UFS Lecture 6: Soliton Perturbation Theory

3 Nonlinear Pulse Propagation continued

- 3.4 Universality of the NSE
- 3.5 Soliton Perturbation Theory
- 3.6 Soliton Instabilities by Periodic Perturbations

3.4 Universality of NSE

$$j\frac{\partial A(z,t)}{\partial z} = -D_2\frac{\partial^2 A}{\partial t^2} + \delta|A|^2 A$$

2nd order dispersion:
Lowest order non-trivial
linear effect in wave propagation

SPM: lowest order
nonlinear effect in a
homogeneous medium

NSE describes many phenomena related to nonlinear wave propagation:

Self-focusing
Langmuir waves in Plasma Physics
Waves on Protein Molecules, ….

3.5 Soliton Perturbation Theory

$$\underbrace{\frac{\partial A(z,t)}{\partial z} = -j\left[|D_2|\frac{\partial^2 A}{\partial t^2} + \delta|A|^2 A\right]}_{\text{Perfect World}} + \underbrace{F(A, A^*, z)}_{\text{Reality: Perturbations}}$$

What happens to the soliton in the presence of perturbations?
Will it fall apart?
Is it just kicked around? If yes, can we understand how it is kicked around?

Ansatz: Solution of perturbed equation is a soliton + a small component:

$$A(z,t) = \left[a(\frac{t}{\tau}) + \Delta A(z,t)\right] e^{-jk_s z}$$

with:

$$a(\frac{t}{\tau}) = A_0 \,\mathrm{sech}(\frac{t}{\tau}) \qquad\qquad k_s = \frac{1}{2}\delta A_0^2$$

Soliton perturbation theory: a very brief introduction

$$\frac{\partial A(z,t)}{\partial z} = -j\left[|D_2|\frac{\partial^2 A}{\partial t^2} + \delta|A|^2 A\right] + F(A, A^*, z)$$

Perfect World

Reality: Perturbations

Without perturbations

$$A_s(z,t) = A_0 \mathrm{sech}(x(z,t))e^{-j\theta(z,t)}$$

$$\left.\begin{aligned} x &= \frac{1}{\tau}(t - 2|D_2|p_0 z - t_0) \\ \theta &= p_0(t - t_0) + |D_2|\left(\frac{1}{\tau^2} - p_0^2\right)z + \theta_0 \\ \frac{\delta A_0^2}{2} &= \frac{|D_2|}{\tau^2} \end{aligned}\right\}$$

Four degrees of freedom:

energy fluence w or amplitude A_0

carrier frequency p_0

phase θ_0

origin t_0

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Soliton perturbation theory: a very brief introduction

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Any deviation ΔA can be decomposed into a contribution that leads to a soliton with a shift in the four soliton parameters and a continuum contribution:

$$\Delta A(z) = \underbrace{\Delta w(z)}_{\text{Energy fluctuation}} f_w + \underbrace{\Delta\theta(z)}_{\text{Optical phase fluctuation}} f_\theta + \underbrace{\Delta p(z)}_{\text{Center frequency fluctuation}} f_p + \underbrace{\Delta t(z)}_{\text{Timing fluctuation}} f_t + \underbrace{a_c(z)}_{\text{Continuum background}}$$

$$f_w = \frac{\partial A}{\partial w}$$

$$f_\theta = \frac{\partial A}{\partial \theta}$$

$$f_p = \frac{\partial A}{\partial p}$$

$$f_t = \frac{\partial A}{\partial t}$$

3.5 Soliton Perturbation Theory

$$\frac{\partial \Delta A}{\partial z} = -jk_s\left[\left(\frac{\partial^2}{\partial x^2}-1\right)\Delta A + 2\mathrm{sech}^2(x)\left(2\Delta A + \Delta A^*\right)\right] + F(A, A^*, z)e^{jk_s z},$$

$$x = t/\tau$$

Decompose into two coupled equations for real and imaginary part or use

$$\mathbf{\Delta A} = \begin{pmatrix} \Delta A \\ \Delta A^* \end{pmatrix}$$

$$\frac{\partial}{\partial z'}\mathbf{\Delta A} = \mathbf{L}\mathbf{\Delta A} + \frac{1}{k_s}\mathbf{F}(A, A^*, z)e^{jz'}$$

$$\mathbf{L} = -j\boldsymbol{\sigma}_3\left[\left(\frac{\partial^2}{\partial x^2}-1\right) + 2\,\mathrm{sech}^2(x)(2+\boldsymbol{\sigma}_1)\right]$$

Pauli matrices:

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}, \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

