UFS Lecture 3: Optical Pulses and Dispersion

2.2 Classical Permittivity / Susceptibility (Review)

2.3 Optical Pulses
2.4 Pulse Propagation
    2.4.1 Dispersion
    2.4.2 Loss andGain
2.5 Sellmeier Equation and Kramers-Kroenig Relations

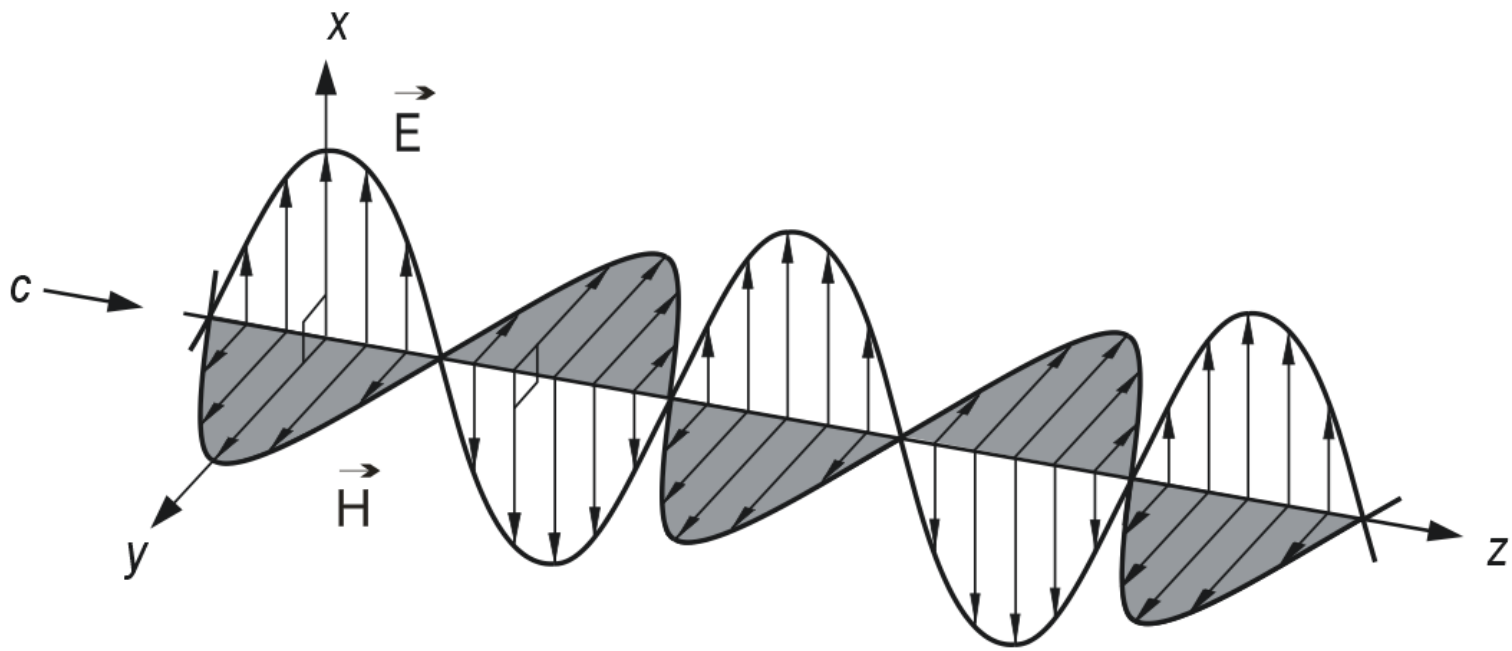


Figure 2.1: Transverse electromagnetic wave (TEM) [2]

Real and Imaginary Part of the Susceptibility

$$\underline{\tilde{\chi}}(\omega) = \tilde{\chi}_r(\omega) + j\tilde{\chi}_i(\omega)$$

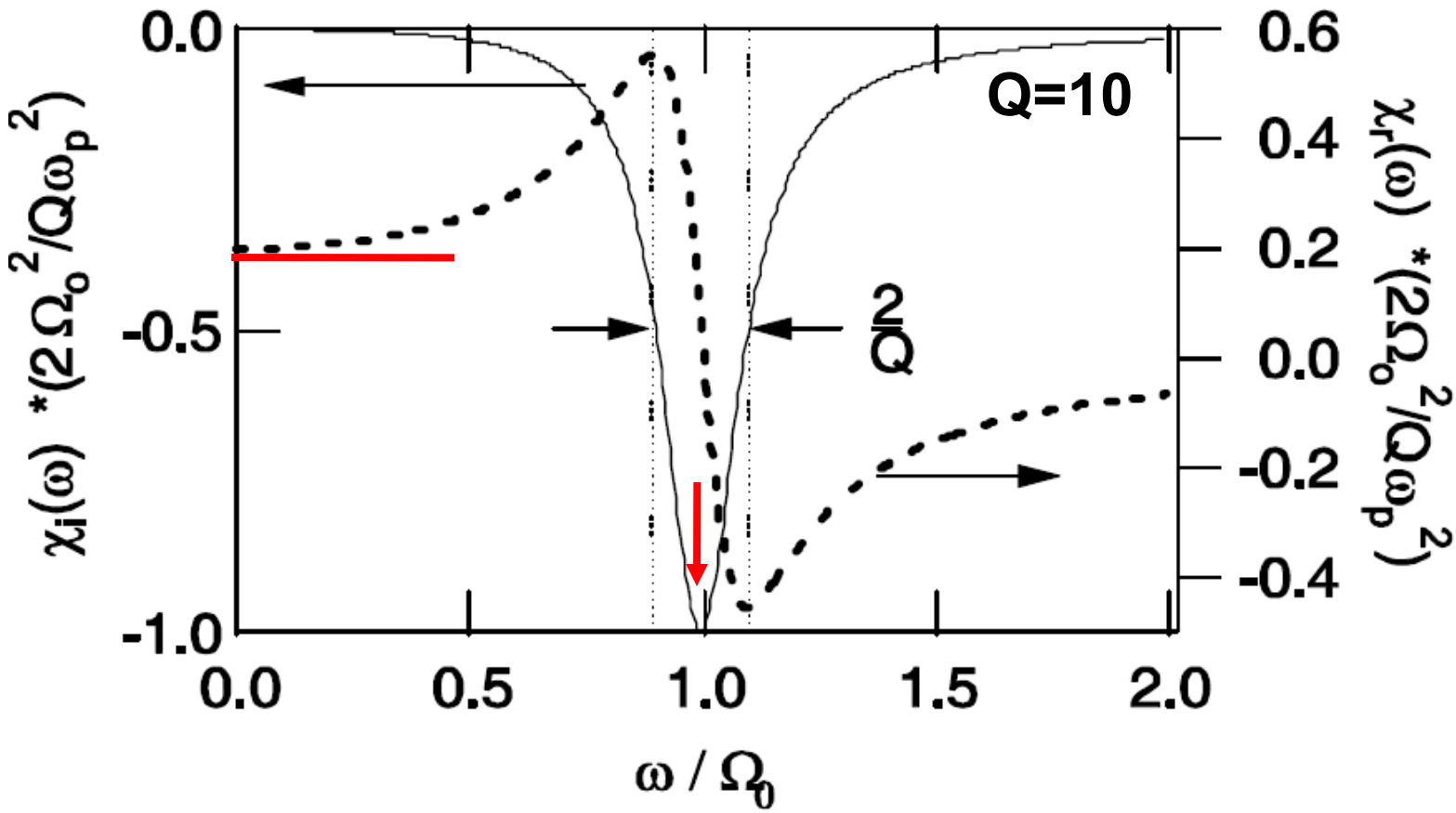


Figure 2.3: Real part (dashed line) and imaginary part (solid line) of the susceptibility of the classical oscillator model for the dielectric polarizability

Real and Imaginary Part of the Susceptibility

$$\underline{\tilde{\chi}}(\omega) = \tilde{\chi}_r(\omega) + j\tilde{\chi}_i(\omega)$$

Example: EM-Wave polarized along x-axis and propagation along z-direction:

$$\underline{\vec{E}}(z,t) = \underline{E}_0 e^{j(\omega t - kz)}\ \vec{e}_x$$

In general:

$$k(\omega) = \frac{\omega}{c_0}\tilde{n}(\omega) = \frac{\omega}{c_0}\sqrt{1+\tilde{\chi}(\omega)} = \frac{\omega}{c_0}\left(\tilde{n}_r(\omega) + j\tilde{n}_i(\omega)\right) = k_r(\omega) - j\alpha(\omega)$$

$$\boxed{\underline{\vec{E}}(z,t) = \underline{E}_0 e^{-\alpha\cdot z}\ e^{j(\omega t - k_r z)}\ \vec{e}_x}$$

damping

$$k(\omega) = \frac{\omega}{c_0}\sqrt{1+\tilde{\chi}(\omega)}$$

for: $\tilde{\chi}(\omega) \ll 1$

$$= \frac{\omega}{c_0}\left(1 + \frac{1}{2}\tilde{\chi}(\omega)\right) = \frac{\omega}{c_0}\left(1 + \frac{1}{2}\tilde{\chi}_r(\omega) + \frac{1}{2}j\tilde{\chi}_i(\omega)\right)$$

2.1.5 Optical Pulses (propagating along z-axis)

$$\underline{\vec{E}}(\vec{r},t) = \int_0^\infty \frac{d\Omega}{2\pi} \underline{\tilde{E}}(\Omega) e^{\mathrm{j}(\Omega t - K(\Omega) z)} \ \vec{e}_x$$

$$\underline{\vec{H}}(\vec{r},t) = \int_0^\infty \frac{d\Omega}{2\pi Z_F(\Omega)} \underline{\tilde{E}}(\Omega) e^{\mathrm{j}(\Omega t - K(\Omega) z)} \ \vec{e}_y$$

$$\vec{E}(\vec{r},t) = \frac{1}{2}\left(\underline{\vec{E}}(\vec{r},t) + \underline{\vec{E}}(\vec{r},t)^*\right)$$

$$\vec{H}(\vec{r},t) = \frac{1}{2}\left(\underline{\vec{H}}(\vec{r},t) + \underline{\vec{H}}(\vec{r},t)^*\right)$$

$|\underline{\tilde{E}}(\Omega)| e^{\mathrm{j}\varphi(\Omega)}$: Wave amplitude and phase

$K(\Omega) = \Omega / c(\Omega) = n(\Omega)\Omega / c_0$: Wave number

$c(\Omega) = \dfrac{c_0}{n(\Omega)}$: Phase velocity of wave

$$\tilde{n}^2(\Omega) = 1 + \tilde{\chi}(\Omega)$$

At z=0

$$\underline{E}(z=0,t) = \frac{1}{2\pi}\int_0^{\infty} \underline{\tilde{E}}(\Omega)e^{j\Omega t}d\Omega$$

For Example:
Optical Communication; 10Gb/s
Pulse length: 20 ps
Center wavelength : λ=1550 nm.
Spectral width: ~ 50 GHz,
Center frequency: 200 THz,

