UFS Lecture 4: Nonlinear Pulse Propagation

Overview Nonlinear Optics

3 Nonlinear Pulse Propagation

3.1 The Optical Kerr-effect

3.2 Self-Phase Modulation (SPM)

3.3 The Nonlinear Schrödinger Equation

Interaction between EM waves and materials

Light wave perturbs material

Perturbed material alters the light wave

$$\longrightarrow P = \varepsilon_0 \chi E$$

$$\longrightarrow (\nabla^2 - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2}) E = \mu_0 \frac{\partial^2 P}{\partial t^2}$$

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Examples of changes to light wave: Frequency, amplitude, phase, polarization state, direction of propagation, transverse profile

Output of a linear optical system with multiple inputs is simply the field summation of the outputs for each individual input.



How does Nonlinear Optics work?

P: Polarization (Dipole moment / unit volume)

p: dipole moment per atom or molecule*N*: Number density

$$\mathbf{P} = N\mathbf{p}$$

q: charge that is displaced x: displacement

$$\mathbf{p} = \mathbf{q} \cdot \mathbf{x}$$



A simple atom model explaining the effect of an optical electric field on the induced polarization in an atom: (a) without field, (b) with field. Compare with Lorentz Model: For strong fiels the spring becomes nonlinear

Perturbation Expansion

p: nonlinear dipole moment of atom or molecule

$$\mathbf{p} = \mathbf{q} \cdot \mathbf{x} = q \left\{ \alpha^{(1)} \left(\frac{E}{E_a} \right) + \alpha^{(2)} \left(\frac{E}{E_a} \right)^2 + \alpha^{(3)} \left(\frac{E}{E_a} \right)^3 + \cdots \right\} \frac{\mathbf{E}}{|\mathbf{E}|}.$$

 $\alpha^{(i)}$: typical excursion of electron cloud at the critical field is on the order of the Bohr radius

$$\alpha^{(i)} = d_a = 10^{-10} \mathrm{m}$$

 E_a : critical field where perturbation theory breaks down: ionization field strength

$$E_a = \frac{e_0}{4\pi\epsilon_0 d_a^2} = 1.4 \cdot 10^{11} \frac{V}{m} = 1.4 \, GV/cm,$$

 $\epsilon_0 = 8.854 \cdot 10^{-12}$ F/m the vacuum dielectric constant

Perturbative Nonlinear Optics

Wave equation: $\left(\Delta - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2}\right) \vec{E} = \mu_0 \frac{\partial^2}{\partial t^2} \vec{P}$. Source term describing light-matter interaction

Typical solid density: $N = 6 \cdot 10^{29} m^{-3}$ $E_a = 0.14 V/pm$

Mixing of two sine waves

 $E = \frac{1}{2} \left[\tilde{E}(\omega_1) e^{j\omega_1 t} + \tilde{E}(\omega_2) e^{j\omega_2 t} + \text{c.c.} \right] \longrightarrow \text{Input electric field}$

 $P_{\rm NL}^{(2)} = \frac{\varepsilon_0}{4} \Big\{ \chi^{(2)}(2\omega_1 : \omega_1, \omega_1) \tilde{E}^2(\omega_1) e^{j2\omega_1 t} + \chi^{(2)}(2\omega_2 : \omega_2, \omega_2) \tilde{E}^2(\omega_2) e^{j2\omega_2 t} \Big\}$

Second-harmonic generation (SHG)

 $+ 2\chi^{(2)}(\omega_1 + \omega_2 : \omega_1, \omega_2)\tilde{E}(\omega_1)\tilde{E}(\omega_2)e^{j(\omega_1 + \omega_2)t}$

Sum-frequency generation (SFG)

$$+2\chi^{(2)}(\omega_1-\omega_2:\omega_1,-\omega_2)\tilde{E}(\omega_1)\tilde{E}^*(\omega_2)e^{j(\omega_1-\omega_2)t}$$

Difference-frequency generation (DFG)

 $+ \chi^{(2)}(0:\omega_1,-\omega_1)\tilde{E}(\omega_1)\tilde{E}^*(\omega_1) + \chi^{(2)}(0:\omega_2,-\omega_2)\tilde{E}(\omega_2)\tilde{E}^*(\omega_2) \Big\}$

Optical rectification

 $\chi^{(2)}$ may be both complex and frequency dependent.