3.5 Soliton Perturbation Theory

Right sided eigen solutions or main solutions:

$$\mathbf{L}\,\mathbf{f} = \lambda \mathbf{f} \qquad \mathbf{L}^{n}\,\mathbf{f}_{j} = \lambda \mathbf{f}$$

Discrete solutions:

$$\begin{aligned}
\mathrm{f}_w(x) &= \frac{1}{w}(1 - x\tanh x)a(x)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\mathrm{f}_\theta(x) &= -ja(x)\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \\
\mathrm{f}_p(x) &= -j\,x\tau a(x)\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \\
\mathrm{f}_t(x) &= \frac{1}{\tau}\tanh(x)\,a(x)\begin{pmatrix} 1 \\ 1 \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
\mathbf{L}\mathrm{f}_w &= \frac{1}{w}\mathrm{f}_\theta, \\
\mathbf{L}\mathrm{f}_\theta &= 0, \\
\mathbf{L}\mathrm{f}_p &= -2\tau^2\mathrm{f}_t, \\
\mathbf{L}\mathrm{f}_t &= 0.
\end{aligned}$$

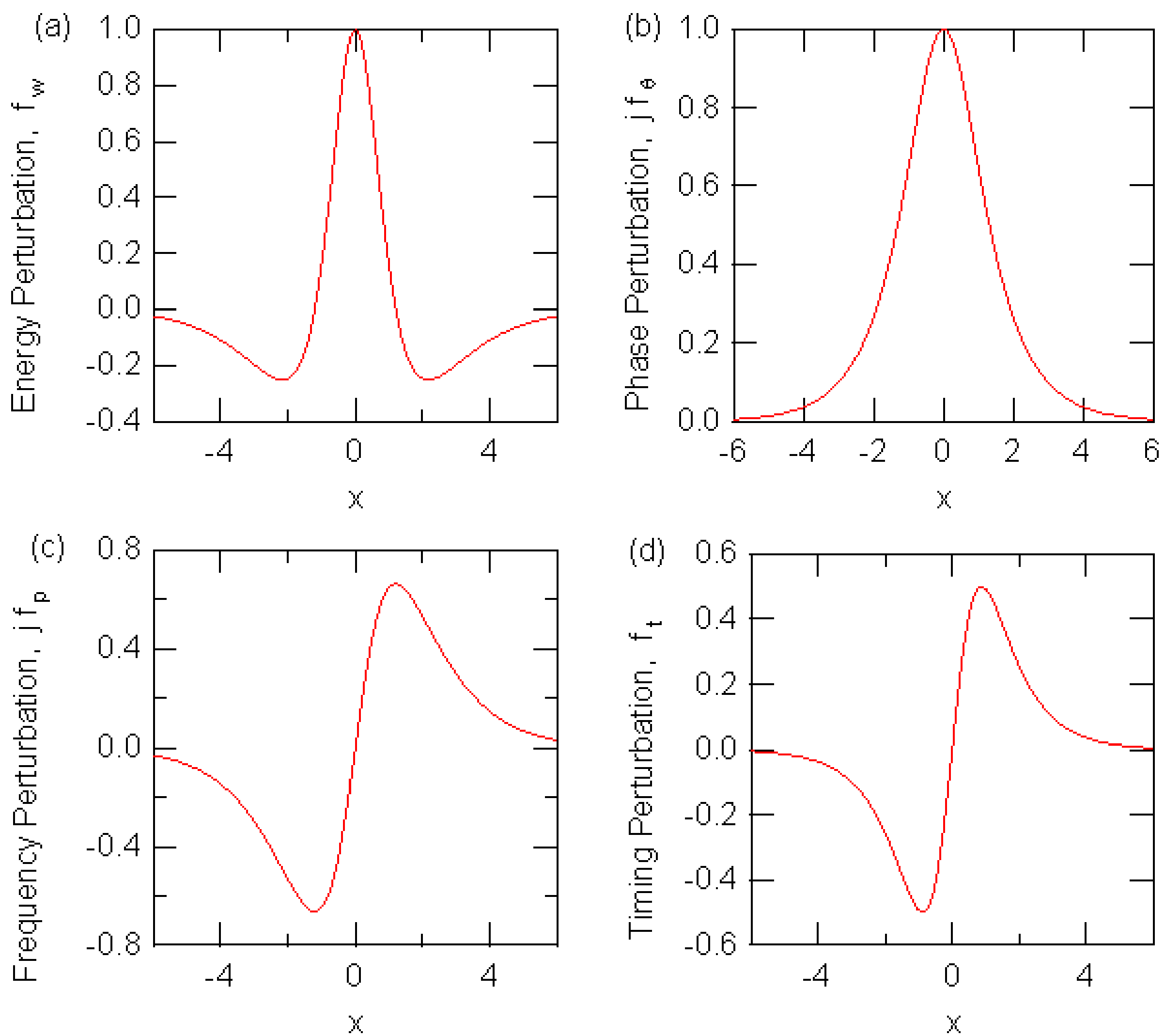


Figure 3.9: Perturbations in soliton amplitude (a), phase (b), frequency (c), and timing (d)