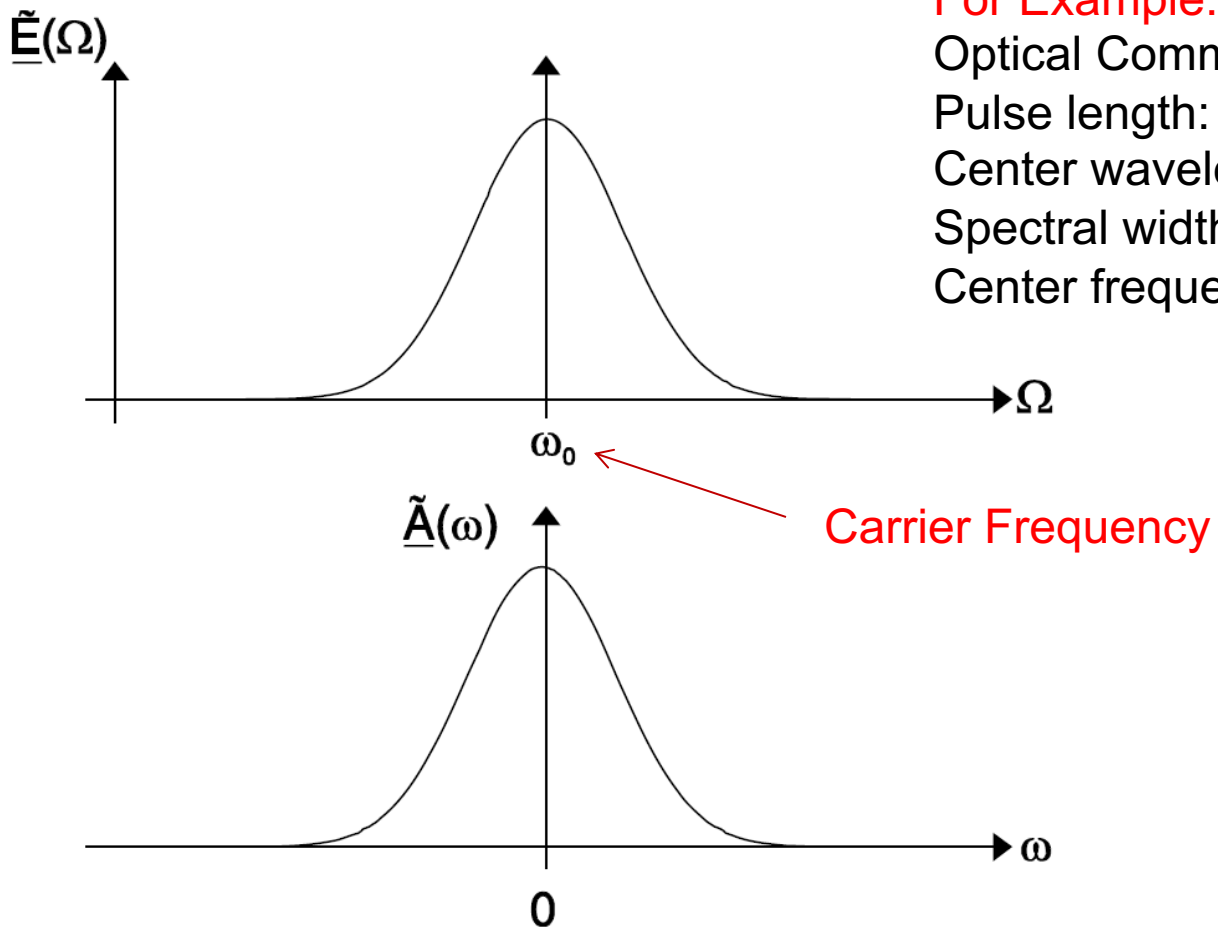


Figure 2.4: Spectrum of an optical wave packet described in absolute and relative frequencies

Carrier and Envelope

$$\underline{E}(z = 0, t) = \frac{1}{2\pi}\int_{-\omega_0}^{\infty} \underline{\tilde{E}}(\omega_0 + \omega) e^{j(\omega_0+\omega)t} d\omega$$

$$= \frac{1}{2\pi} e^{j\omega_0 t} \int_{-\omega_0}^{\infty} \underline{\tilde{E}}(\omega_0 + \omega) e^{j\omega t} d\omega$$

$$A(t) e^{j\omega_0 t}.$$

Carrier Frequency

Envelope:

$$\underline{A}(t) = \frac{1}{2\pi}\int_{-\omega_0 \to -\infty}^{\infty} \underline{\tilde{A}}(\omega) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underline{\tilde{A}}(\omega) e^{j\omega t} d\omega,$$

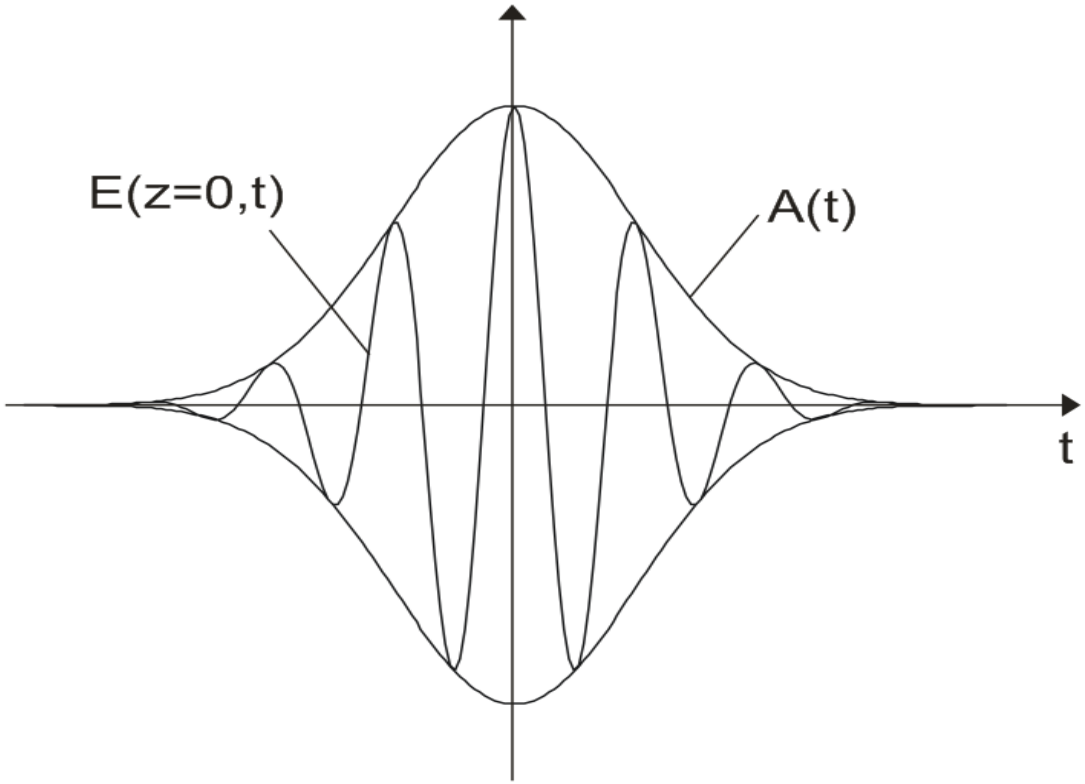


Figure 2.5: Electric field and envelope of an optical pulse

Pulse width: Full Width at Half Maximum of $|A(t)|^2$

Spectral width : Full Width at Half Maximum of $|\underline{\tilde{A}}(\omega)|^2$

Often Used Pulses

Pulse Shape	Fourier Transform	Pulse Width	Time-Band-width Product
$\underline{A}(t)$	$\underline{A}(\omega) = \int_{-\infty}^{\infty} a(t) e^{-j\omega t} dt$	Δt	$\Delta t \cdot \Delta f$
Gaussian: $e^{-\frac{t^2}{2\tau^2}}$	$\sqrt{2\pi}\tau e^{-\frac{1}{2}\tau^2\omega^2}$	$2\sqrt{\ln 2}\tau$	0.441
Hyperbolic Secant: $\mathrm{sech}(\frac{t}{\tau})$	$\frac{\tau}{2}\,\mathrm{sech}\left(\frac{\pi}{2}\tau\omega\right)$	$1.7627\ \tau$	0.315
Rect-function: $\begin{cases} 1, & \lvert t\rvert \le \tau/2 \\ 0, & \lvert t\rvert > \tau/2 \end{cases}$	$\tau \frac{\sin(\tau\omega/2)}{\tau\omega/2}$	τ	0.886
Lorentzian: $\frac{1}{1+(t/\tau)^2}$	$2\pi\tau e^{-\lvert\tau\omega\rvert}$	$1.287\ \tau$	0.142
Double-Exp.: $e^{-\left\lvert\frac{t}{\tau}\right\rvert}$	$\frac{\tau}{1+(\omega\tau)^2}$	$\ln 2\ \tau$	0.142

Table 2.2: Pulse shapes, corresponding spectra and time bandwidth products.

Pulse width and spectral width: FWHM

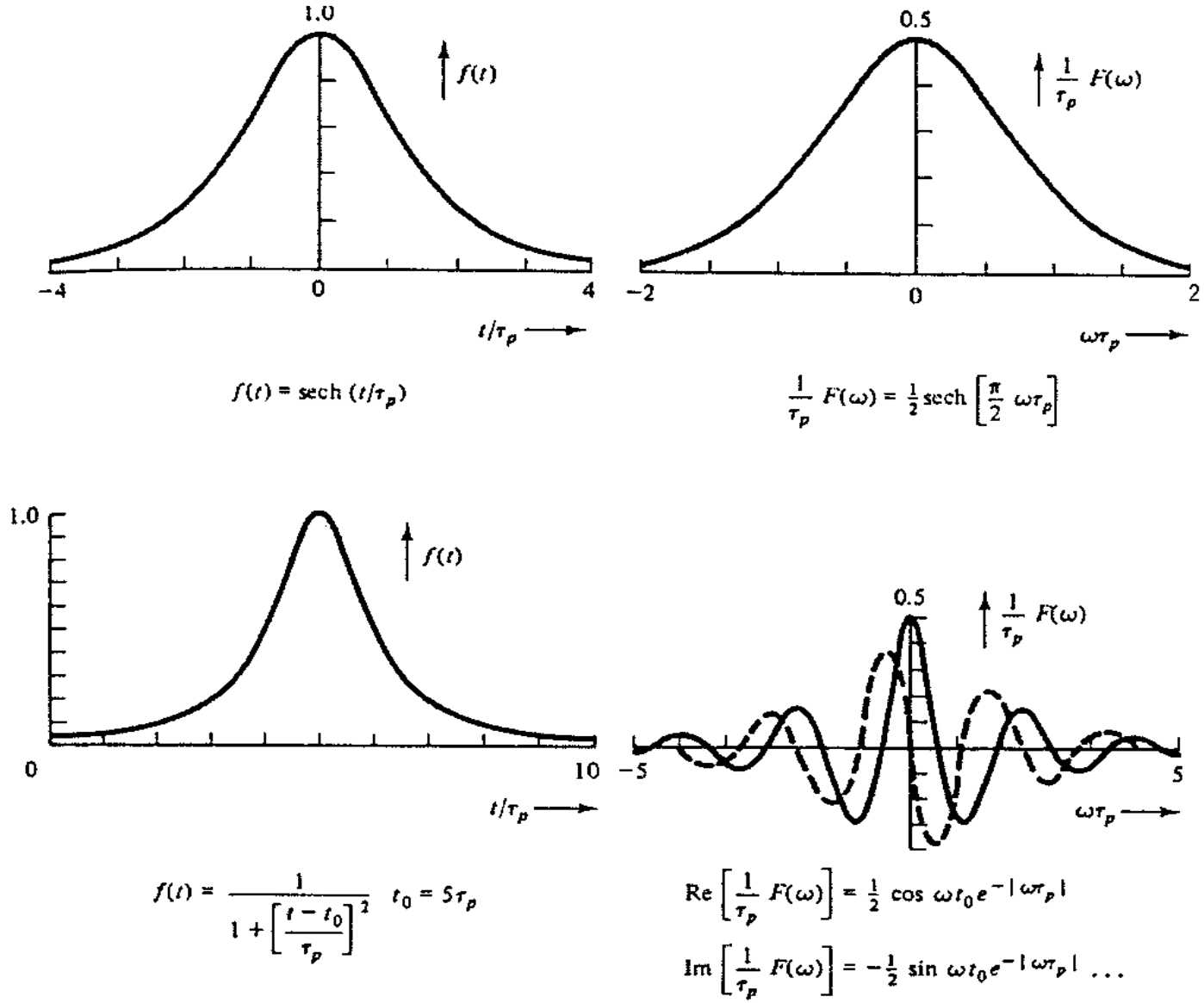


Figure 2.6: Fourier transforms to pulse shapes listed in table 2.2 [16]

