UFS Lecture 9: Laser Dynamics

Review: coherent and incoherent light-matter interaction

Laser Rate Equations

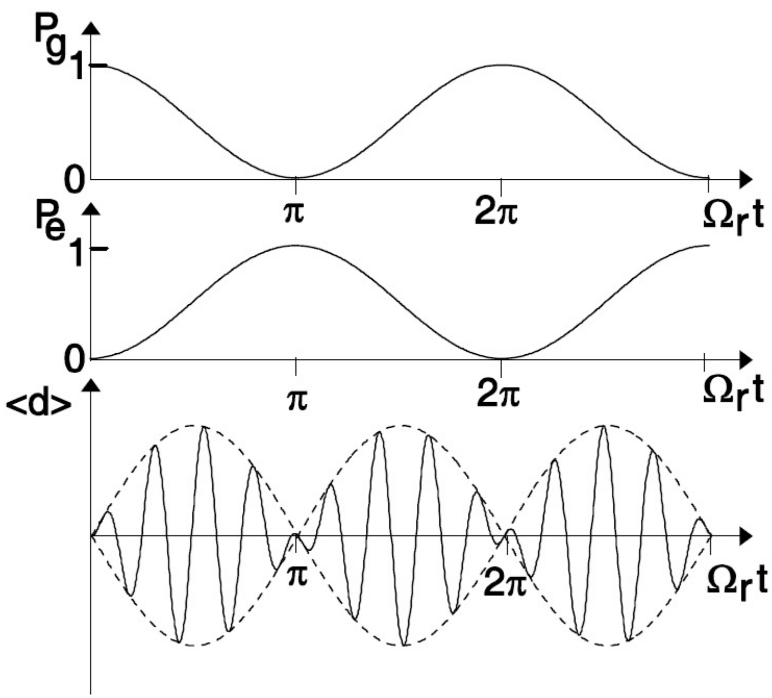
Built-up of Laser Oscillation and Continuous Wave Operation

Stability and Relaxation Oscillations

Laser Efficiency

Steady-state solution

For moderate field strength E_0, the magnitude of the Rabi-frequency is much smaller than the optical frequency, $|\Omega_r| << \omega$, the inversion and dipole moment do not change much within an optical cycle of the field.



If the optical pulse duration is longer than energy relaxation time constant T_1, implying that the temporal variation of the EM field is slow than the energy decay, we can assume that population inversion and dipole moment are always at the steady-state though the steady state value adjust following the amplitude variation of the EM field.

$$\boxed{\dot{\underline{d}} = 0 \qquad \dot{w} = 0}$$

$$\underline{d}_s = \frac{\mathrm{j}}{2\hbar} \frac{\left(\vec{M}_{eg}^{*} \cdot \vec{e}\right) w_s}{1/T_2 + \mathrm{j}(\omega - \omega_{eg})} \underline{E}_0$$

$$w_s = \frac{w_0}{1 + \frac{T_1}{\hbar^2} \frac{1/T_2 \, |\vec{M}_{eg}^{*} \cdot \vec{e}|^2}{(1/T_2)^2 + (\omega_{eg} - \omega)^2} |\underline{E}_0|^2}$$

Inversion saturation

We introduce the normalized lineshape function, which is in this case a Lorentzian:

$$L(\omega) = \frac{(1/T_2)^2}{(1/T_2)^2 + (\omega_{eg} - \omega)^2}$$

Intensity: $I = \frac{1}{2Z_F} |\underline{E}_0|^2$

Unsaturated inversion

Steady state inversion: $w_s = \dfrac{w_0}{1 + \frac{I}{I_s} L(\omega)}$

Stationary inversion depends on the intensity of the incident light

Saturated inversion

Saturation intensity:

$$I_s = \left[\frac{2T_1 T_2 Z_F}{\hbar^2} |\vec{M}^*_{eg} \cdot \vec{e}|^2 \right]^{-1}$$

Dielectric Susceptibility

Expectation value of the dipole moment $\left\langle \vec{d} \right\rangle = \vec{M}_{eg} \underline{d}\, e^{j\omega t} + c.c.$

Multiplication with the number of atoms per unit volume, N, relates the dipole moment of the atom to the macroscopic polarization P

$$\vec{P}(t) = \frac{1}{2}\left(\underline{\vec{P}}_0 e^{j\omega t} + \underline{\vec{P}}_0^{*} e^{-j\omega t}\right) = N \vec{M}_{eg} \underline{d}_s\, e^{j\omega t} + c.c.$$

$$\Downarrow$$

$$\underline{\vec{P}}_0 = 2N\vec{M}_{eg}\underline{d}_s$$

Definition of the complex susceptibility $\underline{\vec{P}}_0 = \epsilon_0 \chi(\omega) \vec{e}\, \underline{E}_0$

Linear susceptibility of the medium

$$\chi(\omega) = \vec{M}_{eg}\vec{M}_{eg}^{*} \frac{jN}{\hbar\epsilon_0} \frac{w_s}{1/T_2 + j(\omega - \omega_{eg})}$$

Linear susceptibility

If the incident EM field is weak $\quad \dfrac{I}{I_s}L(\omega) \ll 1 \longrightarrow w_s \approx w_0$

Linear susceptibility derived using semi-classical model

$$\chi(\omega) = \frac{1}{3}|\vec{M}_{eg}|^2 \frac{\mathrm{j}N}{\hbar\epsilon_0} \frac{w_0}{1/T_2 + \mathrm{j}(\omega - \omega_{eg})}$$

Linear susceptibility derived using classical harmonic oscillator model

$$\tilde{\chi}(\omega) = \frac{N\frac{e_0^2}{m}\frac{1}{\epsilon_0}}{(\Omega_0^2 - \omega^2) + 2\mathrm{j}\omega\frac{\Omega_0}{Q}} \xrightarrow{\omega \approx \Omega_0} \tilde{\chi}(\omega) = \frac{-\mathrm{j}N\frac{e_0^2}{m}\frac{1}{\epsilon_0}/(2\Omega_0)}{\mathrm{j}(\omega - \Omega_0) + \frac{\Omega_0}{Q}}$$

As the EM field has a frequency close to the oscillator's intrinsic frequency and define $Q = T_2\omega_{eg}$, the shape of the susceptibility computed quantum mechanically agrees well with the classical susceptibility derived from the harmonic oscillator model.