 $\chi^{(2)}(\omega_a + \omega_b : \omega_a, \omega_b) \quad \substack{\text{keeps track of the input and output frequencies}\\ \text{involved in a particular interaction.}} 6$

Important nonlinear processes

susceptibility	nonlinear process	
$\chi^{(2)}\left(2\omega_1;\omega_1,\omega_1\right)$	frequency doubling	
$\chi^{(2)}\left(\omega_3;\omega_1,\pm\omega_2\right)$	sum- and difference-frequency generation,	
	2-photon absorption, saturable absorption	
$\chi^{(2)}\left(\omega_1;\omega_1,0\right)$	linear electro-optic effect, SFG process	
	Pockels effect ω_2	
$\chi^{(2)}(0;\omega_1,-\omega_1)$	optical rectification ω_3	
$\chi^{(3)}(\omega_1;\omega_1,0,0)$	DC Kerr effect	
$\chi^{(3)}\left(\omega_1;\omega_1,\omega_1,-\omega_1\right)$	self-phase modulation, self-focusing	
	2-photon absorption, saturable absorption	
$\chi^{(3)}(2\omega_1;\omega_1,\omega_1,0)$	field-induced second-harmonic generation	
$\chi^{(3)}\left(3\omega_1;\omega_1,\omega_1,\omega_1\right)$	frequency tripling	
$\chi^{(3)}\left(\omega_2;\omega_2,\omega_1,-\omega_1\right)$	stimulated Raman scattering $(\omega_{vib} = \omega_2 - \omega_1)$	
$\chi^{(3)}(2\omega_1 - \omega_2; \omega_1, \omega_1, -\omega_2)$	four-wave mixing, CARS ($\omega_{vib} = \omega_2 - \omega_1$)	

Table 1.3: Important nonlinear optical susceptibilities and corresponding nonlinear optical processes. The first argument in the susceptibility gives the frequency of the generated wave and the other arguments after the semicolon give the frequency components of the input waves.

Second-order nonlinear optical effects

$$P = \varepsilon_0 [\chi^{(1)}E + \chi^{(2)}E^2] \qquad \qquad \chi^{(2)} \longrightarrow 2^{nd} \text{ order susceptibility}$$

Example: Pockels Effect

$$E = A_0 + A_1 \cos(\omega t)$$
 — Input electric field

The total polarization at frequency ω is:

$$P^{(\omega)} = \varepsilon_0 [\chi^{(1)} + 2\chi^{(2)}A_0]A_1 \cos(\omega t)$$

New refractive index:
$$n = \sqrt{1 + \chi^{(1)} + 2\chi^{(2)}A_0}$$

The Pockels effect is used to make optical switch (or modulator) using an electrical field to control the interaction between an optical crystal and the optical field propagating in it.

Intensity dependent nonlinear refractive index

In general, a medium responses nonlinearly to an optical field. Here we are interested in intensity dependent nonlinear refractive index arising from the third-order nonlinearity:

$$P = \varepsilon_0 \left[\chi^{(1)} E + \chi^{(3)} |E|^2 E \right] = \varepsilon_0 \left[\chi^{(1)} + \chi^{(3)} |E|^2 \right] E$$

Substituting the polarization into the wave equation

So the

$$\frac{\partial^{2} E}{\partial z^{2}} - \frac{1}{c_{0}^{2}} \frac{\partial^{2} E}{\partial t^{2}} = \mu_{0} \varepsilon_{0} \left[\chi^{(1)} + \chi^{(3)} \left| E \right|^{2} \right] \frac{\partial^{2} E}{\partial t^{2}}$$

$$\frac{\partial^{2} E}{\partial z^{2}} - \frac{\left[1 + \chi^{(1)} + \chi^{(3)} \left| E \right|^{2} \right]}{c_{0}^{2}} \frac{\partial^{2} E}{\partial t^{2}} = 0 \quad \text{since} \quad \mu_{0} \varepsilon_{0} = 1/c_{0}^{2}$$
refractive index is:
$$n = \sqrt{1 + \chi^{(1)} + \chi^{(3)} \left| E \right|^{2}}$$