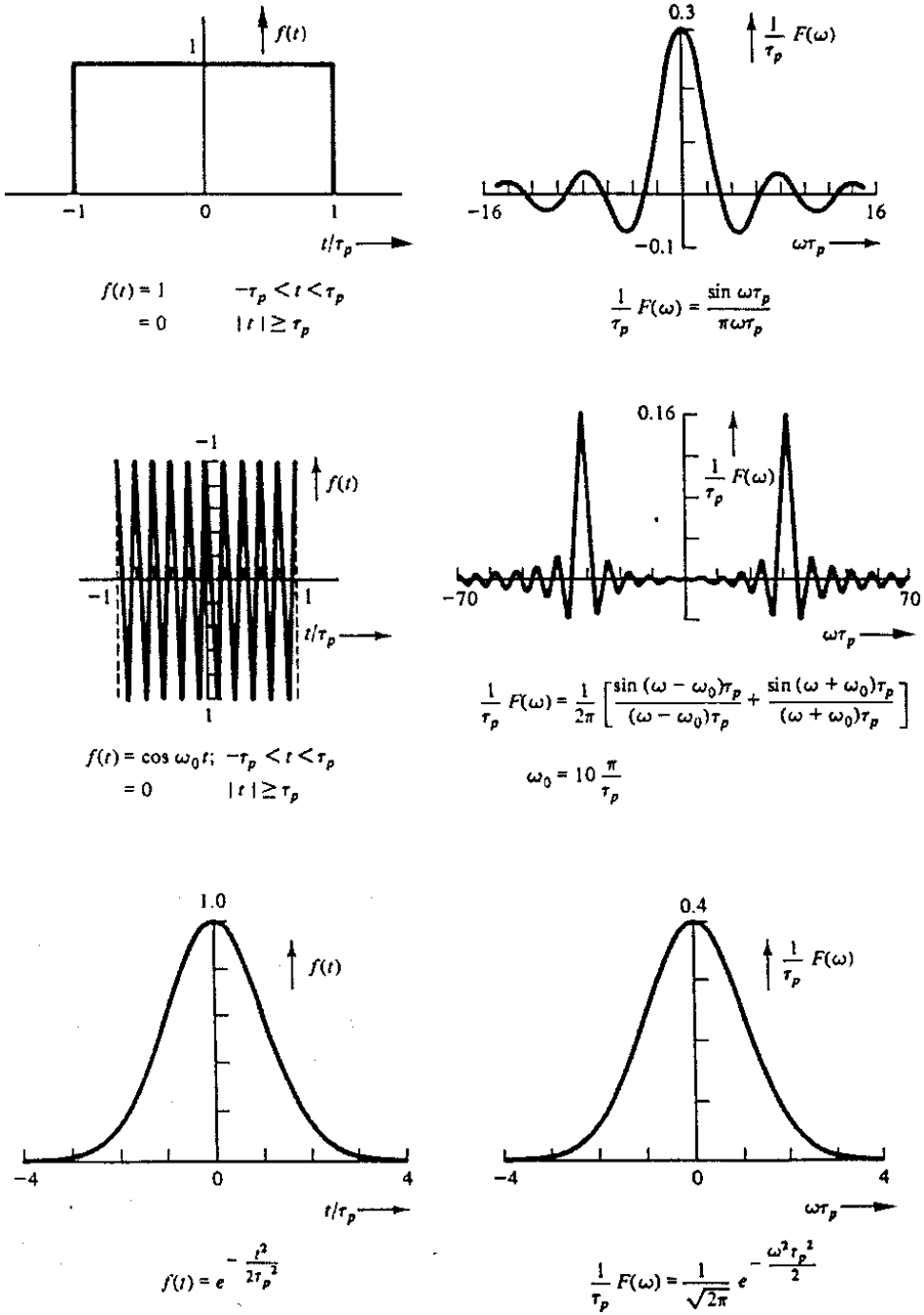


Figure 2.7: Fourier transforms to pulse shapes listed in table 2.2, continued [16]

2.4 Pulse Propagation

$$\underline{E}(z,t) = \frac{1}{2\pi}\int_0^\infty \underline{\tilde{E}}(\Omega)e^{\mathrm{j}(\Omega t - K(\Omega)z)}d\Omega$$

$$\underline{E}(z,t) = \underline{A}(z,t)e^{\mathrm{j}(\omega_0 t - K(\omega_0)z)}$$

Envelope + Carrier Wave

$$\begin{aligned}
\omega &= \Omega - \omega_0, \\
k(\omega) &= K(\omega_0 + \omega) - K(\omega_0), \\
\underline{\tilde{A}}(\omega) &= \underline{\tilde{E}}(\Omega = \omega_0 + \omega).
\end{aligned}$$

$$\underline{E}(z,t) = \underbrace{\frac{1}{2\pi}\int_{-\infty}^\infty \underline{\tilde{A}}(\omega)e^{\mathrm{j}(\omega t - k(\omega)z)}d\omega}\ e^{\mathrm{j}(\omega_0 t - K(\omega_0)z)}$$

$$\underline{A}(z,t) = \frac{1}{2\pi}\int_{-\infty}^\infty \underline{\tilde{A}}(\omega)e^{\mathrm{j}(\omega t - k(\omega)z)}d\omega$$

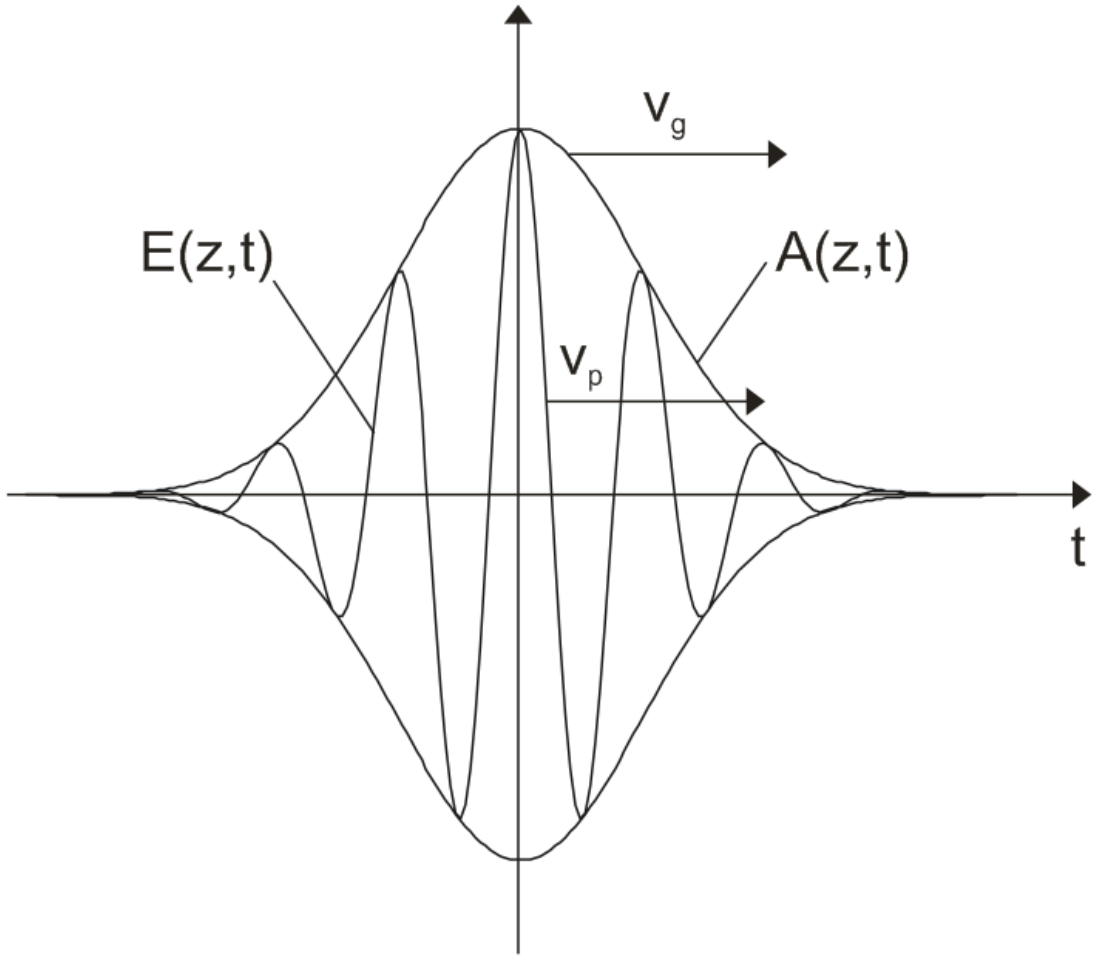


Figure 2.8: Electric field and pulse envelope in time domain

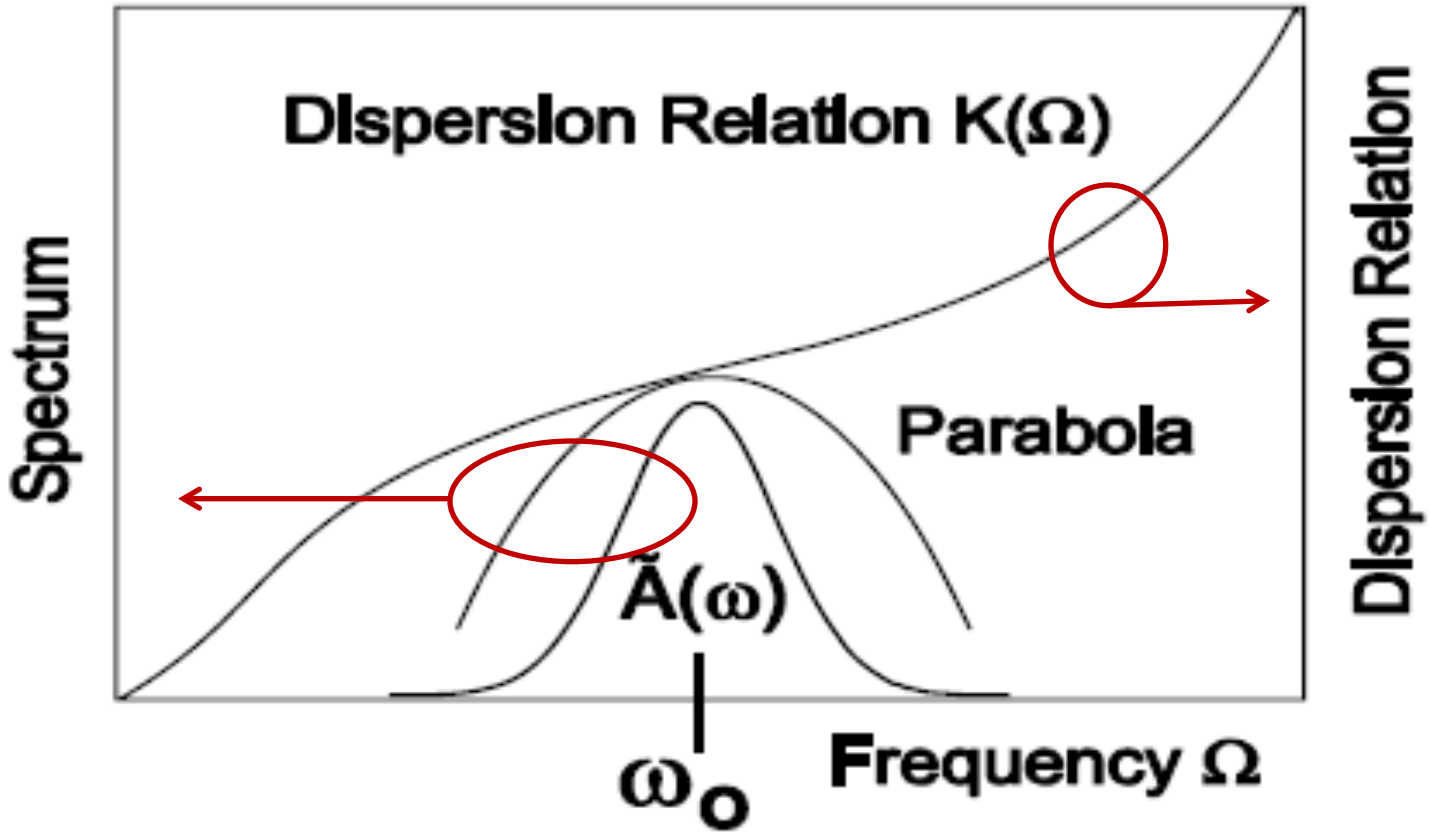


Figure 2.9: Taylor expansion of dispersion relation at the center frequency of the wave packet