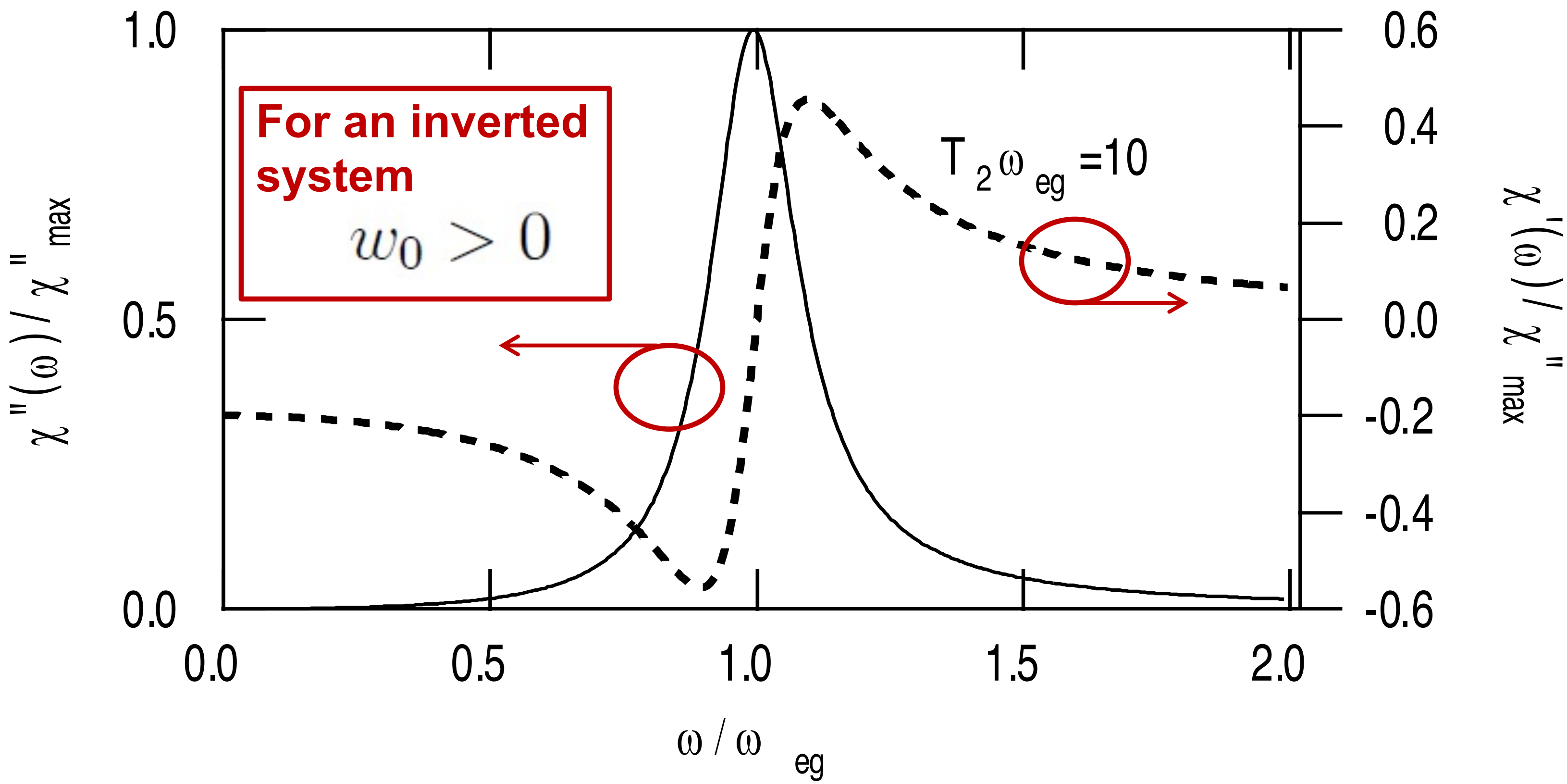
Linear susceptibility

Real and imaginary part of the susceptibility $\chi(\omega) = \chi'(\omega) + j\chi''(\omega)$

$$\chi'(\omega) = -\frac{|\vec{M}_{eg}|^2 N w_s T_2^2(\omega_{eg}-\omega)}{3\hbar\epsilon_0} L(\omega)$$

$$\chi''(\omega) = \frac{|\vec{M}_{eg}|^2 N w_s T_2}{3\hbar\epsilon_0} L(\omega).$$

Positive imaginary susceptibility indicates exponential growth of an EM wave traveling in the medium.



Linear susceptibility: semi-classical versus classical

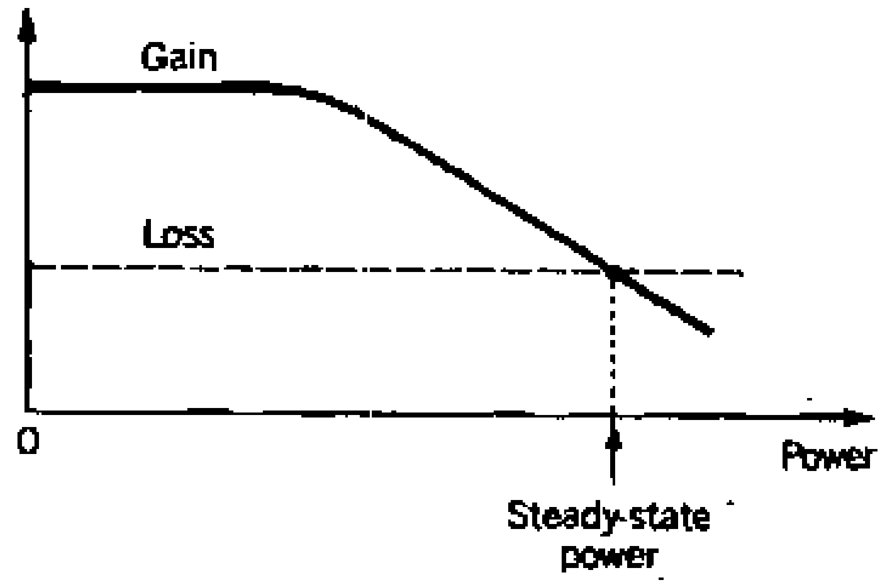
The phase relaxation rate $1/T_2$ of the dipole moment determines the width of the absorption line or the bandwidth of the amplifier.