Intensity dependent nonlinear refractive index

The refractive index in the presence of linear and nonlinear polarizations:

$$n = \sqrt{1 + \chi^{(1)} + \chi^{(3)} |E|^2}$$

Now, the usual refractive index (which we'll call n_0) is: $n_0 = \sqrt{1 + \chi^{(1)}}$

So:

$$n = \sqrt{n_0^2 + \chi^{(3)} |E|^2} = n_0 \sqrt{1 + \chi^{(3)} |E|^2 / n_0^2}$$

Assume that the nonlinear term << n₀:

So:

$$n \approx n_0 \left[1 + \chi^{(3)} |E|^2 / 2n_0^2 \right]$$

$$n \approx n_0 + \chi^{(3)} |E|^2 / 2n_0$$

Usually we define a "nonlinear refractive index", $n_{2,L}$:

since:
$$I \propto \left| E \right|^2$$

 $n = n_0 + n_{2,L}I$

Kerr effect: refractive index linearly dependent on light intensity.

Who is Kerr?

John Kerr (1824-1907) was a Scottish physicist. He was a student in Glasgow from 1841 to 1846, and at the Theological College of the Free Church of Scotland, in Edinburgh, in 1849. Starting in 1857 he was mathematical lecturer at the Free Church Training College in Glasgow.

He is best known for the discovery in 1875 of what is now called Kerr effect—the first nonlinear optical effect to be observed. In the Kerr effect, a change in refractive index is proportional to the square of the electric field. The Kerr effect is exploited in the *Kerr cell*, which is used in applications such as shutters in high-speed photography, with shutter-speeds as fast as 100 ns.



John Kerr, c. 1860, photograph by Thomas Annan

Magnitude of nonlinear refractive index

Material	Refractive index n	$n_{2,L}[cm^2/W]$
Sapphire (Al_2O_3)	1.76 @ 850 nm	$3 \cdot 10^{-16}$
Fused Quarz	1.45 @ 1064 nm	$2.46 \cdot 10^{-16}$
Glass (LG-760)	1.5 @ 1064 nm	$2.9 \cdot 10^{-16}$
$YAG (Y_3Al_5O_{12})$	1.82 @ 1064 nm	$6.2 \cdot 10^{-16}$
YLF (LiYF ₄), n_e	1.47 @ 1047 nm	$1.72 \cdot 10^{-16}$
Si	3.3 @ 1550 nm	$4 \cdot 10^{-14}$

- 1) A variety of effects give rise to a nonlinear refractive index.
- 2) Those that yield a large n_2 typically have a slow response.
- 3) Nonlinear coefficient can be negative.

Kerr effect for an optical beam: self focusing





Focusing an optical beam at three different powers red: low power green: near critical power blue: above critical power Due to diffraction, a lens can only focus a beam to a finite size

$$r = \frac{0.61}{\sin \theta_0} \lambda$$

For $n_2 > 0$, Kerr effect in a medium acts as a positive lens for a Gaussian beam—the beam's center experiences larger refractive index than the edge. If the peak power exceeds a critical power, self focusing overtakes diffraction and the beam converges rapidly leading to material damage.

$$P_{cr} = \frac{\pi (0.61)^2 \lambda_0^2}{8n_0 n_2}$$

Example: self focusing critical power in fused silica

$$\lambda_0 = 1.03 \,\mu m$$
 $n_0 = 1.45$ $n_2 = 2.7 \times 10^{-20} \, m^2 \, / W$
 $P_{cr} = 4 M W$

Kerr effect for an optical pulse: self-phase modulation

In a purely one dimensional propagation problem, the intensity dependent refractive index imposes an additional self-phase shift on the pulse envelope during propagation, which is proportional to the instantaneous intensity of the pulse:

Note that here the pulse profile has been re-normalized so that its square gives intensity:

Pulse shape does not change, but the pulse acquires nonlinear phase:

Nonlinear phase modulation of a pulse, caused by its own intensity via the Kerr effect.

$$A(z,t) = A(0,t)e^{j\varphi_{NL}}$$
$$= A(0,t)e^{-j\delta|A(z,t)|^2 z}$$

$$|A(z,t)| = |A(0,t)|$$
$$\varphi_{NL} = -\delta |A(z,t)|^2 z$$

SPM induces positive chirp



SPM modifies spectrum



Spectral bandwidth is proportional to the amount of nonlinear phase accumulated inside the fiber.

$$\phi_{\scriptscriptstyle NL} \approx (M - \frac{1}{2}) \times \pi$$

M is the number of spectral peaks.



Input: Gaussian pulse, Pulse duration = 100 fs, Peak power = 1 kW

Pulse propagation: pure dispersion Vs pure SPM

• Pure dispersion $j\frac{\partial A(z,t)}{\partial z} = -D_2 \frac{\partial^2 A}{\partial t^2}$ $D_2 = \frac{\beta_2}{2} \xrightarrow{} \text{GVD}$

(1) Pulse's spectrum acquires phase.

(2) Spectrum profile does not change.

- (3) In the time domain, pulse may be stretched or compressed depending on its initial chirp .
- Pure SPM $j\frac{\partial A(z,t)}{\partial z} = \delta |A|^2 A$

(1) Pulse acquires phase in the time domain.

- (2) Pulse profile does not change.
- (3) In the frequency domain, pulse's spectrum may be broadened or narrowed depending on its initial chirp.

Nonlinear Schrödinger Equation (NLSE)

$$j\frac{\partial A(z,t)}{\partial z} = -D_2\frac{\partial^2 A}{\partial t^2} + \delta |A|^2 A \qquad D_2 = \frac{\beta_2}{2}$$

Positive GVD (normal dispersion) + SPM:

GVD and SPM both act to shift the red frequency to the front of the pulse. Therefore the pulse will spread faster than it would in the purely linear case.

Negative GVD (anomalous dispersion) + SPM:

GVD and SPM shift frequency in the opposite direction. At a certain condition, the dispersive spreading of the pulse is exactly balanced by the compression due to the opposite chirp induced by SPM. A steadystate pulse can propagate without changing its shape. (i.e. soliton regime)

NLSE has soliton solution.

General properties of soliton

In mathematics and physics, a **soliton** is a self-reinforcing solitary wave (a wave packet or pulse) that maintains its shape while it travels at constant speed. Solitons are caused by a cancellation of nonlinear and dispersive effects in the medium. ---Wiki

- When two solitons get closer, they gradually collide and merge into a single wave packet.
- This packet soon splits into two solitons with the same shape and velocity before "collision".



Who discovered soliton?

John Scott Russell (1808 – 1882) was a Scottish civil engineer, naval architect and shipbuilder.

In 1834, while conducting experiments to determine the most efficient design for canal boats, John Scott Russell made a remarkable scientific discovery, leading to a conference paper— Report on Waves.

<u>Report of the fourteenth meeting of the</u> <u>British Association for the Advancement of</u> <u>Science, York, September 1844 (London</u> <u>1845), pp 311-390, Plates XLVII-LVII).</u>



John Scott Russell (1808-1882)

Russell's report

- "I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its the course along channel apparently without change of form or diminution of speed."
- "I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation."

<u>Report of the fourteenth meeting of the British Association for the Advancement of Science.</u> York, September 1844 (London 1845), pp 311-390, Plates XLVII-LVII).

Water wave soliton in Scott Russell's Aqueduct



89.3m long, 4.13m wide, 1.52m deep, On the union Canal, Near Heroit-Watt Univ.

www.spsu.edu/math/txu/research/presentations/soliton/talk.ppt

Water wave soliton in Scott Russell Aqueduct



www.spsu.edu/math/txu/research/presentations/soliton/talk.ppt