2.4.1 Dispersion

In the frequency domain:

$$\underline{\tilde{A}}(z,\omega) = \underline{\tilde{A}}(z=0,\omega)e^{-\mathrm{j}k(\omega)z}$$

Taylor expansion of dispersion relation:

$$k(\omega) = k'\omega + \frac{k''}{2}\omega^2 + \frac{k^{(3)}}{6}\omega^3 + O(\omega^4)$$

Equation of motion in frequency domain:

$$\frac{\partial \underline{\tilde{A}}(z,\omega)}{\partial z} = -\mathrm{j}k(\omega)\underline{\tilde{A}}(z,\omega)$$

Equation of motion in time domain:

$$\frac{\partial \underline{A}(z,t)}{\partial z} = -\mathrm{j}\sum_{n=1}^{\infty}\frac{k^{(n)}}{n!}\left(-\mathrm{j}\frac{\partial}{\partial t}\right)^n \underline{A}(z,t) \qquad \textbf{(2.63)}$$

i) Keep only linear term:

$$k(\omega) = k'\omega + \cancel{\frac{k''}{2}}\omega^2 + \cancel{\frac{k^{(3)}}{6}}\omega^3 + \cancel{\mathcal{O}}(\omega^4)$$

$$\underline{\tilde{A}}(z,\omega) = \underline{\tilde{A}}(z=0,\omega)e^{-\mathrm{j}k'\omega z}$$

Time domain:

$$\underline{A}(z,t) = \underline{A}(0, t - z/v_{g0})$$

Group velocity:

$$v_{g0} = 1/k' = \left(\left.\frac{dk(\omega)}{d\omega}\right|_{\omega=0}\right)^{-1} = \left(\left.\frac{dK(\Omega)}{d\Omega}\right|_{\Omega=\omega_0}\right)^{-1}$$

Compare with phase velocity:

$$v_{p0} = \omega_0/K(\omega_0) = \left(\frac{K(\omega_0)}{\omega_0}\right)^{-1}$$

Retarded time: $t' = t - z/v_{g0}$

$$\underline{A}(z,t) = \underline{A}(0,t')$$

Or start from (2.63)

$$\frac{\partial \underline{A}(z,t)}{\partial z} + \frac{1}{v_{g0}} \frac{\partial \underline{A}(z,t)}{\partial t} = 0$$

Substitute:

$$\begin{aligned} z' &= z, \\ t' &= t - z/v_{g0}, \end{aligned} \qquad \begin{aligned} \frac{\partial}{\partial z} &= \frac{\partial}{\partial z'} - \frac{1}{v_{g0}} \frac{\partial}{\partial t'} \\ \frac{\partial}{\partial t} &= \frac{\partial}{\partial t'} \end{aligned}$$

$$\boxed{\frac{\partial \underline{A}(z',t')}{\partial z'} = 0}$$

ii) Keep up to second order term:

$$k(\omega) = k'\omega + \frac{k''}{2}\omega^2 + \cancel{\frac{k^{(3)}}{6}\omega^3} + \cancel{O(\omega^4)}$$

$$\frac{\partial \underline{A}(z,t')}{\partial z} = \mathrm{j}\frac{k''}{2}\frac{\partial^2 \underline{A}(z,t')}{\partial t'^2}$$

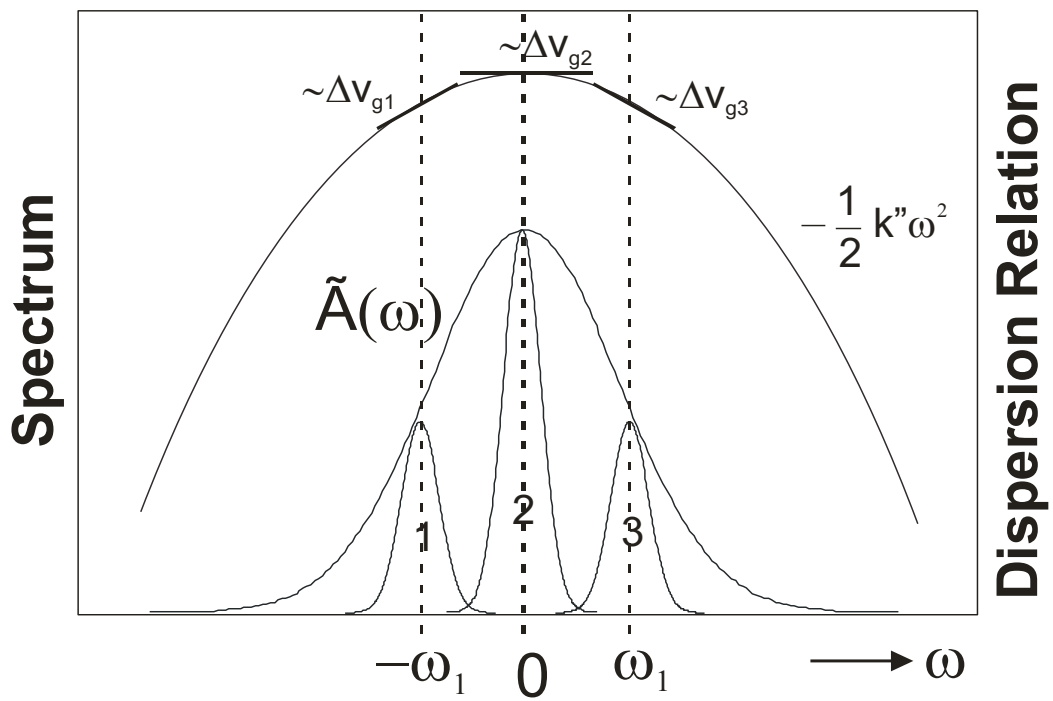


Figure 2.10: Decomposition of a pulse into wave packets with different center frequency. In a medium with dispersion the wave packets move at different relative group velocity

Gaussian Pulse:

$$\underline{E}(z = 0, t) = \underline{A}(z = 0, t)e^{j\omega_0 t}$$

$$\underline{A}(z = 0, t = t') = \underline{A}_0 \exp\left[-\frac{1}{2}\frac{t'^2}{\tau^2}\right]$$

Pulse width

$$\frac{\partial \underline{\tilde{A}}(z,\omega)}{\partial z} = -\mathrm{j}\frac{k''\omega^2}{2}\underline{\tilde{A}}(z,\omega)$$

Substitute:

$$\underline{\tilde{A}}(z,\omega) = \underline{\tilde{A}}(z = 0,\omega)\exp\left[-\mathrm{j}\frac{k''\omega^2}{2}z\right]$$

Gaussian Integral:

$$\boxed{\frac{1}{\sqrt{2\pi\sigma}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma}}e^{-jx\varsigma}dx = e^{-\frac{\sigma}{2}\varsigma^2} \text{ for } \mathrm{Re}\{\sigma\} \geq 0}$$

Apply to (2.79)

$$\underline{\tilde{A}}(z = 0,\omega) = A_0\sqrt{2\pi}\tau\exp\left[-\frac{1}{2}\tau^2\omega^2\right]$$

Propagation:

$$\underline{\tilde{A}}(z,\omega) = A_0\sqrt{2\pi}\tau \exp\left[-\frac{1}{2}\left(\tau^2 + \mathrm{j}k''z\right)\omega^2\right]$$

$$\underline{A}(z,t') = A_0\left(\frac{\tau^2}{(\tau^2 + \mathrm{j}k''z)}\right)^{1/2} \exp\left[-\frac{1}{2}\frac{t'^2}{(\tau^2 + \mathrm{j}k''z)}\right]$$

Exponent Real and Imaginary Part:

$$\underline{A}(z,t') = A_0\left(\frac{\tau^2}{(\tau^2 + \mathrm{j}k''z)}\right)^{1/2} \exp\left[-\frac{1}{2}\frac{\tau^2 t'^2}{\left(\tau^4 + (k''z)^2\right)} + \mathrm{j}\frac{1}{2}k''z\frac{t'^2}{\left(\tau^4 + (k''z)^2\right)}\right]$$

z-dependent phase shift **determines pulse width** **chirp**

FWHM Pulse width:

$$\exp\left[-\frac{\tau^2(\tau'_{FWHM}/2)^2}{\left(\tau^4 + (k''z)^2\right)}\right] = 0.5$$

Initial pulse width:

$$\tau_{FWHM} = 2\sqrt{\ln 2}\,\tau$$

Initial pulse width:

$$\tau_{FWHM} = 2\sqrt{\ln 2}\,\tau$$

After propagation over a distance z=L:

$$\tau'_{FWHM} = 2\sqrt{\ln 2}\,\tau\sqrt{1+\left(\frac{k''L}{\tau^2}\right)^2} = \tau_{FWHM}\sqrt{1+\left(\frac{k''L}{\tau^2}\right)^2}$$

For large distances:

$$\tau'_{FWHM} = 2\sqrt{\ln 2}\,\left|\frac{k''L}{\tau}\right| \quad \text{for} \quad \left|\frac{k''L}{\tau^2}\right| \gg 1$$

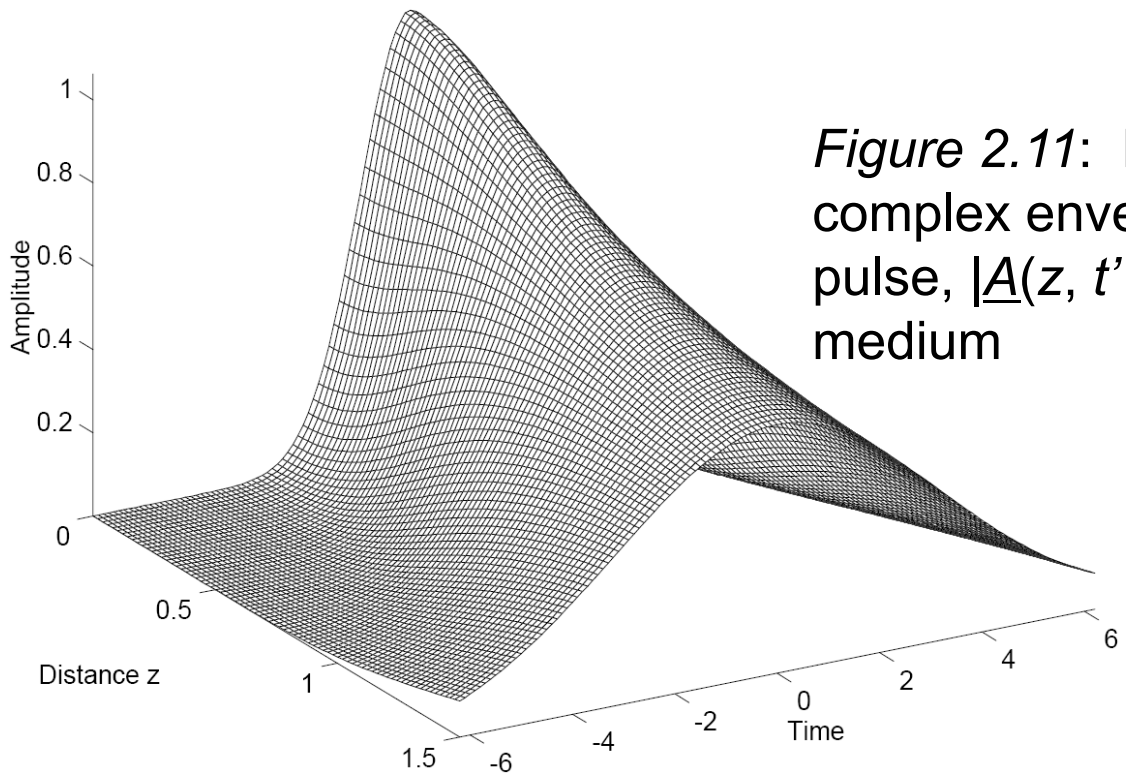


Figure 2.11: Magnitude of the complex envelope of a Gaussian pulse, $|\underline{A}(z, t')|$, in a dispersive medium