Continuum of solutions:

$$\mathbf{L}\mathbf{f}_k = \lambda_k \mathbf{f}_k,$$
$$\lambda_k = j(k^2+1),$$
$$\mathbf{f}_k(x) = e^{-jkx}\begin{pmatrix} (k - j\tanh x)^2 \\ \mathrm{sech}^2 x \end{pmatrix}$$

and:

$$\mathbf{L}\bar{\mathbf{f}}_k = \bar{\lambda}_k \bar{\mathbf{f}}_k,$$
$$\bar{\lambda}_k = -j(k^2+1)$$
$$\bar{\mathbf{f}}_k = \sigma_1 \mathbf{f}_k.$$

Define inner product: (Norm: Signal Energy)

$$< \mathbf{u}|\mathbf{v} >= \frac{1}{2}\int_{-\infty}^{+\infty} \mathbf{u}^+(x)\mathbf{v}(x)dx$$

Then, the adjoint operator is:

$$\mathbf{L}^+ = -\sigma_3 \mathbf{L} \sigma_3$$

Note: L is not self-adjoint, because the linearized NSE does not conserve energy

$\mathbf{L^+}$ has its own set of eigen solutions $\mathbf{f}^{(+)}$ and eigen values $\lambda^{(+)}$

$$
\begin{aligned}
< \mathbf{f}_k^{(+)} | \mathbf{f}_{k'} > &= \delta(k-k'), \qquad < \bar{\mathbf{f}}_k^{(+)} | \bar{\mathbf{f}}_{k'} >= \delta(k-k') \\
< \bar{\mathbf{f}}_k^{(+)} | \mathbf{f}_{k'} > &= < \mathbf{f}_k^{(+)} | \bar{\mathbf{f}}_{k'} >= 0.
\end{aligned}
$$

Completeness Relation:

$$
\begin{aligned}
\delta(x-x') &= \int_{-\infty}^{\infty} dk \left[|\mathbf{f}_k >< \mathbf{f}_k^{(+)}| + |\bar{\mathbf{f}}_k >< \bar{\mathbf{f}}_k^{(+)}| \right] \\
&+ |\mathbf{f}_w >< \mathbf{f}_w^{(+)}| + |\mathbf{f}_\theta >< \mathbf{f}_\theta^{(+)}| \\
&+ |\mathbf{f}_p >< \mathbf{f}_p^{(+)}| + |\mathbf{f}_t >< \mathbf{f}_t^{(+)}|.
\end{aligned}
$$

Any perturbation can be written as:

$$
\mathbf{\Delta A}(z') = \Delta w(z')\mathbf{f}_w + \Delta\theta(z')\mathbf{f}_\theta + \Delta p(z')\mathbf{f}_p + \Delta t(z')\mathbf{f}_t + \mathbf{a}_c(z')
$$

Soliton, perturbed in amplitude, phase, center frequency and timing + Continuum

$$
\mathbf{a}_c = \int_{-\infty}^{\infty} dk \left[g(k)\mathbf{f}_k(x) + \bar{g}(k)\bar{\mathbf{f}}_k(x) \right]
$$

Substitution into (3.20)

$$\int \mathbf{f}_j^{(+)^{+}} \cdots dx \quad \left| \quad \begin{aligned} &\frac{\partial \Delta w}{\partial z'}\mathbf{f}_w + \frac{\partial \Delta \theta}{\partial z'}\mathbf{f}_\theta + \frac{\partial \Delta p}{\partial z'}\mathbf{f}_p + \frac{\partial \Delta t}{\partial z'}\mathbf{f}_t + \frac{\partial}{\partial z'}\mathbf{a}_c = \\ &\mathbf{L}\left(\Delta w(z')\mathbf{f}_w + \Delta p(z')\mathbf{f}_p + \mathbf{a}(z')_c\right) + \frac{1}{k_s}\mathbf{F}(A, A^*, z')e^{-iz'} \end{aligned} \right.$$