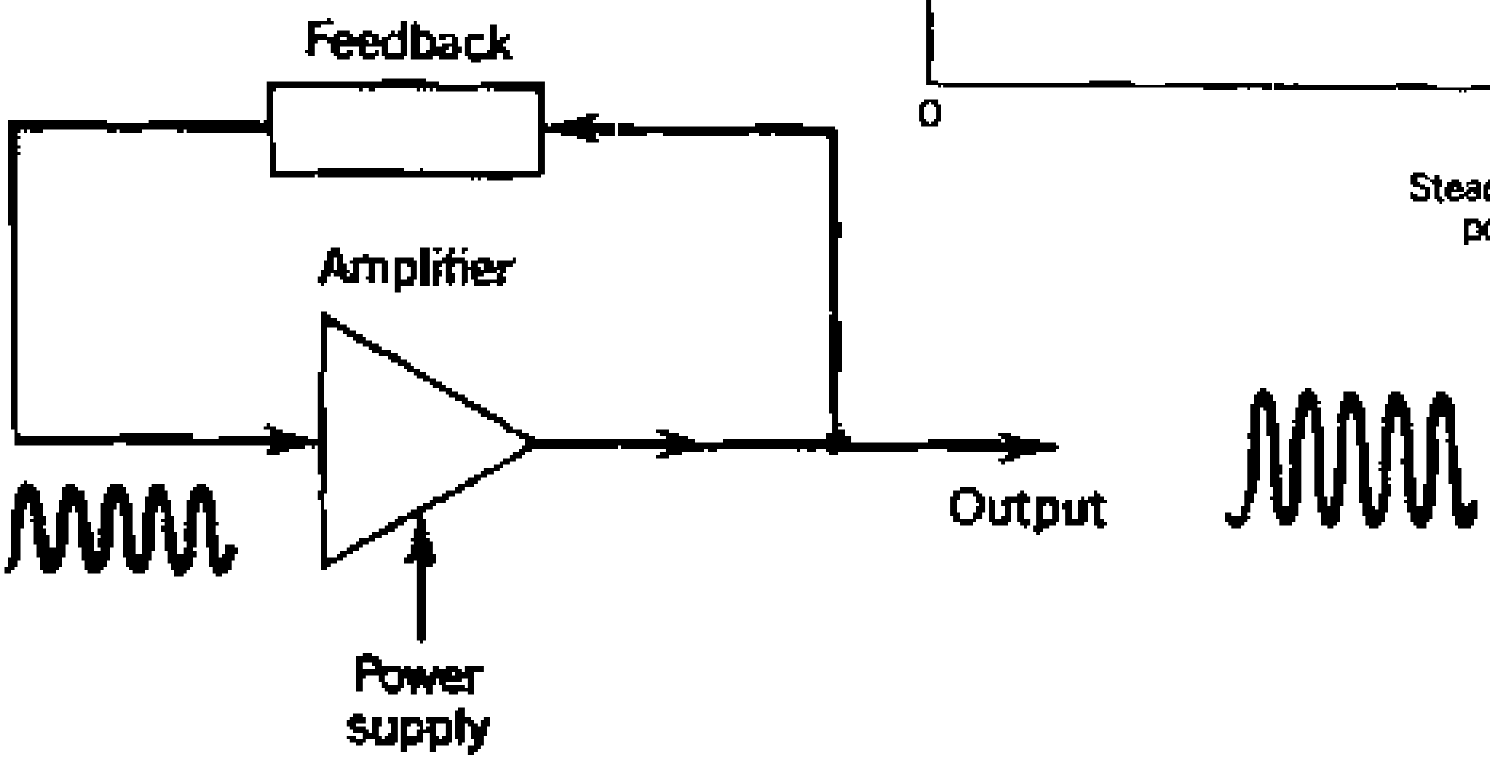
The amplification can not occur forever, because the amplifier saturates when the intensity reaches the saturation intensity. This is a strong deviation from the linear susceptibility derived from the classical oscillator model.

- **Light can not extract more energy from the atoms than the energy stored in them, i.e., energy conservation holds.**
- **Induced dipole moment in a two-level atom is limited by the maximum value of the matrix element.**
- **In contrast, the induced dipole moment in a classical oscillator growth proportionally to the applied field without limits.**

Gain saturation is critical in laser operation

Initially, unstable feedback loop. Oscillation builds up until amplifier saturates such that there is zero net roundtrip gain.





The Laser (Oscillator) Concept

Self-consistent in steady state

Radiation field *E*
classical field, no photons, Maxwell's equations, mainly in the SVE-approximation of the wave equation

Optical resonator/amplifier
The field interacts with the electronic system (atoms or molecules) inside an optical resonator or amplifier.

Atoms/ molecules
are approximated by a two-level system, interacting with the field in the dipole approximation. Schrödinger's equation delivers a macroscopic polarization ***P***.

Polarization *P*
is the source of a new coherent electromagnetic field.

Spontaneous emission
is introduced phenomenologically and produces a stochastic field which is added to the coherent field.

self-consistent in steady state

V.A. Lopota and H. Weber, "fundamentals of the semiclassical laser theory"

Three regimes of solving Bloch equations

$$\dot{w} = -\frac{w - w_0}{T_1} + \mathrm{j}\Omega_r^* \, \underline{d} - \mathrm{j}\Omega_r \, \underline{d}^*$$

$$\underline{\dot{d}} = -\left(\frac{1}{T_2} - \mathrm{j}\left(\omega_{eg} - \omega\right)\right)\underline{d} + \mathrm{j}\frac{\Omega_r}{2} \, w$$

$$\Omega_r = \frac{\vec{M}_{eg}^* \cdot \vec{e}}{\hbar}\left(\underline{E}_0 + \underline{E}_0^* e^{-\mathrm{j}2\omega t}\right)$$

Coherent equations:
Rabi oscillation

$$\boxed{\begin{aligned} \dot{w} &= \mathrm{j}\Omega_r^* \, \underline{d} - \mathrm{j}\Omega_r \, \underline{d}^* \\ \underline{\dot{d}} &= \mathrm{j}\left(\omega_{eg} - \omega\right)\underline{d} + \mathrm{j}\frac{\Omega_r}{2} \, w \end{aligned}}$$

Steady state equations:
Optical pulse duration >> T_1, T_2

$$\boxed{\underline{\dot{d}} = 0 \qquad \dot{w} = 0}$$

Adiabatic equations:
T_2 << T_1, polarization is in equilibrium with the applied field. No transient oscillations of the electronic system.

$$\boxed{\underline{\dot{d}} = 0 \qquad \dot{w} \neq 0}$$

e.g. semiconductors: T_2~50 fs

Adiabatic equations: induced transitions

$$\dot{w} = -\frac{w - w_0}{T_1} + \mathrm{j}\Omega_r^* \, \underline{d} - \mathrm{j}\Omega_r \, \underline{d}^*$$

$$\underline{\dot{d}} = -\left(\frac{1}{T_2} - \mathrm{j}\,(\omega_{eg} - \omega)\right)\underline{d} + \mathrm{j}\frac{\Omega_r}{2} w$$

Adiabatic equations: $T_2 << T_1$

$$\boxed{\underline{\dot{d}} = 0 \qquad \dot{w} \neq 0}$$

$$\dot{w} = -\underbrace{\frac{w(t) - w_0}{T_1}}_{\text{energy relaxation (e.g., spontaneous emission)}} - \underbrace{\frac{w(t)}{T_1 I_s} L(\omega) I(t)}_{\text{Induced transitions (absorption, stimulated emission)}}$$

Light intensity:

$$I(t) = |E_0(t)|^2 / (2Z_F)$$

Resonant interaction between atom and EM field: $\omega = \omega_{eg}$ $L(\omega) = 1$

$$\dot{w}\big|_{induced} = -\frac{w}{T_1 I_s} I = -\underbrace{\sigma}_{\text{Interaction cross section}} w I_{ph}$$