2.4.2 Loss and Gain

$$\underline{\tilde{n}}(\Omega) = n_r(\Omega) + \mathrm{j}n_i(\Omega)$$

Refractive index + gain and/or loss

$$\underline{\tilde{n}}(\Omega) = \sqrt{1 + \underline{\tilde{\chi}}(\Omega)}$$

for: $\underline{\tilde{\chi}}(\Omega) \ll 1$

$$\underline{\tilde{n}}(\Omega) \approx 1 + \frac{\underline{\tilde{\chi}}(\Omega)}{2}$$

Complex Lorentzian close to resonance : $\Omega \approx \Omega_0$

$$\underline{\chi}(\omega) = \frac{\omega_p^2}{(\Omega_0^2 - \omega^2) + 2\mathrm{j}\omega\frac{\Omega_0}{Q}} \longrightarrow \underline{\tilde{\chi}}(\Omega) = \frac{-\mathrm{j}\chi_0}{1 + \mathrm{j}Q\frac{\Omega - \Omega_0}{\Omega_0}}$$

Maximum absorption: $\chi_0 = Q\frac{\omega_p^2}{2\Omega_0^2}$

Half Width Half Maximum linewidth (HWHM): $\Delta\Omega = \frac{\Omega_0}{Q}$

Real and imaginary parts:

$$\widetilde{\chi}_r(\Omega) = \frac{-\chi_0 \frac{(\Omega-\Omega_0)}{\Delta\Omega}}{1+\left(\frac{\Omega-\Omega_0}{\Delta\Omega}\right)^2},$$

$$\widetilde{\chi}_i(\Omega) = \frac{-\chi_0}{1+\left(\frac{\Omega-\Omega_0}{\Delta\Omega}\right)^2},$$

Complex wave number in lossy medium:

$$\underline{\tilde{K}}(\Omega) = \frac{\Omega}{c_0}\left(1+\frac{1}{2}\left(\widetilde{\chi}_r(\Omega)+\mathrm{j}\widetilde{\chi}_i(\Omega)\right)\right)$$

Redefine group velocity: e.g. at line center:

$$v_g^{-1} = \left.\frac{\partial K_r(\Omega)}{\partial\Omega}\right|_{\Omega_0} = \frac{1}{c_0}\left(1-\frac{\chi_0}{2}\frac{\Omega_0}{\Delta\Omega}\right)$$

Change in group velocity can be positive or negative

Absorption:

$$K = \frac{\Omega}{c_0} \qquad \alpha(\Omega) = -\frac{K}{2}\tilde{\chi}_i(\Omega)$$

For a wavepacket (optical pulse) with carrier frequency $\omega_0 = \Omega_0 \quad K_0 = \frac{\Omega_0}{c_0}$

$$\left.\frac{\partial \underline{\tilde{A}}(z,\omega)}{\partial z}\right|_{(loss)} = -\alpha(\Omega_0+\omega)\tilde{A}(z,\omega) = \frac{-\chi_0 K_0/2}{1+\left(\frac{\omega}{\Delta\Omega}\right)^2}\underline{\tilde{A}}(z,\omega)$$

Parabolic loss or gain approximation:

$$\left.\frac{\partial \underline{A}(z,t')}{\partial z}\right|_{(loss)} = -\frac{\chi_0 K_0}{2}\left(1+\frac{1}{\Delta\Omega^2}\frac{\partial^2}{\partial t^2}\right)\underline{A}(z,t')$$

Gain: $g = -\frac{\chi_0 K_0}{2}$

$$\left.\frac{\partial \underline{A}(z,t')}{\partial z}\right|_{(gain)} = g\left(1+\frac{1}{\Omega_g^2}\frac{\partial^2}{\partial t^2}\right)\underline{A}(z,t')$$

HWHM – gain bandwidth

2.5 Sellmeier Equations and Kramers-Kroenig Relations

Causality of medium impulse response: $\chi(t) = 0$, for $t < 0$

Leads to relationship between real and imaginary part of susceptibility

$$\chi_r(\Omega) = \frac{2}{\pi}\int_0^\infty \frac{\omega\chi_i(\omega)}{\omega^2 - \Omega^2}d\omega = n_r^2(\Omega) - 1$$

$$\chi_i(\Omega) = -\frac{2}{\pi}\int_0^\infty \frac{\Omega\chi_r(\omega)}{\omega^2 - \Omega^2}d\omega.$$

Approximation for absorption spectrum in a medium:

$$\chi_i(\Omega) = \sum A_i\delta\left(\omega - \omega_i\right)$$

$$n^2(\Omega) = 1 + \underbrace{\sum_i A_i\frac{\omega_i}{\omega_i^2 - \Omega^2} = 1 + \sum_i a_i\frac{\lambda}{\lambda^2 - \lambda_i^2}}_{\chi_r(\Omega)}$$

Example: Sellmeier Coefficients for Fused Quartz and Sapphire

	Fused Quartz	Sapphire
a_1	0.6961663	1.023798
a_2	0.4079426	1.058364
a_3	0.8974794	5.280792
λ_1^2	$4.679148 \cdot 10^{-3}$	$3.77588 \cdot 10^{-3}$
λ_2^2	$1.3512063 \cdot 10^{-2}$	$1.22544 \cdot 10^{-2}$
λ_3^2	$0.9793400 \cdot 10^{2}$	$3.213616 \cdot 10^{2}$

Table 2.3: Table with Sellmeier coefficients for fused quartz and sapphire.

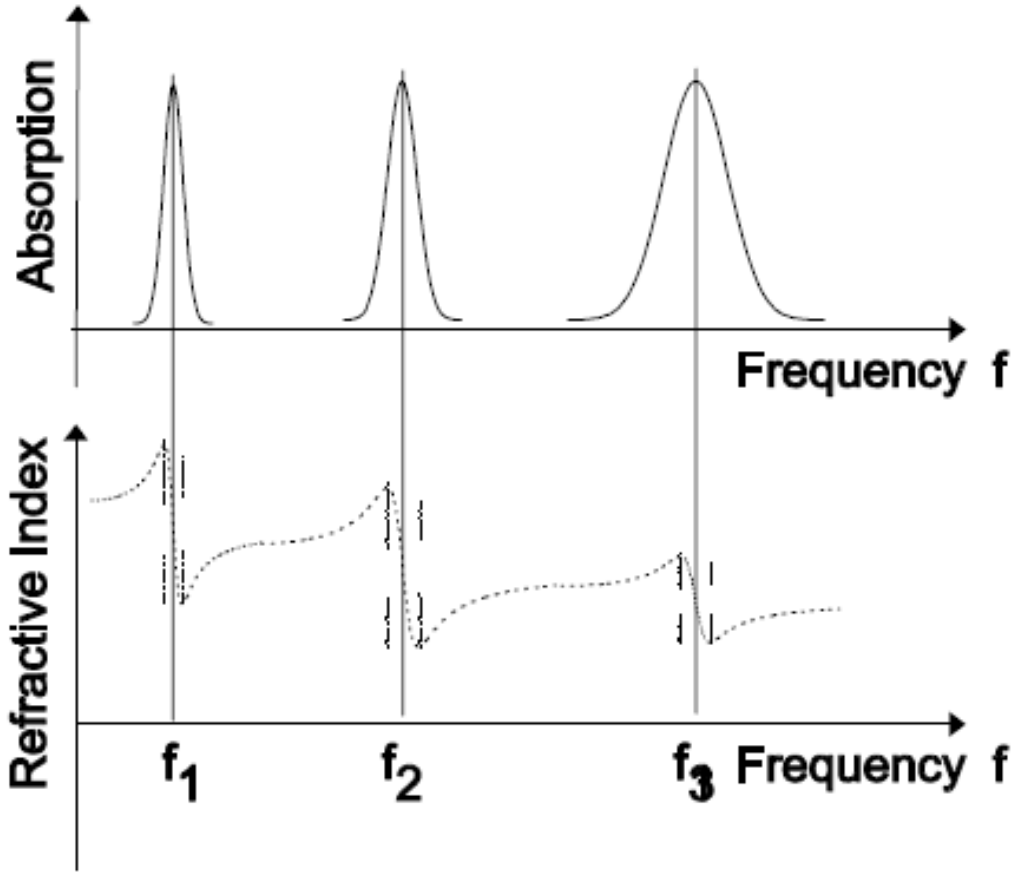


Figure 2.14: Contribution of absorption lines to index changes

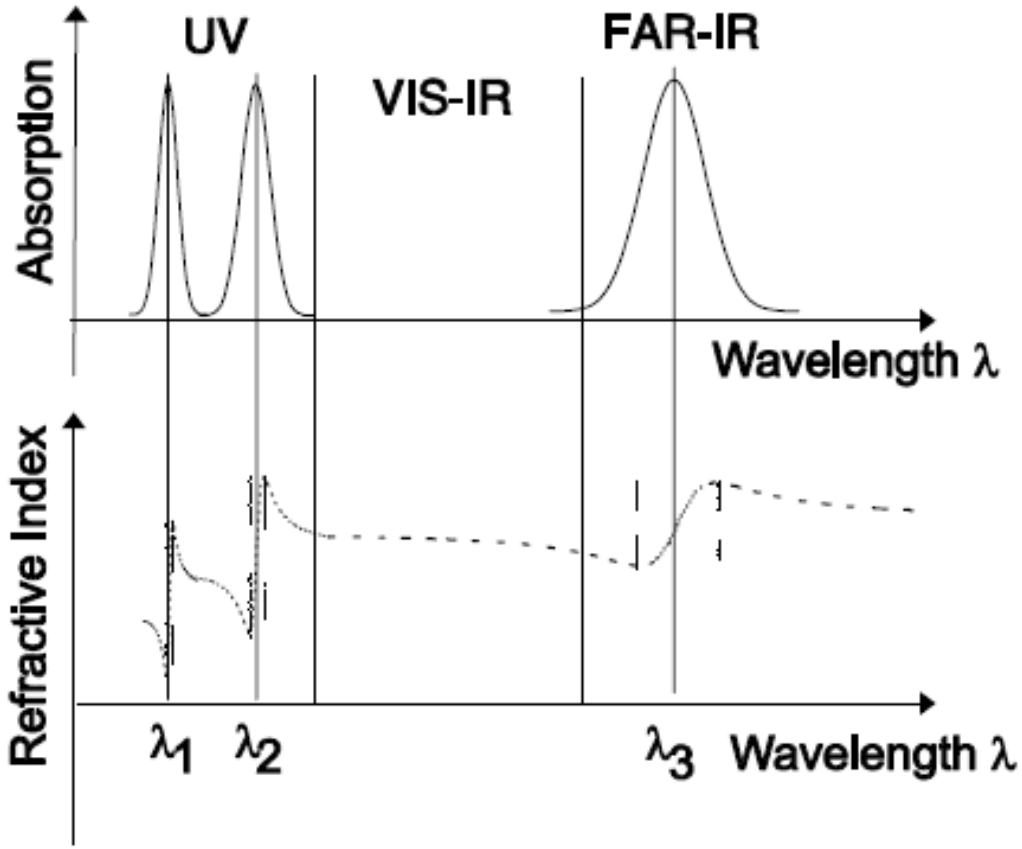


Figure 2.15: Typical distribution of absorption lines in medium transparent in the visible.