Projecting out with the eigensolutions of the adjoint operator:

$$\begin{aligned}
\frac{\partial}{\partial z'}\Delta w &= \frac{1}{k_s} < \mathbf{f}_w^{(+)}|\mathbf{F}e^{jz'} >, \\
\frac{\partial}{\partial z'}\Delta \theta &= \frac{\Delta w}{w} + \frac{1}{k_s} < \mathbf{f}_\theta^{(+)}|\mathbf{F}e^{jz'} >, \\
\frac{\partial}{\partial z'}\Delta p &= \frac{1}{k_s} < \mathbf{f}_p^{(+)}|\mathbf{F}e^{jz'} >, \\
\frac{\partial}{\partial z'}\Delta t &= 2\tau\Delta p + \frac{1}{k_s} < \mathbf{f}_t^{(+)}|\mathbf{F}e^{jz'} >, \\
\frac{\partial}{\partial z'}g(k) &= j(1+k^2)g(k) + \frac{1}{k_s} < \mathbf{f}_k^{(+)}|\mathbf{F}(A, A^*, z')e^{jz'} >
\end{aligned}$$

Gordon's associated function:

$$G(x) = \int_{-\infty}^{\infty} g(k)\, e^{ikx} dk.$$

Only dispersion:

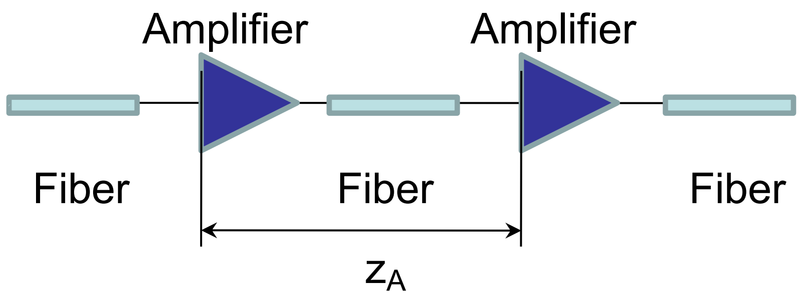
$$\frac{\partial G(z', x)}{\partial z'} = -j \left(1 + \frac{\partial^2}{\partial x^2}\right) G(z', x)$$

One can show that:

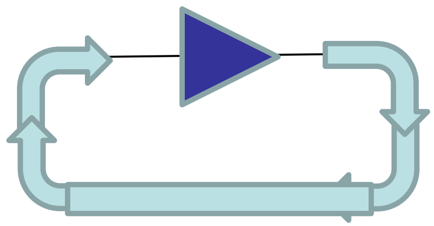
$$a_c = -\frac{\partial^2 G(x)}{\partial x^2} + 2 \tanh(x) \frac{\partial G(x)}{\partial x} - \tanh^2(x)\, G(x) + G^*(x) \mathrm{sech}^2(x)$$

3.6 Soliton Instabilities by Periodic Perturbations

Long haul opt. communication link



Modelocked fiber laser



$$F(A, A^*, z) = \xi \sum_{n=-\infty}^{\infty} \delta(z - n z_A) A(z, t).$$

$$\frac{\partial}{\partial z} g(k) = j k_s (1 + k^2) g(k) + < \mathbf{f}_k^{(+)} | \mathbf{F}(A, A^*, z) e^{j k_s z} > .$$

$$< \mathbf{f}_k^{(+)}|\mathbf{F}(A,A^*,z)e^{jk_s z} >= \xi \sum_{n=-\infty}^{\infty} \delta(z - nz_A)\frac{1}{2} \cdot \tag{3.67}$$

$$\int_{-\infty}^{+\infty} \frac{1}{\pi(k^2+1)^2} e^{jkx} \begin{pmatrix} (k+j\tanh x)^2 \\ -\text{sech}^2 x \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} A_0 \,\text{sech} x \; dx$$

$$= \xi \sum_{n=-\infty}^{\infty} \delta(z - nz_A) \cdot \tag{3.68}$$

$$\int_{-\infty}^{+\infty} \frac{A_0}{2\pi(k^2+1)^2} e^{jkx} \left(k^2 + 2jk\tanh x - 1\right) \cdot \text{sech} x \; dx$$

$\frac{d}{dx}\text{sech} x = -\text{sech} x \tanh x$ use partial Integration

Using $\sum_{n=-\infty}^{\infty} \delta(z - nz_A) = \frac{1}{z_A} \sum_{m=-\infty}^{\infty} e^{jm\frac{2\pi}{z_A}z}$

$$\boxed{\frac{\partial}{\partial z} g(k) = jk_s(1+k^2)g(k) - \frac{\xi}{z_A} \sum_{m=-\infty}^{\infty} e^{jm\frac{2\pi}{z_A}z} \frac{A_0}{4(k^2+1)} \,\text{sech}\left(\frac{\pi k}{2}\right)}$$