Photon flux density $I_{ph} = I / \hbar\omega_{eg}$

Laser rate equations

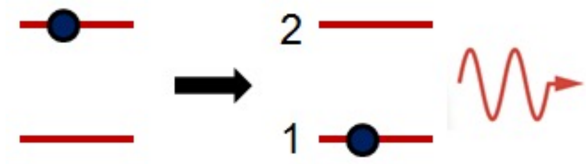
Interaction cross section: [Unit: cm^2]

$$\sigma = \frac{\hbar\omega_{eg}}{T_1 I_s} = \frac{2\omega_{eg}T_2 Z_F}{\hbar}|\vec{M}_{eg}^* \cdot \vec{e}|^2$$

$$\dot{w}\big|_{induced} = -\frac{w}{T_1 I_s} I = -\sigma w I_{ph}$$

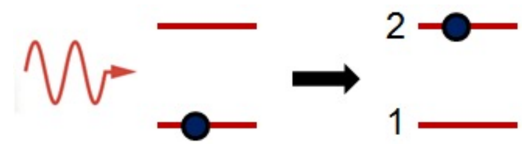
- Interaction cross section is the probability that an interaction will occur between EM field and the atomic system.
- Interaction cross section only depends on the dipole matrix element and the linewidth of the transition

$$\dot{w} = -\frac{w(t) - w_0}{T_1} - \sigma w I_{ph}$$



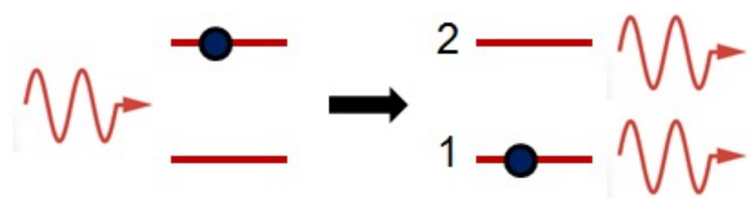
$$\frac{dN_2}{dt} = -AN_2 = -\frac{N_2}{\tau_{21}}$$