$$\frac{dn}{d\lambda} < 0 : \text{normal dispersion (blue refracts more than red)}$$

$$\frac{dn}{d\lambda} > 0 : \text{abnormal dispersion}$$

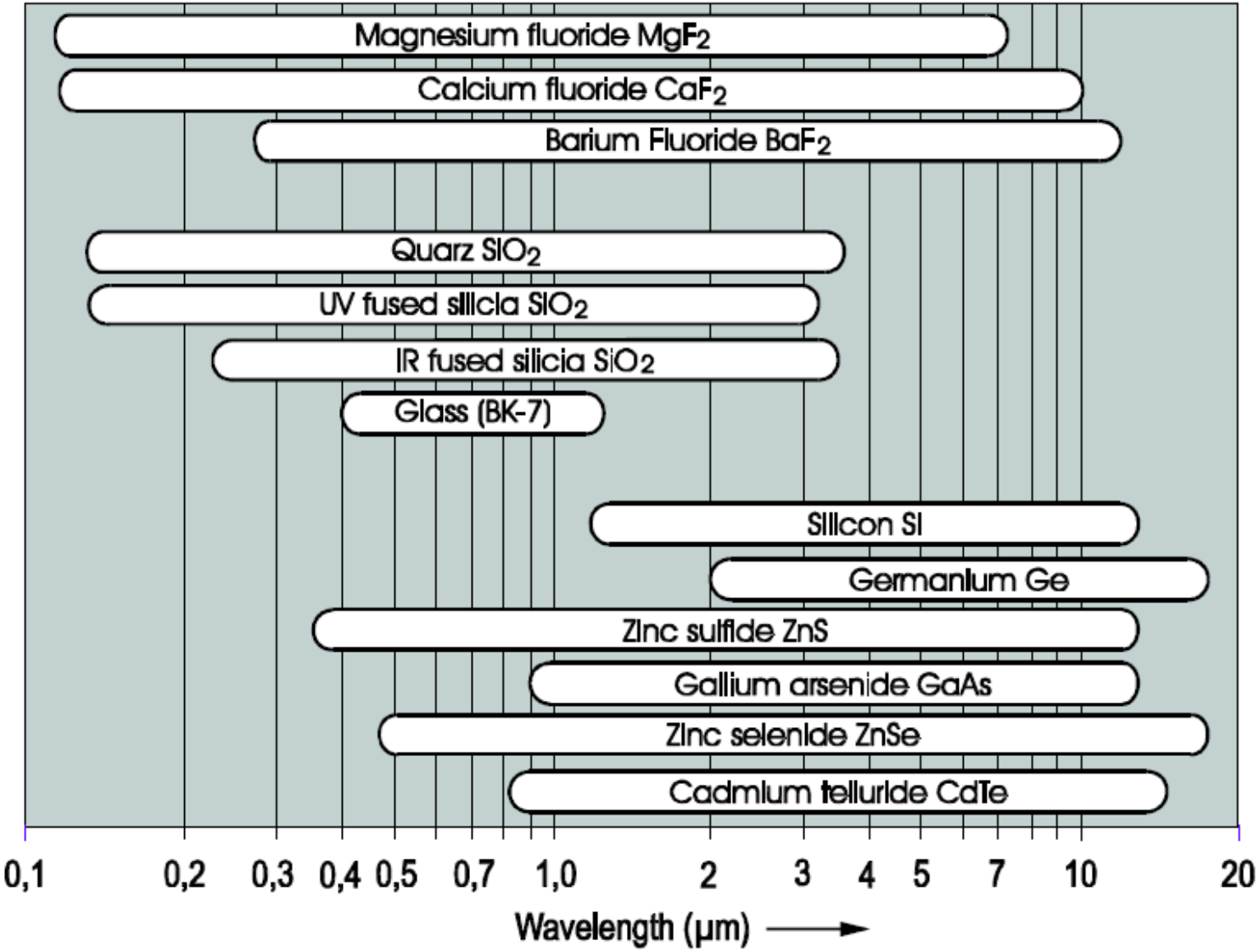


Figure 2.16: Transparency range of some materials according to Saleh and Teich, Photonics p. 175.

Group Velocity and Group Delay Dispersion

$$GVD = \left.\frac{d^2k(\omega)}{d\omega^2}\right|_{\omega=0} = \left.\frac{d}{d\omega}\frac{1}{v_g(\omega)}\right|_{\omega=0}$$

$$GDD = \left.\frac{d^2k(\omega)}{d\omega^2}\right|_{\omega=0} L = \left.\frac{d}{d\omega}\frac{L}{v_g(\omega)}\right|_{\omega=0} = \frac{d}{d\omega}T_g(\omega)|_{\omega=0}$$

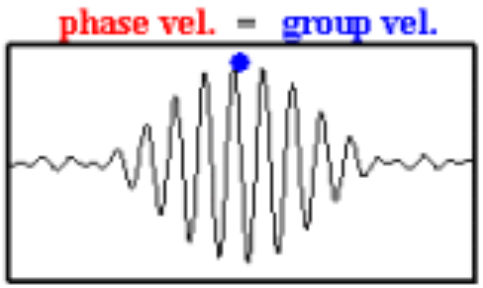
Group Delay: $T_g(\omega) = L/v_g(\omega)$

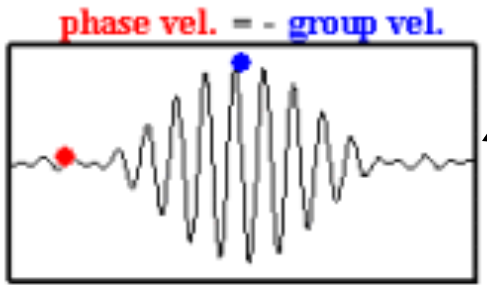
Dispersion Characteristic	Definition	Comp. from $n(\lambda)$
medium wavelength: λ_n	$\frac{\lambda}{n}$	$\frac{\lambda}{n(\lambda)}$
wavenumber: k	$\frac{2\pi}{\lambda_n}$	$\frac{2\pi}{\lambda}n(\lambda)$
phase velocity: v_p	$\frac{\omega}{k}$	$\frac{c_0}{n(\lambda)}$
group velocity: v_g	$\frac{d\omega}{dk}; d\lambda = \frac{-\lambda^2}{2\pi c_0}d\omega$	$\frac{c_0}{n}\left(1 - \frac{\lambda}{n}\frac{dn}{d\lambda}\right)^{-1}$
group velocity dispersion: GVD	$\frac{d^2k}{d\omega^2}$	$\frac{\lambda^3}{2\pi c_0^2}\frac{d^2n}{d\lambda^2}$
group delay: $T_g = \frac{L}{v_g} = \frac{d\phi}{d\omega}$	$\frac{d\phi}{d\omega} = \frac{d(kL)}{d\omega}$	$\frac{n}{c_0}\left(1 - \frac{\lambda}{n}\frac{dn}{d\lambda}\right)L$
group delay dispersion: GDD	$\frac{dT_g}{d\omega} = \frac{d^2(kL)}{d\omega^2}$	$\frac{\lambda^3}{2\pi c_0^2}\frac{d^2n}{d\lambda^2}L$

Table 2.4: Table with important dispersion characteristics and how to compute them from the wavelength dependent refractive index $n(\lambda)$.

Group velocity Vs phase velocity

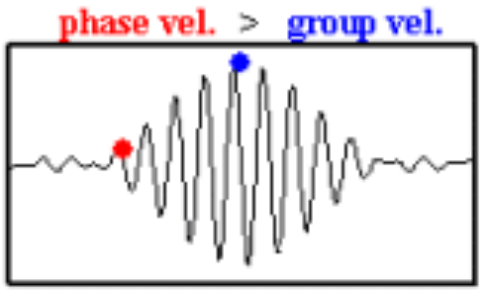
In vacuum

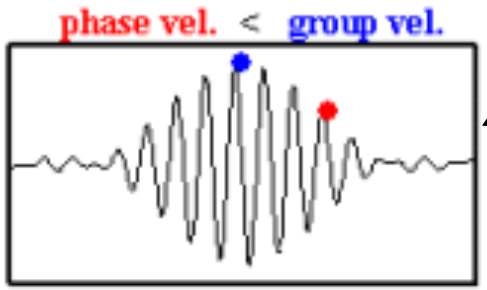




Unrealistic

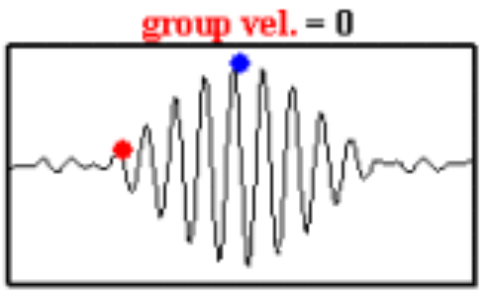
Most common case

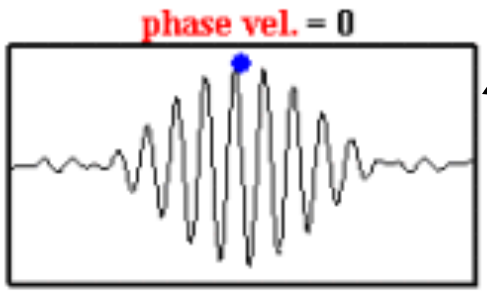




Possible

Unrealistic





Unrealistic

isvr

Adapted from Rick Trebino's course slides

Group-velocity dispersion (GVD)

$$k^{(2)} = \frac{d}{d\omega}\left(\frac{1}{v_g}\right) = -\frac{1}{{v_g}^2(\omega_0)}\frac{dv_g}{d\omega} = \left(\frac{\lambda}{2\pi c}\right)\frac{\lambda^2}{c}\frac{d^2n}{d\lambda^2}$$

Positive GVD or normal dispersion

$$k^{(2)} > 0$$

$$\frac{dv_g}{d\omega} < 0$$

Low frequency travels faster

Negative GVD or anomalous dispersion

$$k^{(2)} < 0$$

$$\frac{dv_g}{d\omega} > 0$$

High frequency travels faster

Instantaneous frequency and chirp

$$\underline{A}(z,t') = A_0 \underbrace{\left(\frac{\tau^2}{(\tau^2 + \mathrm{j}k''z)}\right)^{1/2}}_{\text{z-dependent phase shift, independent on time}} \exp\left[\underbrace{-\frac{1}{2}\frac{\tau^2 t'^2}{\left(\tau^4 + (k''z)^2\right)}}_{\text{determines pulse width}} + \underbrace{\mathrm{j}\frac{1}{2}k''z\frac{t'^2}{\left(\tau^4 + (k''z)^2\right)}}_{\text{temporal quadratic phase}}\right]$$

After propagation of L distance: $\omega(z=L,t') = \frac{\partial}{\partial t'}\phi(L,t') = \frac{k''L}{\left(\tau^4 + (k''L)^2\right)}t'$

$$E(L,t') = \underline{A}(L,t')\exp(j\omega_0 t') \propto \exp[j\omega_0 t' + j\phi(L,t')]$$

$$\phi(z=L,t') = -\frac{1}{2}\arctan\left[\frac{k''L}{\tau^2}\right] + \frac{1}{2}k''L\frac{t'^2}{\left(\tau^4 + (k''L)^2\right)}$$