3.6 Soliton Instabilities by Periodic Perturbations

$$g(k,z) = C(k,z) e^{jk_s(1+k^2)z}$$

$$\frac{\partial}{\partial z} C(k,z) = -\frac{\xi}{z_A} \sum_{m=-\infty}^{\infty} \frac{A_0}{4(k^2+1)} \operatorname{sech}\left(\frac{\pi k}{2}\right) e^{-j\left(k_s(1+k^2) - m\frac{2\pi}{z_A} z\right)}$$

initial conditions $C(z=0)=0$

$$\boxed{\begin{aligned} C(k,z) \quad = \quad & -\frac{\xi}{z_A}\frac{A_0}{4(k^2+1)} \operatorname{sech}\left(\frac{\pi k}{2}\right) \cdot \\ & \sum_{m=-\infty}^{\infty} \frac{e^{j\left(-k_s(1+k^2)+m\frac{2\pi}{z_A}\right)z} - 1}{m\frac{2\pi}{z_A} - k_s(1+k^2)}. \end{aligned}}$$

Resonance catastrophy: $$m\frac{2\pi}{z_A} - k_s(1+k_m^2) \quad = \quad 0$$

$$\text{or} \qquad k_m \quad = \quad \pm\sqrt{\frac{m\frac{2\pi}{z_A}}{k_s} - 1}.$$

$$k_m = \pm\sqrt{\frac{m\frac{2\pi}{z_A}}{k_s} - 1}.$$

$$k = \omega\tau,\, k_s = |D_2|\,/\tau^2$$

$$\phi_0 = k_s z_A$$

$$\omega_m = \pm\frac{1}{\tau}\sqrt{\frac{2m\pi}{\phi_0} - 1},$$

$$\begin{aligned} C(\omega, z) &= -j\frac{\xi}{z_A}\frac{A_0}{4((\omega\tau)^2 + 1)}\mathrm{sech}\left(\frac{\pi\omega\tau}{2}\right) \\ &\quad \cdot \sum_{m=-\infty}^{\infty} z_A \frac{e^{j(-k_s(1+(\omega\tau)^2)+m\frac{2\pi}{z_A})z} - 1}{2\pi m - \phi_0(1 + (\omega\tau)^2)}. \end{aligned}$$