Spontaneous emission



$$\frac{dN_1}{dt} = -BN_1 I$$

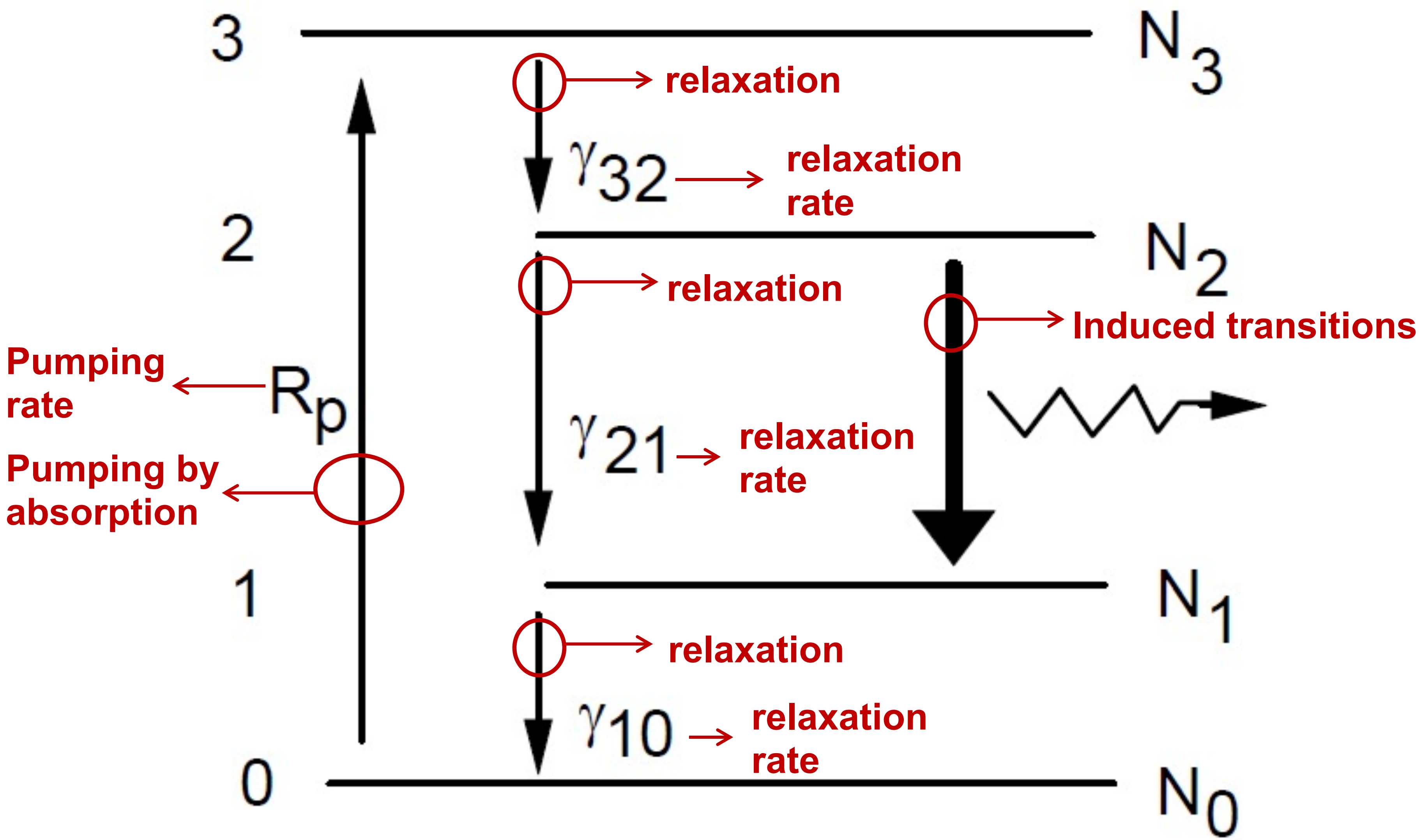
Absorption



$$\frac{dN_2}{dt} = -BN_2 I$$

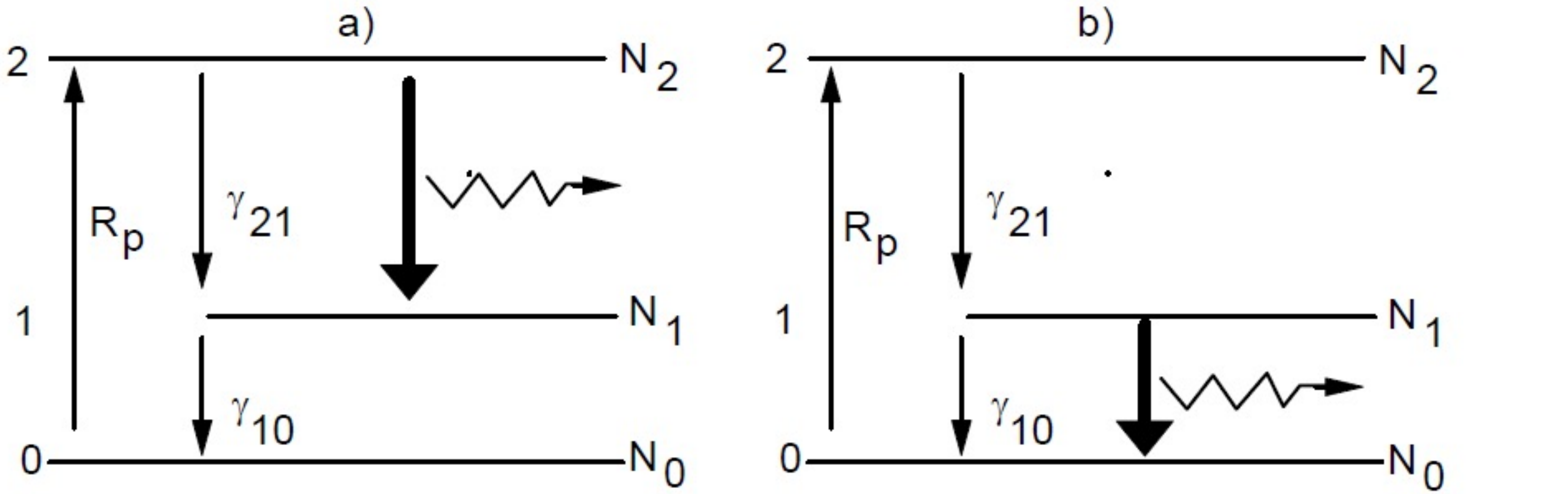
Stimulated emission

How to achieve population inversion?



Four-level gain medium

Laser rate equations for three-level laser medium



For (a):

$$\frac{d}{dt}N_2 = -\gamma_{21}N_2 - \sigma_{21}\left(N_2 - N_1\right)I_{ph} + R_p$$

$$\frac{d}{dt}N_1 = -\gamma_{10}N_1 + \gamma_{21}N_2 + \sigma_{21}\left(N_2 - N_1\right)I_{ph}$$

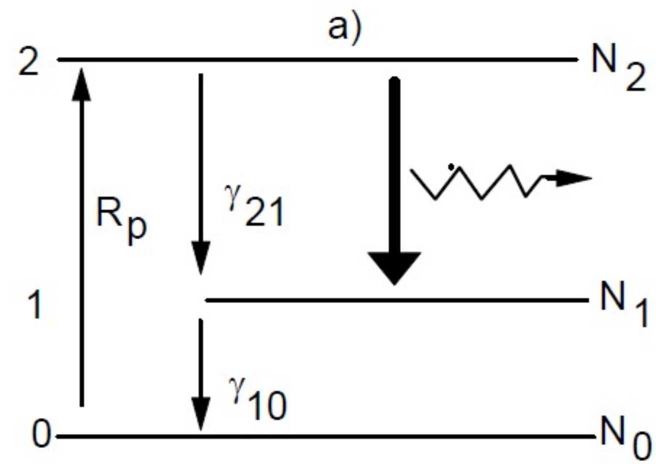
$$\frac{d}{dt}N_0 = \gamma_{10}N_1 - R_p$$

Many atoms are available in the ground state such that optical pumping can never deplete N_0. That is why we can assume a constant pump rate R_p.

σ_{21} is the cross section for stimulated emission between the levels 2 and 1. I_{ph} is the photon flux.

Laser rate equations for three-level laser medium

If the relaxation rate γ_{10} is much faster than γ_{21} and the number of possible stimulated emission events that can occur $\sigma_{21}\left(N_2 - N_1\right) I_{ph}$, we can set $N_1 = 0$ and obtain only a rate equation for the upper laser level:



$$\frac{d}{dt}N_2 = -\gamma_{21}\left(N_2 - \frac{R_p}{\gamma_{21}}\right) - \sigma_{21}N_2 \cdot I_{ph}$$

This equation is identical to the equation for the inversion of the two-level system:

$$\dot{w} = -\frac{w(t) - w_0}{T_1} - \sigma w I_{ph}$$

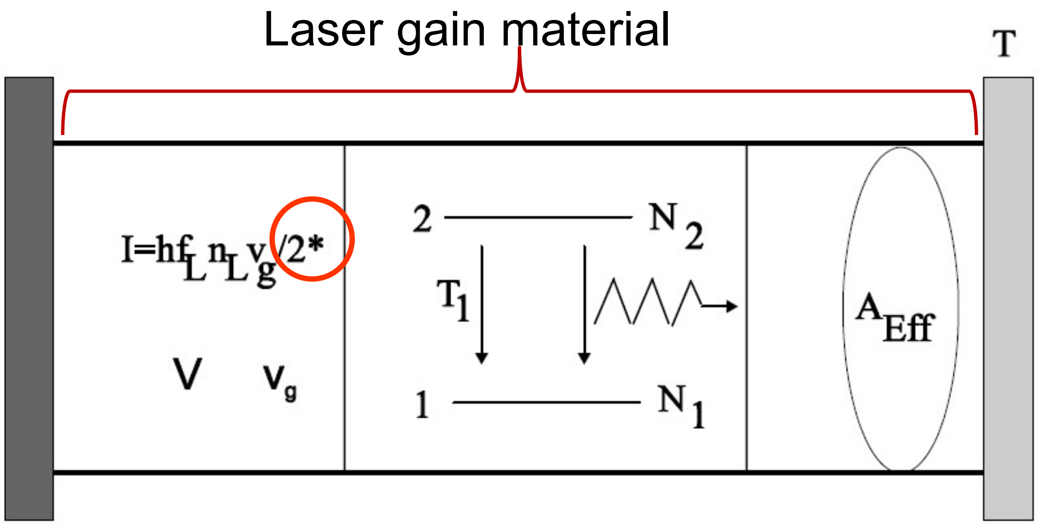
$\frac{R_p}{\gamma_{21}} \longrightarrow$ **equilibrium upper state population w/o photons present**

$\gamma_{21} = \frac{1}{\tau_L} \longrightarrow$ **upper level lifetime due to radiative and non-radiative processes**

Spectroscopic parameters of selected laser materials

Laser Medium	Wave-length $\lambda_0(nm)$	Cross Section σ (cm^2)	Upper-St. Lifetime τ_L (μs)	Linewidth Δf_{FWHM} $\frac{2}{T_2}$(THz)	Typ	Refr. index n
Nd^{3+}:YAG	1,064	$4.1 \cdot 10^{-19}$	1,200	0.210	H	1.82
Nd^{3+}:LSB	1,062	$1.3 \cdot 10^{-19}$	87	1.2	H	1.47
Nd^{3+}:YLF	1,047	$1.8 \cdot 10^{-19}$	450	0.390	H	1.82
$Nd^{3+}:YVO_4$	1,064	$2.5 \cdot 10^{-19}$	50	0.300	H	2.19
Nd^{3+}:glass	1,054	$4 \cdot 10^{-20}$	350	3	H/I	1.5
Er^{3+}:glass	1,55	$6 \cdot 10^{-21}$	10,000	4	H/I	1.46
Ruby	694.3	$2 \cdot 10^{-20}$	1,000	0.06	H	1.76
$Ti^{3+}:Al_2O_3$	660-1180	$3 \cdot 10^{-19}$	3	100	H	1.76
Cr^{3+}:LiSAF	760-960	$4.8 \cdot 10^{-20}$	67	80	H	1.4
Cr^{3+}:LiCAF	710-840	$1.3 \cdot 10^{-20}$	170	65	H	1.4
Cr^{3+}:LiSGAF	740-930	$3.3 \cdot 10^{-20}$	88	80	H	1.4
He-Ne	632.8	$1 \cdot 10^{-13}$	0.7	0.0015	I	$\sim$1
Ar^+	515	$3 \cdot 10^{-12}$	0.07	0.0035	I	$\sim$1
CO_2	10,600	$3 \cdot 10^{-18}$	2,900,000	0.000060	H	$\sim$1
Rhodamin-6G	560-640	$3 \cdot 10^{-16}$	0.0033	5	H	1.33
semiconductors	450-30,000	$\sim 10^{-14}$	$\sim$ 0.002	25	H/I	3 - 4