Instantaneous Frequency:

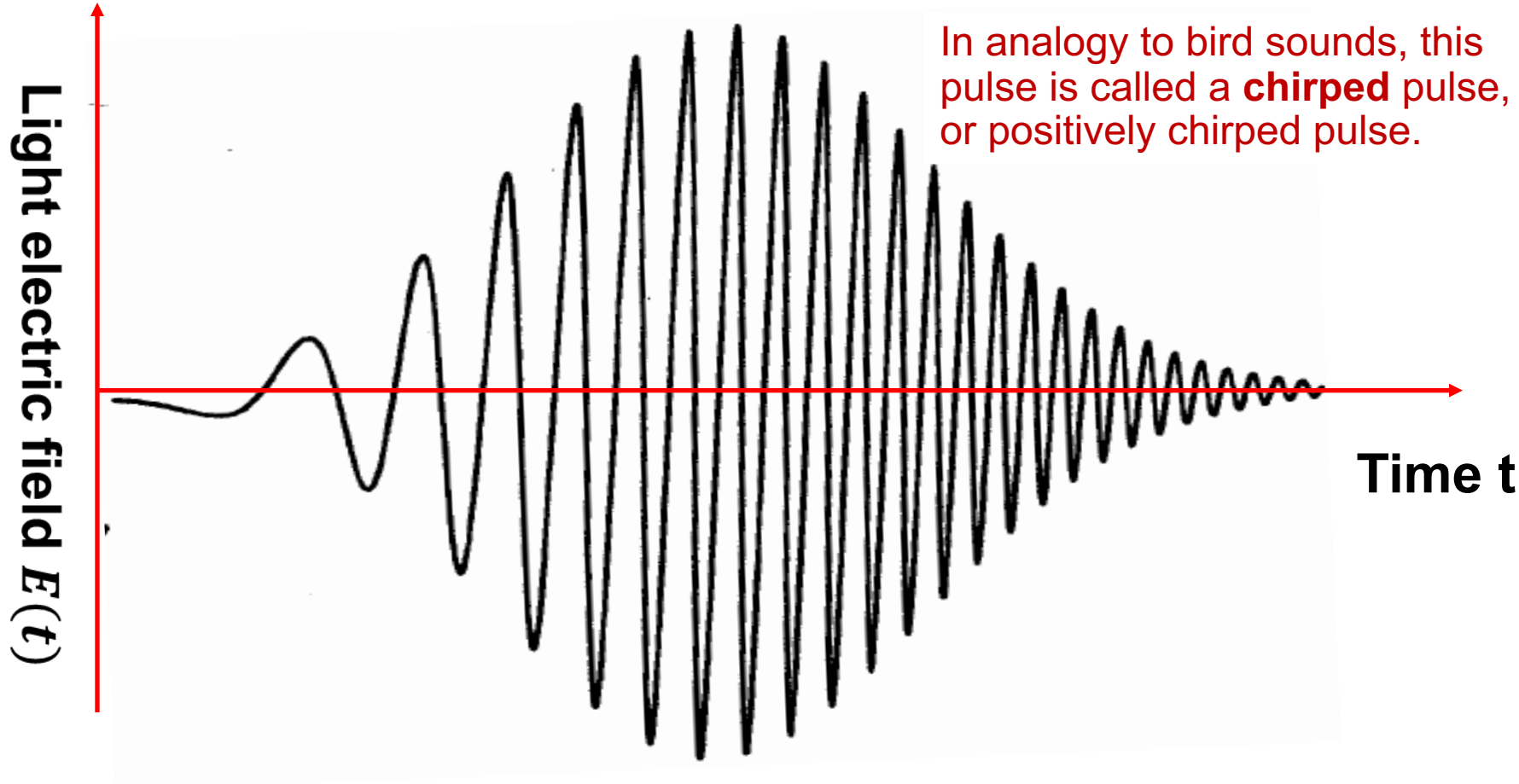
$$\omega_{inst}(t) \equiv \frac{\partial[\omega_0 t' + \phi(L,t')]}{\partial t'} = \omega_0 + \frac{\partial\phi(L,t')}{\partial t'}$$

$$= \omega_0 + \frac{k''L}{\left(\tau^4 + (k''L)^2\right)}t'$$

Linearly chirped Gaussian pulse: positive chirp

$$\omega_{ins}(t') = \omega_0 + \frac{k''L}{\left(\tau^4 + (k''L)^2\right)} t'$$

For positive GVD, i.e., k”>0, lower frequency travels faster, and the instantaneous frequency linearly **INCREASES** with time.

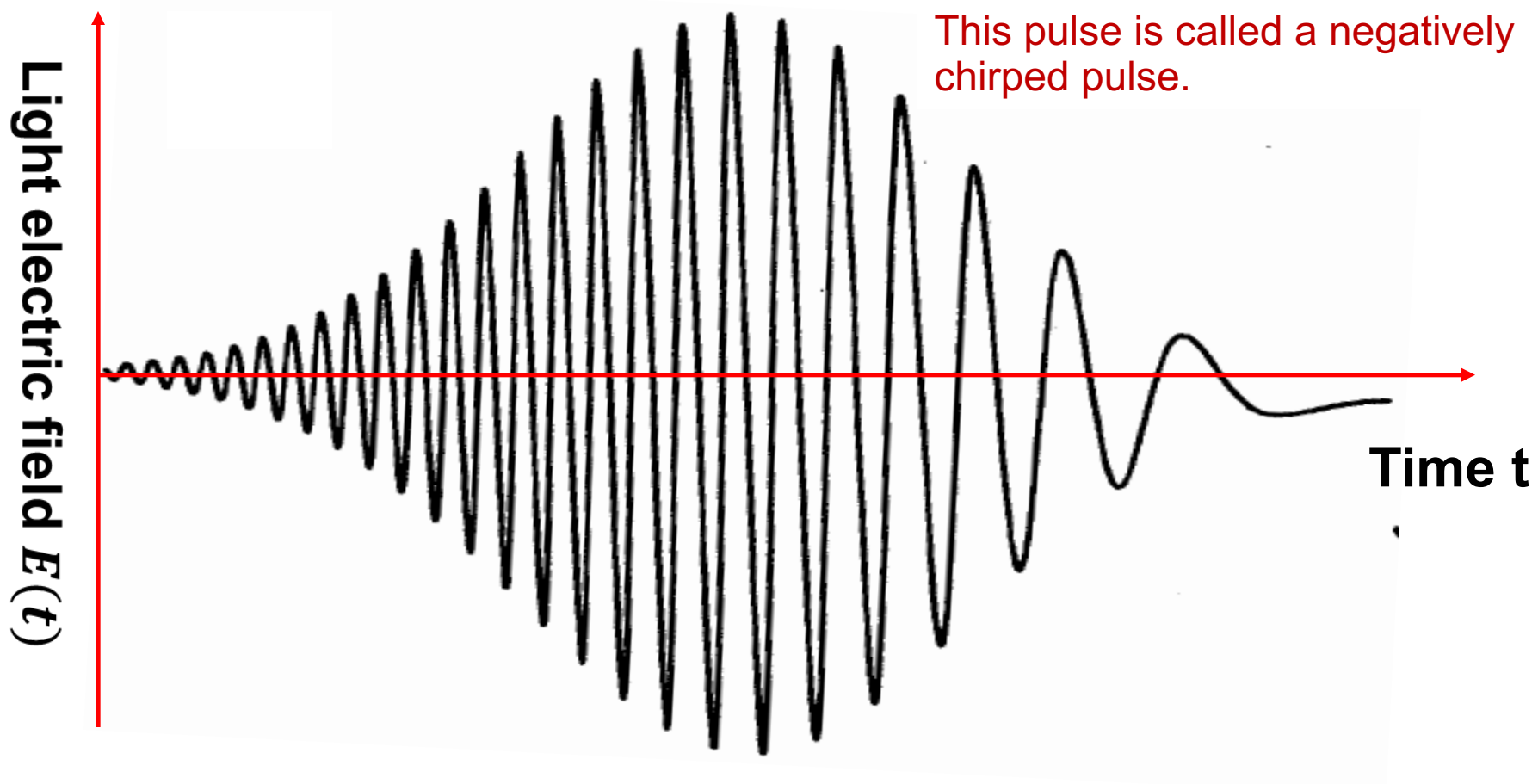


In analogy to bird sounds, this pulse is called a **chirped** pulse, or positively chirped pulse.

Linearly chirped Gaussian pulse: negative chirp

$$\omega_{ins}(t') = \omega_0 + \frac{k''L}{\left(\tau^4 + (k''L)^2\right)} t'$$

For negative GVD, i.e., k”<0, higher frequency travels faster.
The instantaneous frequency linearly **DECREASES** with time.



This pulse is called a negatively chirped pulse.

Transform-limited pulse

$$\tilde{E}(\omega) = \int_{-\infty}^{\infty} E(t)\exp(-j\omega t)dt \qquad E(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{E}(\omega)\exp(j\omega t)d\omega$$

$\left|\tilde{E}(\omega)\right|^2$ has a spectrum bandwidth of $\Delta\nu$

$\left|E(t)\right|^2$ has a pulse duration of Δt

Both are measured at full-width at half-maximum (FWHM).

Uncertainty principle:

$$\Delta\nu\Delta t \geq K$$

Time Bandwidth Product (TBP)

A number depending only on pulse shape

For a given optical spectrum, there exist a lower limit for the pulse duration. If the equality is reached, we say the pulse is a transform-limited pulse.

To get a shorter transform-limited pulse, one needs a broader optical spectrum.

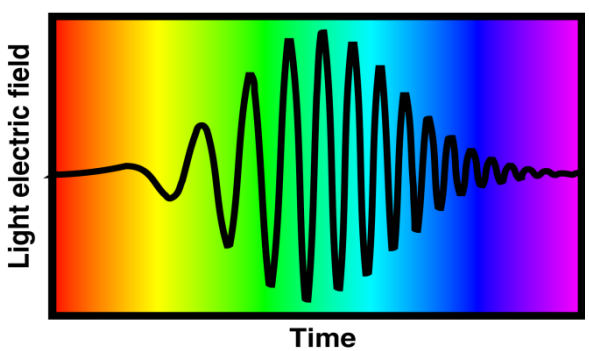
GVD changes the pulse duration and introduces chirp

$$k_2 = \frac{d}{d\omega}\left(\frac{1}{v_g}\right) = -\frac{1}{v_g^{\,2}(\omega_0)}\frac{dv_g}{d\omega} = \left(\frac{\lambda}{2\pi c}\right)\frac{\lambda^2}{c}\frac{d^2n}{d\lambda^2}$$