Kelly Sidebands

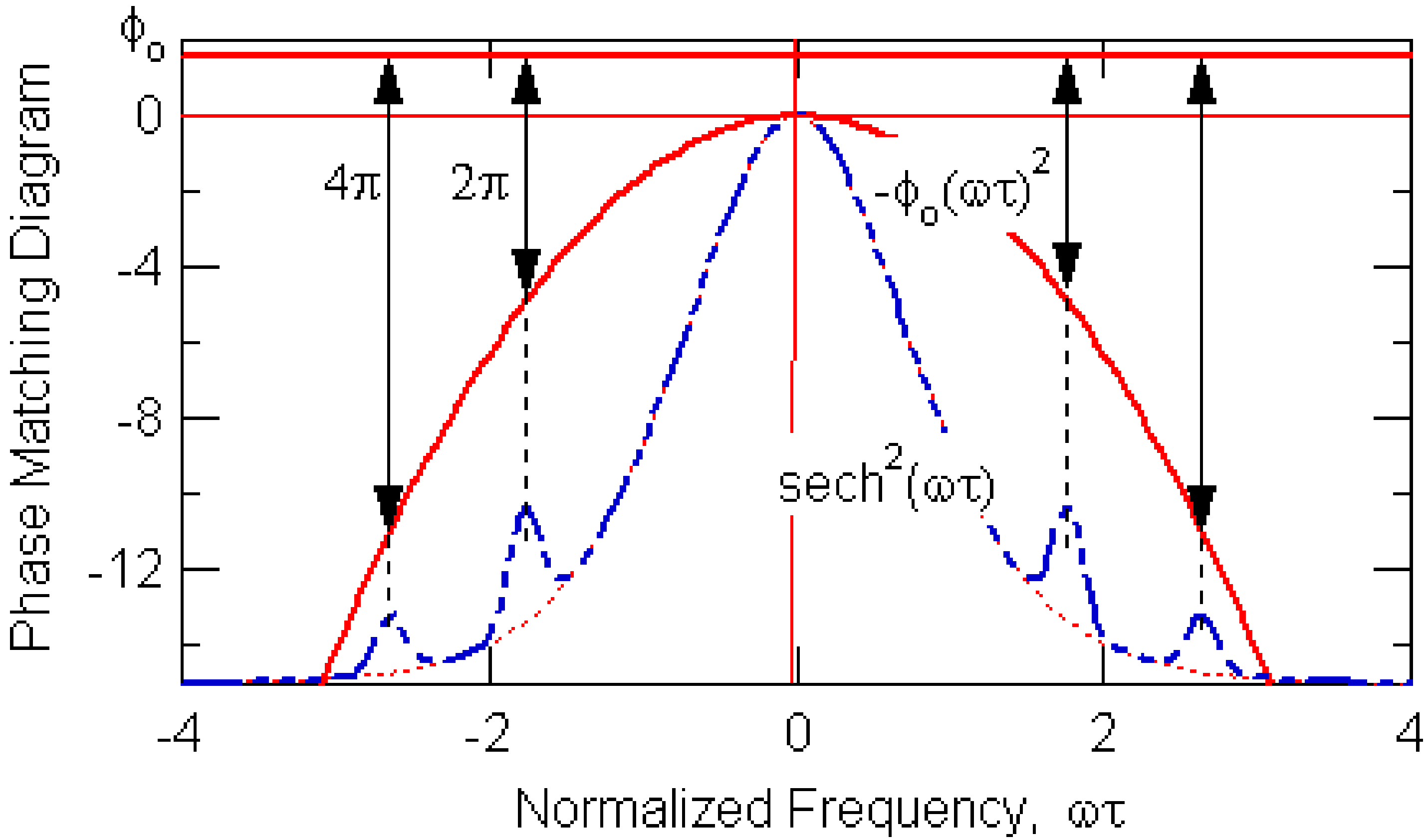


Figure 3.10: Phase matching of soliton and continuum

Avoid resonance catastrophy for: $\omega_m \gg \frac{1}{\tau} \longrightarrow \phi_0 \ll \pi/4$

