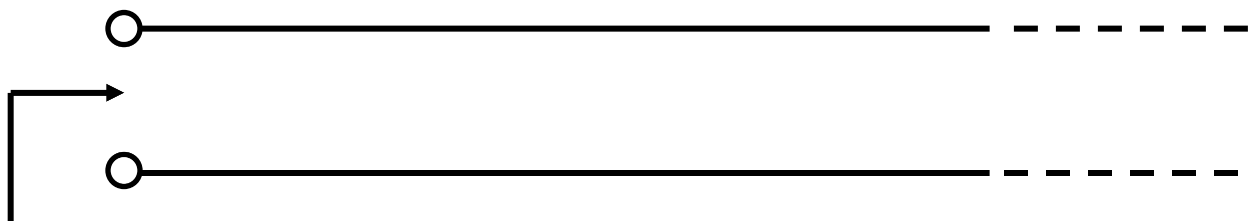
$C_{\xi\xi}(f) = D;\ for\ all\ frequencies: total\ power\ of\ this\ signal\ is\ infinite$

then

$c_{\xi\xi}(\tau) = D\ \delta(\tau)$; not defined at $\tau = 0$;since $c_{\xi\xi}(\tau = 0) =< \xi(t)^2 >$ does not exist.

4 Thermal Noise:

Model of an ideal resistor: the infinite transmission line



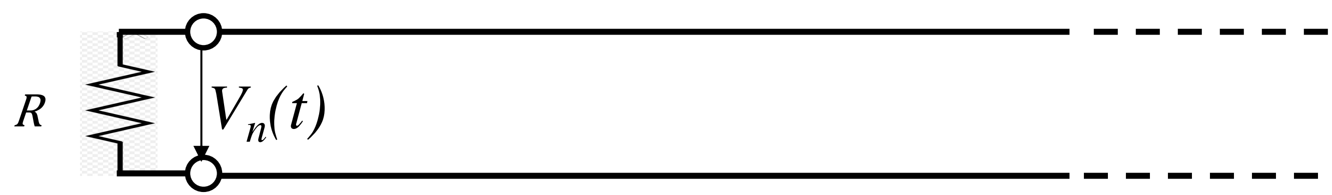
Characteristic impedance of transmission line: $Z_w = R$

All longitudinal modes on this line are in thermal equilibrium: excitation energy kT

Start from finite length L transmission line: modes at each frequency $f_n = \frac{nc}{2L}$

Density of modes: $dn = \frac{2L}{c} df$.

Thermal energy stored on transmission line in frequency interval df: $dw = kT\frac{2L}{c} df$.

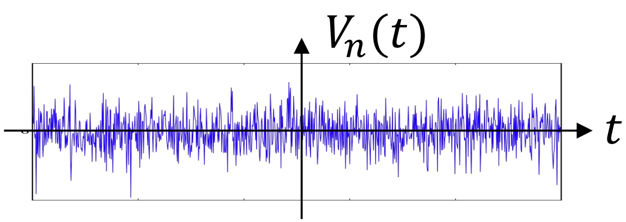


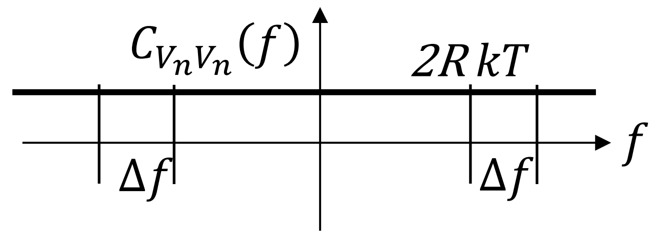
If we connect to the transmission line the resistor R, then the thermal power $dp = \frac{c}{2L} dw = kT\, df$ will flow into the resistor R in the frequency intervall df

The resistor would be heated up by this thermal energy if it would not emit by itself
In thermal equilibrium and equal amount of thermal power into the transmission line:

$dp = \frac{1}{2R} C_{VnVn}(f)\, df = kT\, df. \rightarrow$ $\boxed{C_{VnVn}(f) = 2R\,kT}$ Two sided autocorrelation spectrum of voltage fluctuations at resistor R.

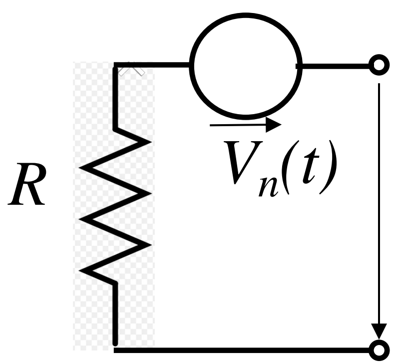
Thermal noise at resistor is white noise!