More on laser rate equations



***V*:= *A_eff* *L* Mode volume**
***f_L*: laser frequency**
***I*: Intensity**
***v_g*: group velocity at laser frequency**
***N_L*: number of photons in mode**
***n_L*: photon density in mode**

Intensity I in a mode propagating at group velocity v_g in one direction with a mode volume V is related to the number of photons N_L stored in the mode with volume V by

$$I = hf_L \frac{N_L}{2^* V} v_g = \frac{1}{2^*} hf_L n_L v_g$$

$$I_{ph} = I / \hbar\omega_{eg} = \frac{N_L}{2^* V} v_g$$

2^* = 2 for a linear laser resonator (then only half of the photons are going in one direction)

2^* = 1 for a ring laser

σ: interaction cross section $\sigma = hf_L / (I_s \tau_L)$

More on laser rate equations

Number of atoms in upper level:

$$\frac{d}{dt}N_2 = -\frac{N_2}{T_1} - \frac{\sigma v_g}{V}N_2 N_L + R_p$$

upper level lifetime

Number of photons in mode:

$$\frac{d}{dt}N_L = -\frac{N_L}{\tau_p} + \frac{\sigma v_g}{V}N_2\left(N_L + 1\right)$$

Photon lifetime in the cavity or cavity decay time

Number of photons spontaneously emitted into laser mode

Laser cavity with a semi-transparent mirror with transmission T produces a small power loss 2l = − ln(1−T) ≈ T (for small T) per roundtrip in the cavity.

Cavity round trip time: $T_R = 2^* L / v_g$

Photon lifetime: $\tau_p = T_R / 2l$

***l*: amplitude loss per roundtrip**

***2l*: power loss per roundtrip**

Rewrite rate equations using measurable quantities

$$\frac{d}{dt}N_2 = -\frac{N_2}{\tau_L} - \sigma v_g N_2 n_L + R_p$$

$$\frac{d}{dt}N_L = -\frac{N_L}{\tau_p} + \frac{\sigma v_g}{V} N_2 \left(N_L + 1\right)$$

Circulating intracavity power

$$P = I \cdot A_{eff} = h f_L \frac{N_L}{T_R}$$

Round trip amplitude gain

$$g = \frac{\sigma v_g}{2V} N_2 T_R$$

$$\frac{d}{dt}g = -\frac{g - g_0}{\tau_L} - \frac{gP}{E_{sat}}$$

$$\frac{d}{dt}P = -\frac{1}{\tau_p}P + \frac{2g}{T_R}\left(P + P_{vac}\right)$$

$$E_{sat} = \frac{h f_L V}{\sigma v_g T_R} = \frac{1}{2^*} I_s A_{eff} \tau_L$$

$$P_{sat} = E_{sat}/\tau_L$$

$$P_{vac} = h f_L / T_R$$

$$g_0 = 2^* \frac{R_p}{2A_{eff}} \sigma \tau_L,$$

Output power: $P_{out} = T \cdot P.$

small signal gain ~ $\sigma\tau_L$ - product

Buildup of laser oscillation

$$\frac{d}{dt}g = -\frac{g-g_0}{\tau_L} - \frac{gP}{E_{sat}} \qquad \frac{d}{dt}P = -\frac{1}{\tau_p}P + \frac{2g}{T_R}(P+P_{vac})$$

During laser buildup,

$$P_{vac} \ll P \ll P_{sat} = E_{sat}/\tau_L$$

we can neglect the spontaneous emission P_{vac}, and the gain is unsaturated: $g = g_0$

$$\frac{d}{dt}P = -\frac{1}{\tau_p}P + \frac{2g}{T_R}(P+P_{vac})$$

$$\downarrow \quad \tau_p = T_R/2l$$

$$\frac{dP}{P} = 2(g_0 - l)\frac{dt}{T_R}$$

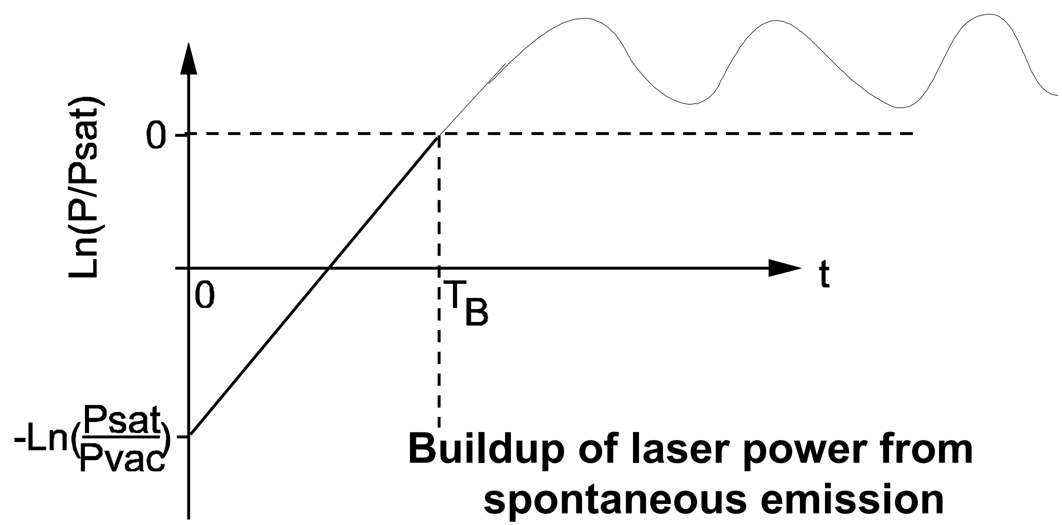
$$\downarrow$$

$$P(t) = P(0)e^{2(g_0-l)\frac{t}{T_R}}$$

The laser power builds up from vacuum fluctuations once the small signal gain surpasses the laser losses: $g_0 > g_{th} = l$

Saturation sets in within the built-up time

$$T_B = \frac{T_R}{2(g_0-l)}\ln\frac{P_{sat}}{P_{vac}} = \frac{T_R}{2(g_0-l)}\ln\frac{A_{eff}T_R}{\sigma\tau_L}$$