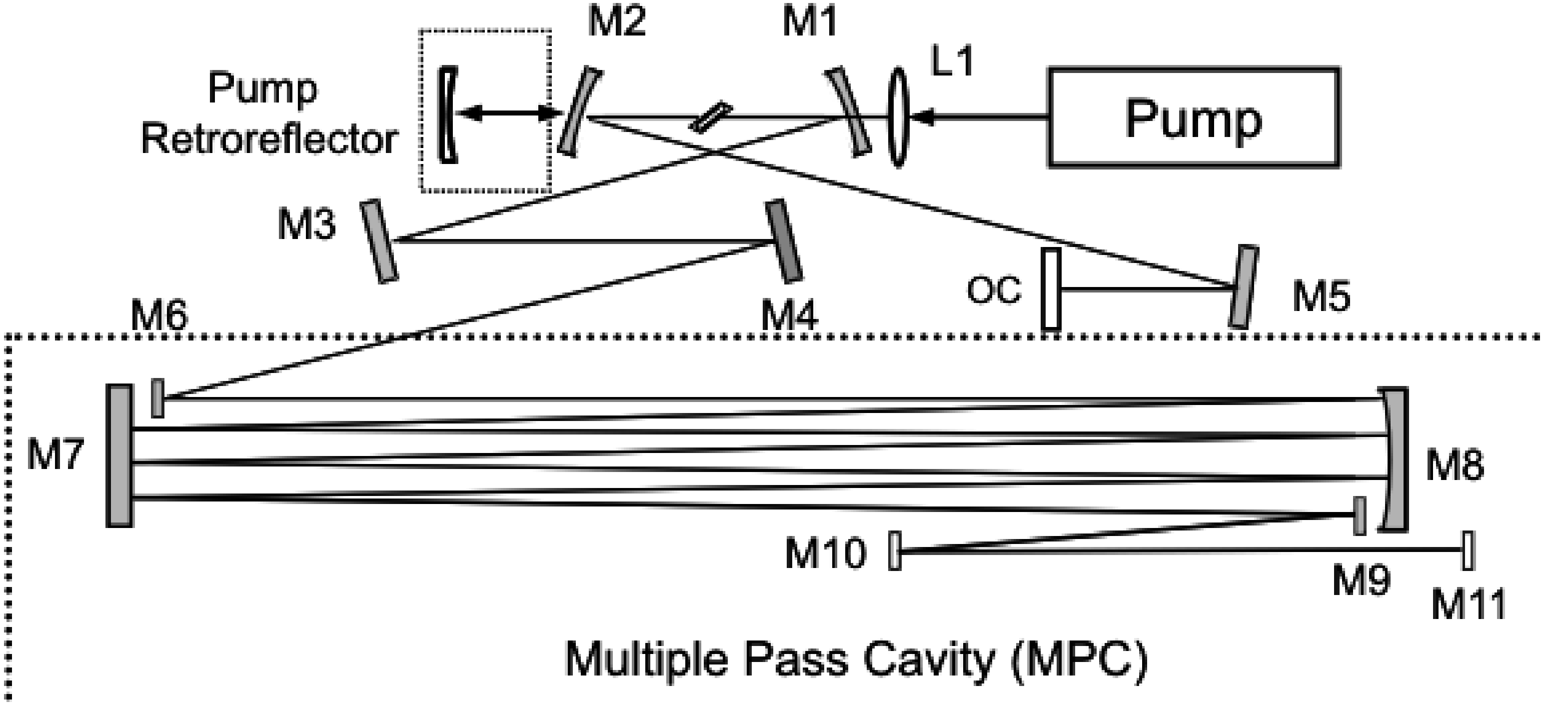


Figure 3.11: Layout of a multiple pass (MPC) cavity laser.

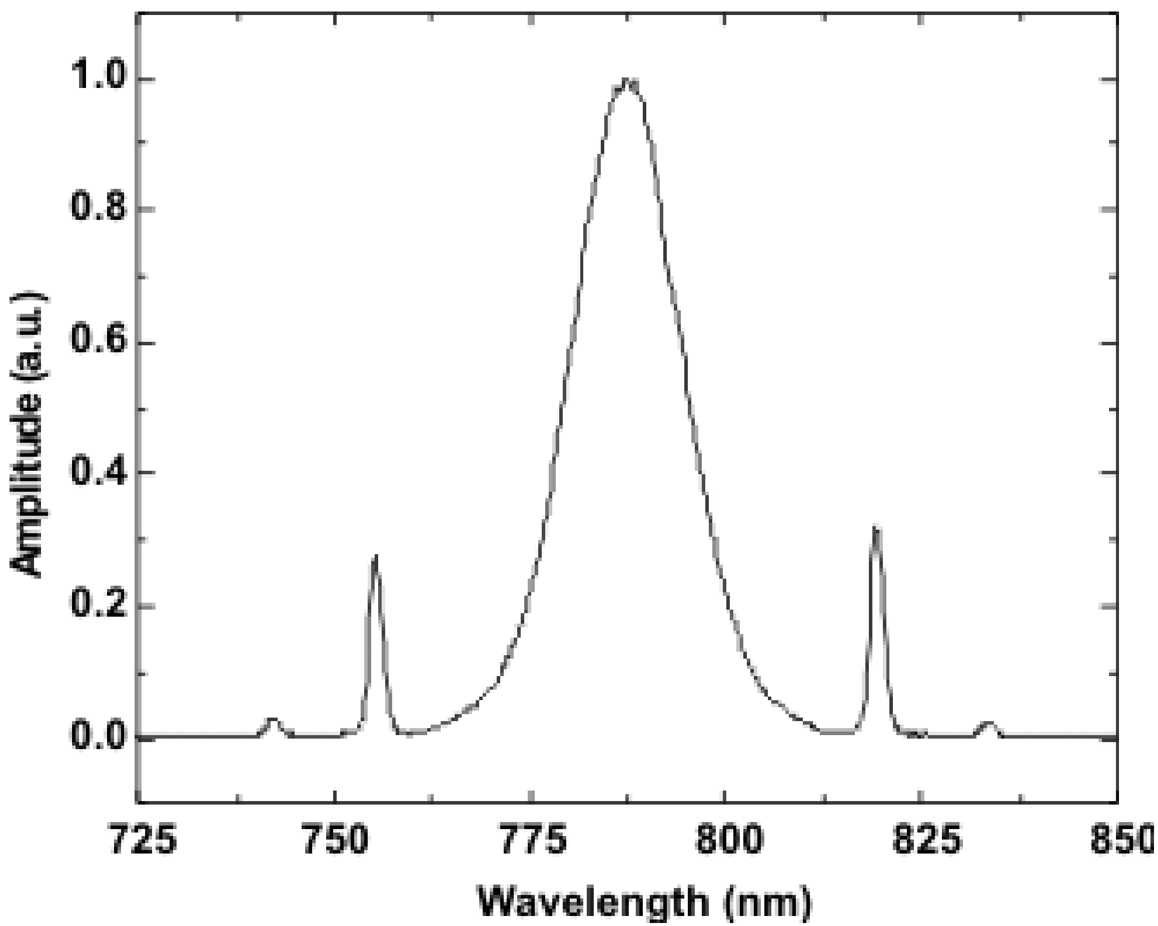


Figure 3.12: Measured modelocked spectrum with Kelly sidebands

Dispersive Wave (Third Order Dispersion)

$$j\frac{\partial A(z,t)}{\partial z} = -D_2\frac{\partial^2 A}{\partial t^2} + jD_3\frac{\partial^3 A}{\partial t^3} + \delta|A|^2 A$$