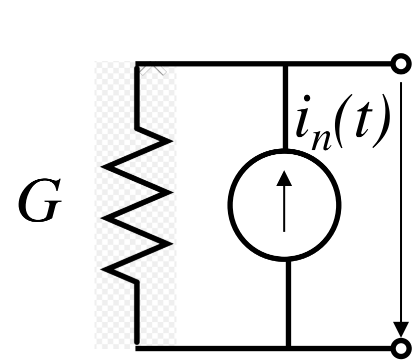




Voltage fluctuations in frequency range Δf: $\quad \langle V_n^2(t) \rangle = 4\,R\,kT\Delta f$

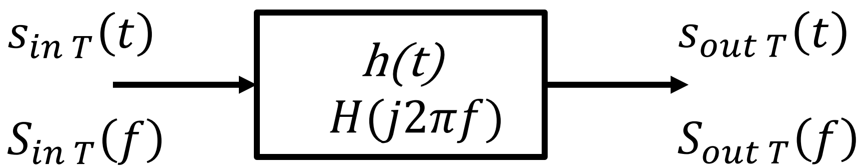
Circuit diagram of resistor with noise:





$C_{i_ni_n}(f) = 2G\,kT$

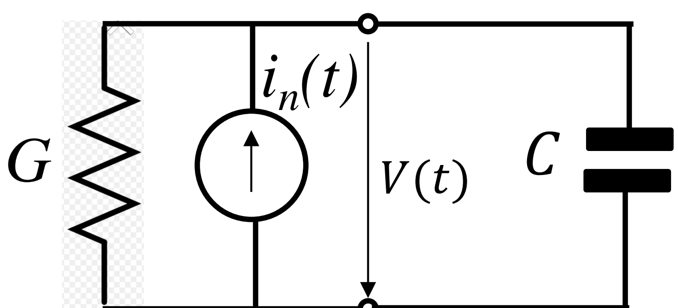
5 Noise in Linear Systems:



$$\boxed{C_{S_{out}S_{out}}(f) = |H(j2\pi f)|^2\, C_{S_{in}S_{in}}(f)}$$

Example IV: Low pass filter

$$C_{i_n i_n}(f) = 2GkT$$



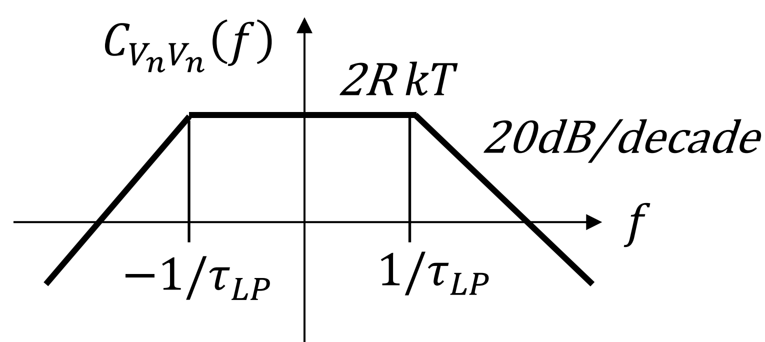
$$(G + j\omega C)\, V(j\omega) = I(j\omega)$$

$$\boxed{H(j\omega) = \frac{R}{(1+j\omega RC)}}$$

$$V(j\omega) = \frac{1}{(G+j\omega C)} I(j\omega)$$
$$= \frac{R}{(1+j\omega RC)} I(j\omega)$$

$$\boxed{C_{VV}(f) = \frac{2RkT}{1+(\omega RC)^2}}$$

$$\tau_{LP} = \frac{1}{2\pi} RC$$



6 Ornstein - Uhlenbeck Process

$$\frac{d}{dt} x(t) = -\gamma\, x(t) + \xi(t)$$ $\xi(t)$: white noise with power spectral density D.

Heavily damped motion driven by white noise!

$$x(t) = \int_{-\infty}^{t} e^{-\gamma(t-t')} \xi(t')\, dt'$$

Compute autocorrelation function: $<\xi(t')\xi(t'') >= D\, \delta(t' - t'')$

$$< x(t)x(t+\tau) >=< \int_{-\infty}^{t} e^{-\gamma(t-t')} \xi(t')\, dt' \int_{-\infty}^{t+\tau} e^{-\gamma(t+\tau-t'')} \xi(t'')\, dt'' >$$

$$= D \int_{-\infty}^{t} e^{-2\gamma(t-t')}\, dt'\, e^{-\gamma\tau} = \frac{D}{2\gamma} e^{-\gamma|\tau|}$$

$$\boxed{C_{xx}(f) = \mathcal{F}\left\{\frac{D}{2\gamma} e^{-\gamma|\tau|}\right\} = \frac{D}{\gamma^2} \frac{1}{1+(\omega/\gamma)^2}}$$

filtered white noise, like in a low pass.