Buildup of laser power from spontaneous emission

Steady state operation: output power vs small signal gain

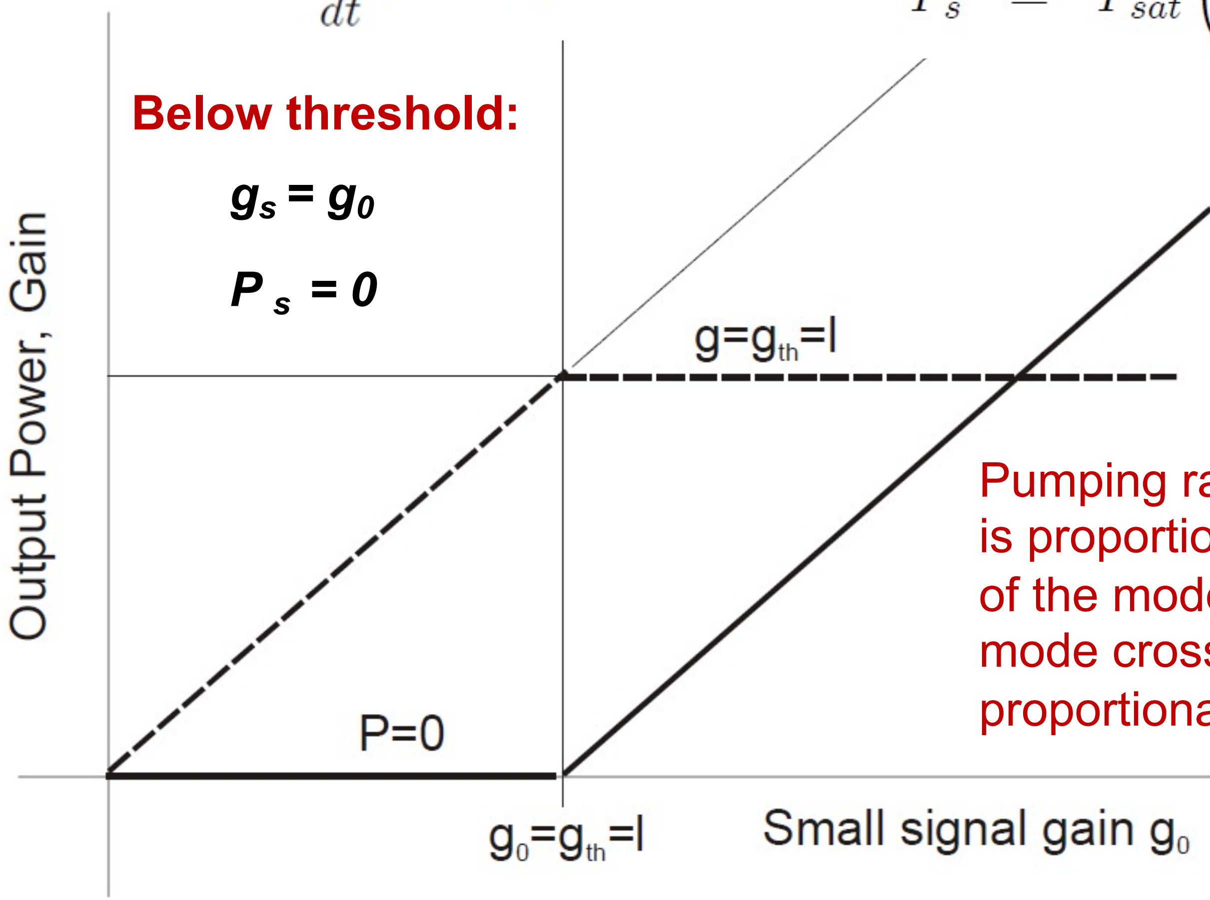
Beyond the gain threshold, some time after the buildup phase, the laser reaches steady state. Neglecting the spontaneous emission, saturated gain and steady state power can be calculated:

$$\begin{aligned}\frac{d}{dt}g &= 0\\ \frac{d}{dt}P &= 0\end{aligned} \quad \longrightarrow \quad \begin{aligned} g_s &= \frac{g_0}{1+\frac{P_s}{P_{sat}}} = l \\ P_s &= P_{sat}\left(\frac{g_0}{l}-1\right)\end{aligned}$$

Below threshold:

$g_s = g_0$

$P_s = 0$



$$\begin{aligned} g_{th} &= l, \\ R_{p,th} &= \frac{2lA_{eff}}{2^{*}\sigma\tau_L}\end{aligned}$$

Pumping rate to reach threshold is proportional to the optical loss of the mode per roundtrip, the mode cross section, and inversely proportional to the $\sigma \cdot \tau_L$.

General description of laser operation

- Pumping begins when a laser is turned on, and the population inversion eventually reaches a steady-state value.

- This steady-state population inversion is determined by the pumping rate and the upper level lifetime, $R_p \tau_L$.

- This steady-state population inversion corresponds to the small signal gain g_0.

- As the gain exceeds the cavity losses, the laser intra-cavity power begins to grow until it eventually reaches the saturation power and begins to extract energy from the medium.

- As the intra-cavity power grows, stimulated emission reduces the population inversion, and consequently the inversion reaches a new, lower steady-state value such that the reduced gain equals the losses in the cavity: $$g_s = \frac{g_0}{1 + \frac{P_s}{P_{sat}}} = l$$

Stability and relaxation oscillations

How does the laser reach steady state, once a perturbation occurs?

$$\begin{aligned} P &= P_s + \Delta P \\ g &= g_s + \Delta g \end{aligned} \quad \Rightarrow \quad \begin{aligned} \frac{d\Delta P}{dt} &= +2\frac{P_s}{T_R}\Delta g \\ \frac{d\Delta g}{dt} &= -\frac{g_s}{E_{sat}}\Delta P - \frac{1}{\tau_{stim}}\Delta g \end{aligned}$$

Stimulated lifetime

$$\frac{1}{\tau_{stim}} = \frac{1}{\tau_L}\left(1 + \frac{P_s}{Psat}\right)$$

The perturbations decay or grow like

$$\begin{pmatrix} \Delta P \\ \Delta g \end{pmatrix} = \begin{pmatrix} \Delta P_0 \\ \Delta g_0 \end{pmatrix} e^{st} \Rightarrow A \begin{pmatrix} \Delta P_0 \\ \Delta g_0 \end{pmatrix} = \begin{pmatrix} -s & 2\frac{P_s}{T_R} \\ -\frac{T_R}{E_{sat} 2\tau_p} & -\frac{1}{\tau_{stim}} - s \end{pmatrix} \begin{pmatrix} \Delta P_0 \\ \Delta g_0 \end{pmatrix} = 0$$

Non-zero solution exists only if the determinant of the coefficient matrix is 0:

$$s\left(\frac{1}{\tau_{stim}} + s\right) + \frac{P_s}{E_{sat}\tau_p} = 0$$

$$\boxed{s_{1/2} = -\frac{1}{2\tau_{stim}} \pm \sqrt{\left(\frac{1}{2\tau_{stim}}\right)^2 - \frac{P_s}{E_{sat}\tau_p}}}$$

Stability and relaxation oscillations

Introducing the pump parameter $r = 1 + \frac{P_s}{P_{sat}} = \frac{g_0}{l}$, which tells us how much we pump the laser over threshold, the eigen frequencies can be rewritten as

$$\boxed{s_{1/2} = -\frac{1}{2\tau_{stim}}\left(1 \pm j\sqrt{\frac{4(r-1)}{r}\frac{\tau_{stim}}{\tau_p} - 1}\right) = -\frac{r}{2\tau_L} \pm j\sqrt{\frac{(r-1)}{\tau_L\tau_p} - \left(\frac{r}{2\tau_L}\right)^2}}$$

(i): The stationary state $(0, g_0)$ for $g_0 < l$ and (P_s, g_s) for $g_0 > l$ are always stable, i.e. $\mathrm{Re}\{s_i\} < 0$.