Modulation Instability

$$j\frac{\partial A(z,t)}{\partial z} = -D_2\frac{\partial^2 A}{\partial t^2} + \delta|A|^2 A$$

$$A(z,t) = \left(\sqrt{P_o} + a(z,t)\right) exp\,(\phi_{NL}(z)) \quad \phi_{NL}(z) = \delta P_o \cdot z$$

Linearized Analyisis: See Agrawal Nonlinear Fiber Optics Page 105

$$j\frac{\partial a(z,t)}{\partial z} = -D_2\frac{\partial^2 a}{\partial t^2} + \delta P_o(a + a^*)$$

$$Ansatz:\ a(z,t) = a_1 cos(Kz - \Omega t) + j a_2 sin(Kz - \Omega t)$$

$$K = \pm|D_2|\Omega\sqrt{\Omega^2 + sign(D_2)\Omega_c^2}$$

$$\Omega_c^2 = \frac{2\delta P_o}{|D_2|} = \frac{2}{|D_2|L_{NL}}. \quad with\ nonlinear\ length\ \ L_{NL} = \frac{1}{\delta P_o}$$

Modulation Instability

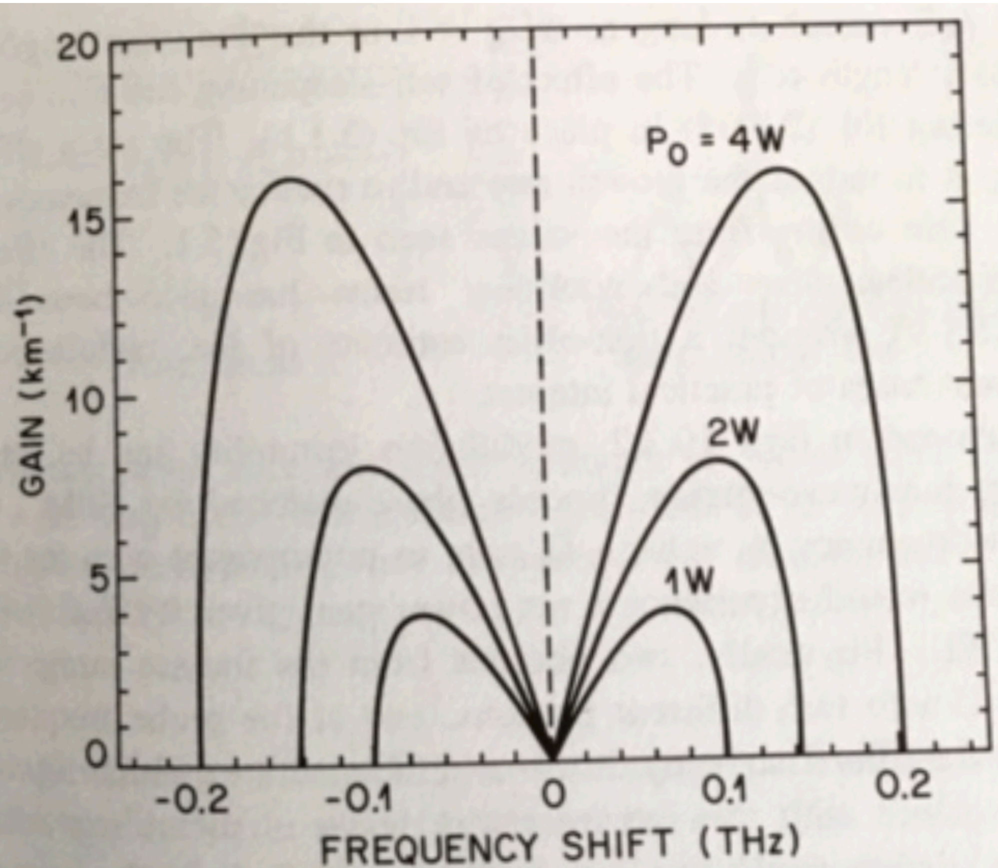


Fig. 5.1 Gain spectra of modulation instability at three power levels. The fibers parameters are $\beta_2 = -20$ ps^2/km and $\gamma = 2$ W^{-1} km^{-1}.

Agrawal Nonlinear Fiber Optics Page 107

Rogue wave

Find more information from New York times:
http://www.nytimes.com/2006/07/11/science/11wave.html

One more Rogue wave