7 Brownian Motion

$$\frac{d}{dt}x(t) = -\gamma\, x(t) + \xi(t)$$

$\xi(t)$: white noise with power spectral density D.

But damping goes to zero: $\gamma \to 0$

$$\frac{d}{dt}x(t) = \xi(t) \qquad\qquad x(t) = \int_{-\infty}^{t} \xi(t')\, dt'$$

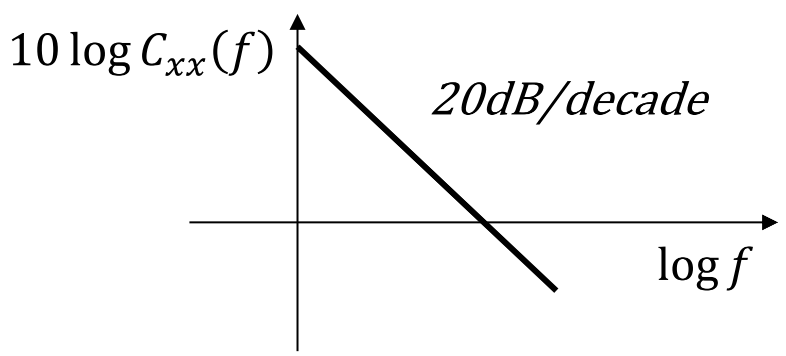
Compute autocorrelation function: $< \xi(t')\xi(t'') >= D\, \delta(t' - t'')$

$$< x(t)x(t+\tau) >= \lim_{\gamma\to 0} \frac{D}{2\gamma} e^{-\gamma|\tau|};$$

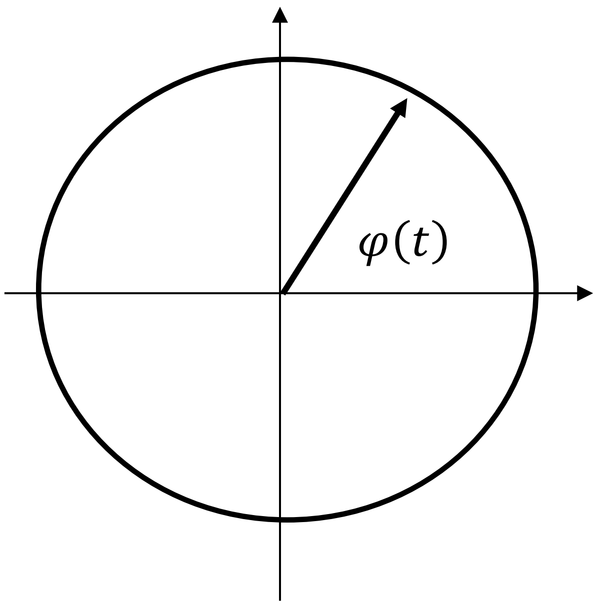
Not so easy?

But:

$$\boxed{C_{xx}(f) = \frac{D}{\omega^2}}$$



Phase Noise of Oscillators



$\xi(t)$: white noise with power spectral density D.

$$\frac{d}{dt}\varphi(t) = \xi(t) \quad \rightarrow \quad \varphi(t) = \int_0^t \xi(t')\, dt'$$

$$<\varphi(t')\varphi(t'')> = D \cdot min\{t'.t''\}$$

Second order moment: $<\varphi(t)^2> = D \cdot t$

Oscillation: $A(t) = A_0 \cdot e^{j\varphi(t)}$

Autocorrelation function: $<A(t)^* A(t+\tau)> = |A_0|^2 <e^{j(\varphi(t+\tau)-\varphi(t))}>$

Statistics of phase: $p(\varphi)=\frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{\varphi^2}{2\sigma}}$ $\quad <A(t)^* A(t+\tau)> = \int_{-\infty}^{+\infty} p(\varphi) e^{j\varphi\zeta}\, d\varphi$

$<A(t)^* A(t+\tau)>$: Characteristic function of phase at. ζ=1

$$c_{AA}(\tau) = <A(t)^* A(t+\tau)> = |A_0|^2\, e^{-\frac{1}{2}\sigma} = |A_0|^2\, e^{-\frac{1}{2}D|\tau|}$$

Phase Noise Spectrum of Oscillators

$$c_{AA}(\tau) = \, <A(t)^{*} A(t+\tau)> \, = |A_0|^2 e^{-\frac{1}{2}\sigma} = |A_0|^2 \, e^{-\frac{1}{2}D|\tau|}$$

Autocorrelation spectrum or power spectral density of oscillator:

$$C_{AA}(f) = \frac{|A_0|^2}{1+(\omega D/2)^2} = \frac{|A_0|^2}{1+(\pi f D)^2}$$

Single-Sideband Phase Noise of oscillator

$$Lf) = 10 \; Log \left(\frac{1}{1+(\pi f D)^2}\right)$$