(ii): For lasers pumped above threshold, $r > 1$, and long upper state lifetimes, i.e. $\frac{r}{4\tau_L} < \frac{1}{\tau_p}$, the relaxation rate becomes complex, i.e. there are relaxation oscillations

$$s_{1/2} = -\frac{1}{2\tau_{stim}} \pm j\omega_R \qquad \omega_R = \sqrt{\frac{1}{\tau_{stim}\tau_p}} \qquad \tau_{stim} = \frac{\tau_L}{r}$$

If the laser can be pumped strong enough, i.e. r can be made large enough so that the stimulated lifetime becomes as short as the cavity decay time, relaxation oscillations vanish.

Relaxation oscillations: a case study

Diode-pumped Nd:YAG-Laser:

$$\lambda_0 = 1064\text{ nm},\ \sigma = 4\cdot 10^{-20} cm^2,\ A_{eff} = \pi\left(100\mu m \times 150\mu m\right)$$

$$r = 50 \quad \tau_L = 1.2\ ms,\ l = 1\%,\ T_R = 10ns$$

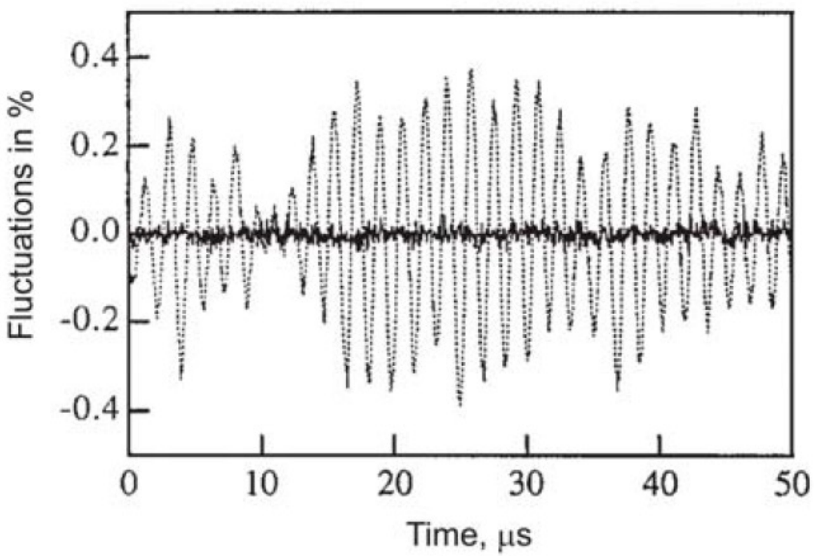
$$I_{sat} = \frac{hf_L}{\sigma\tau_L} = 3.9\frac{kW}{cm^2},\ P_s = 91.5W$$

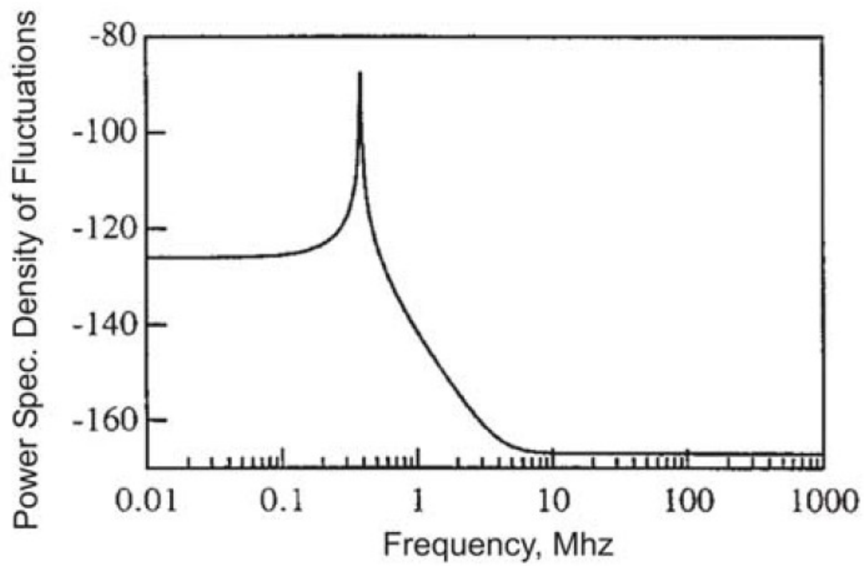
$$P_{sat} = I_{sat}A_{eff} = 1.8\ W,$$

$$\tau_{stim} = \frac{\tau_L}{r} = 24\mu s,\ \tau_p = 1\mu s$$

$$\omega_R = \sqrt{\frac{1}{\tau_{stim}\tau_p}} = 2\cdot 10^5 s^{-1}$$

The physical reason for relaxation oscillations and later instabilities is, that the gain reacts too slow on the light field, i.e. the stimulated lifetime is long in comparison with the cavity decay time.





Typically observed relaxation oscillations in time and frequency domain.

Laser efficiency: how much pump power converted to laser output power

Steady-state intracavity power:

$$P_s = P_{sat}\left(\frac{2g_0}{2l} - 1\right)$$

$$2g_0 = 2^*\frac{R_p}{A_{eff}}\sigma\tau_L,$$

$$P_{sat} = \frac{hf_L}{2^*\sigma\tau_L}A_{eff}$$

Laser power losses include the internal losses $2l_{int}$ and the transmission T through the output coupling mirror: $$2l = 2l_{int} + T$$

Pump photon energy

Laser output power: $$P_{out} = T \cdot P_{sat}\left(\frac{2g_0}{2l_{int}+T} - 1\right)$$

Pump power: $$P_p = R_p hf_P$$

Efficiency: $$\eta = \frac{P_{out}}{P_p}$$

Differential Efficiency: $$\eta_D = \frac{\partial P_{out}}{\partial P_p}$$

If the laser is pumped many times over threshold: $r = 2g_0/2l \to \infty$

$$\eta_D = \eta = \frac{T}{2l_{int}+T}P_{sat}\frac{2^*}{A_{eff}hf_P}\sigma\tau_L = \frac{T}{2l_{int}+T}\cdot\frac{hf_L}{hf_P}$$

Laser efficiency is fundamentally limited by the ratio of output coupling to total losses and the quantum defect in pumping.