Positive GVD or normal dispersion

$$k_2 > 0 \qquad \frac{dv_g}{d\omega} < 0$$

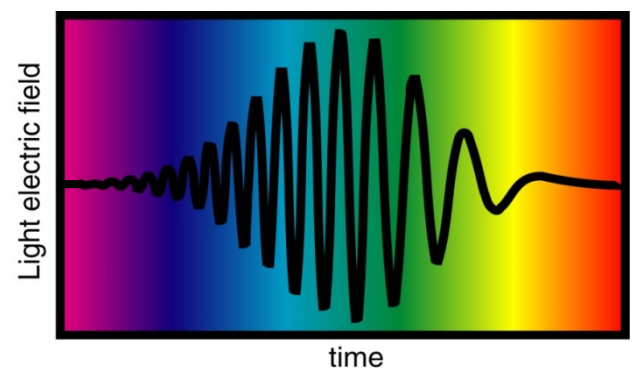
Red faster, positive chirp



Negative GVD or anomalous dispersion

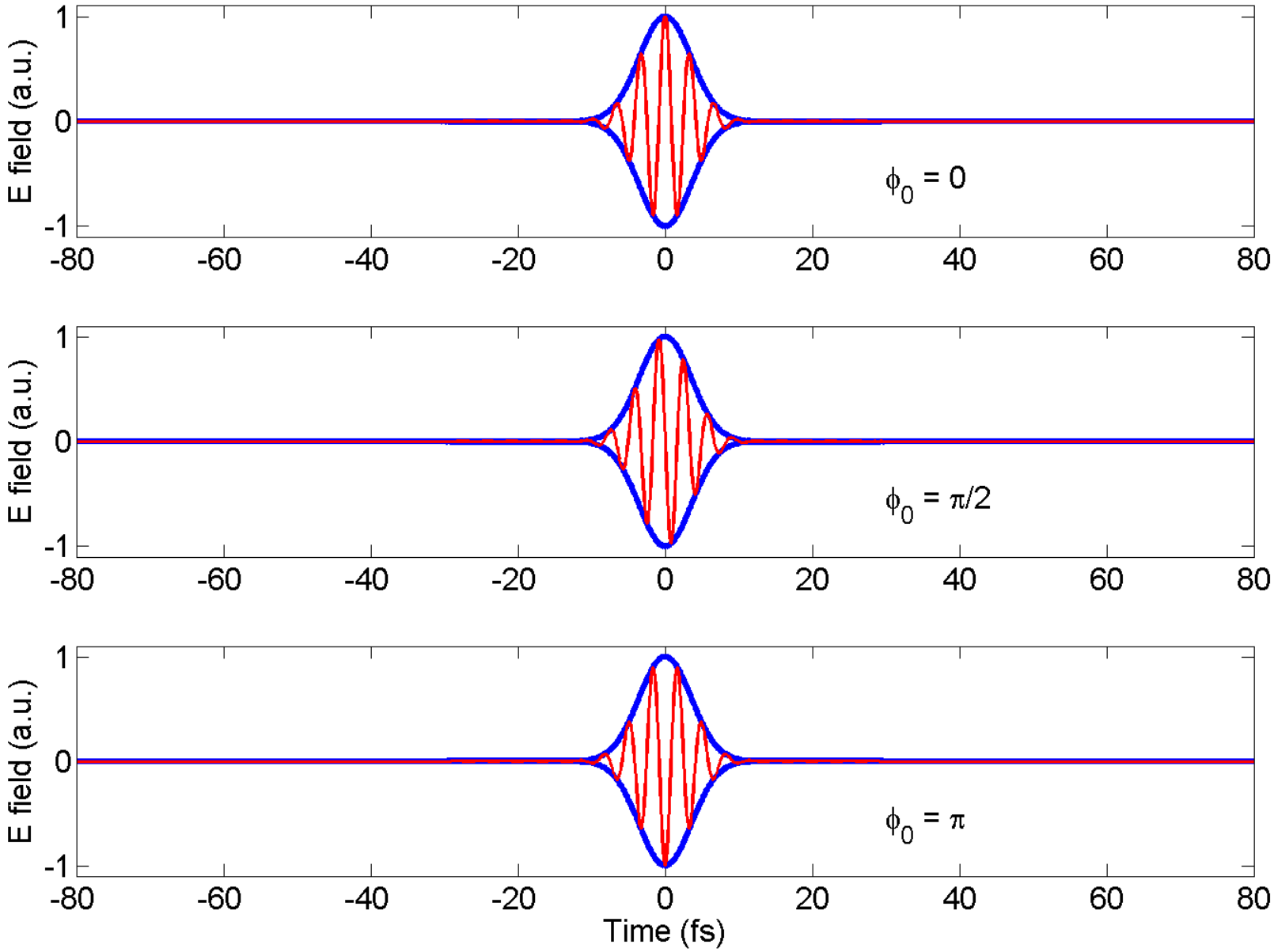
$$k_2 < 0 \qquad \frac{dv_g}{d\omega} > 0$$

Blue faster, negative chirp



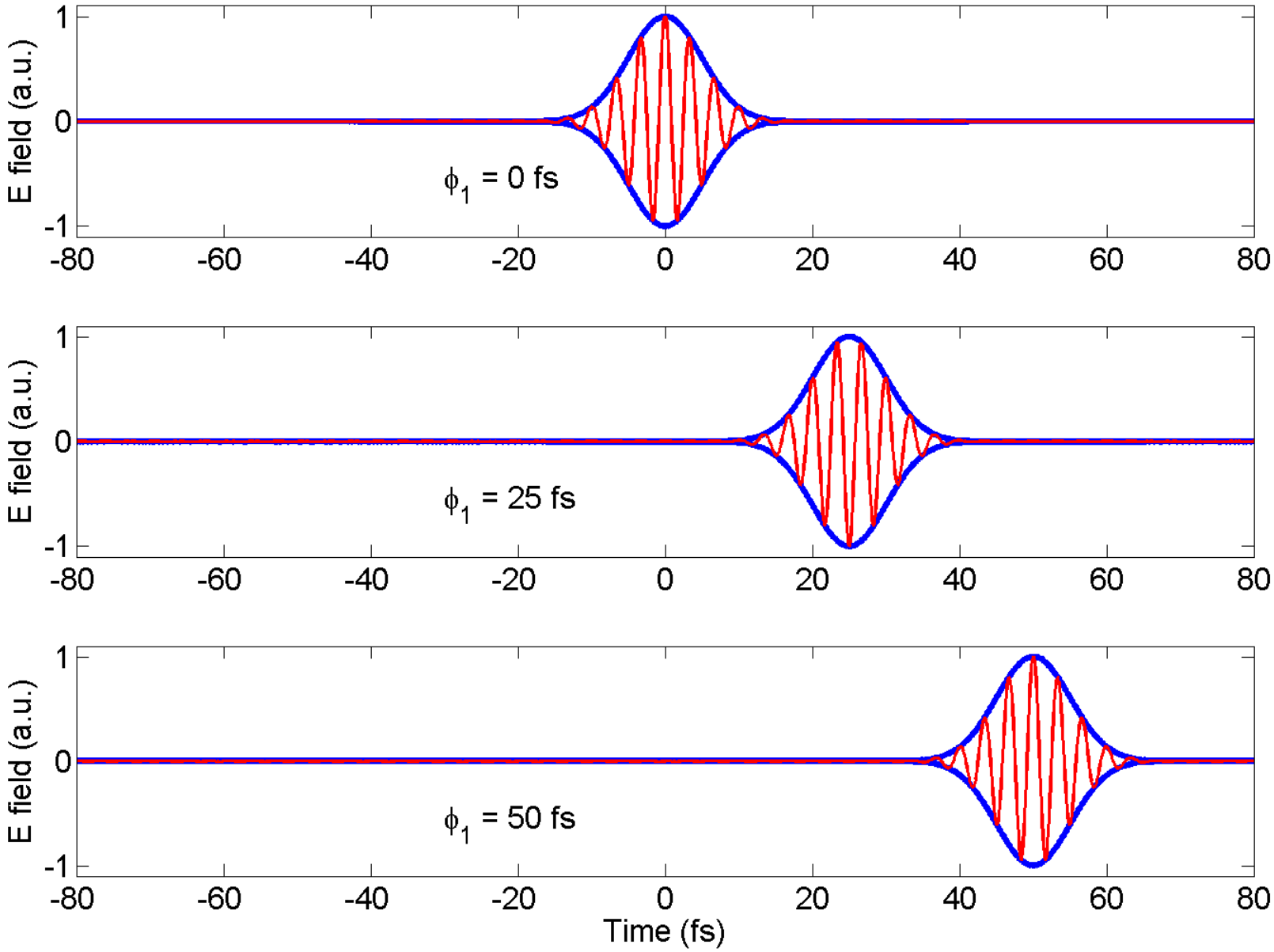
Effect of absolute phase

λ = 1 μm, t_0 = 5 fs.



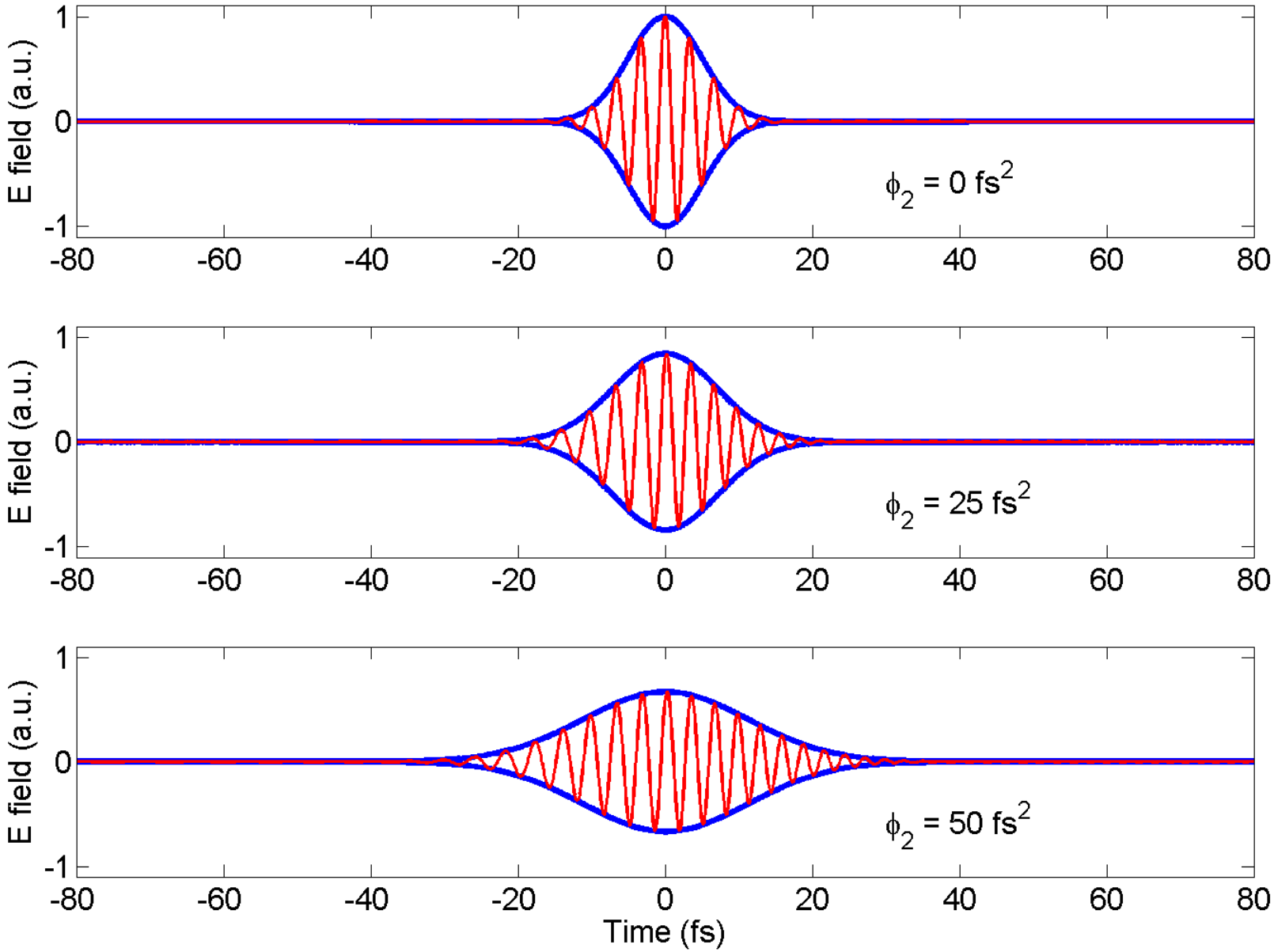
Effect of group delay

λ = 1 μm, t_0 = 5 fs.



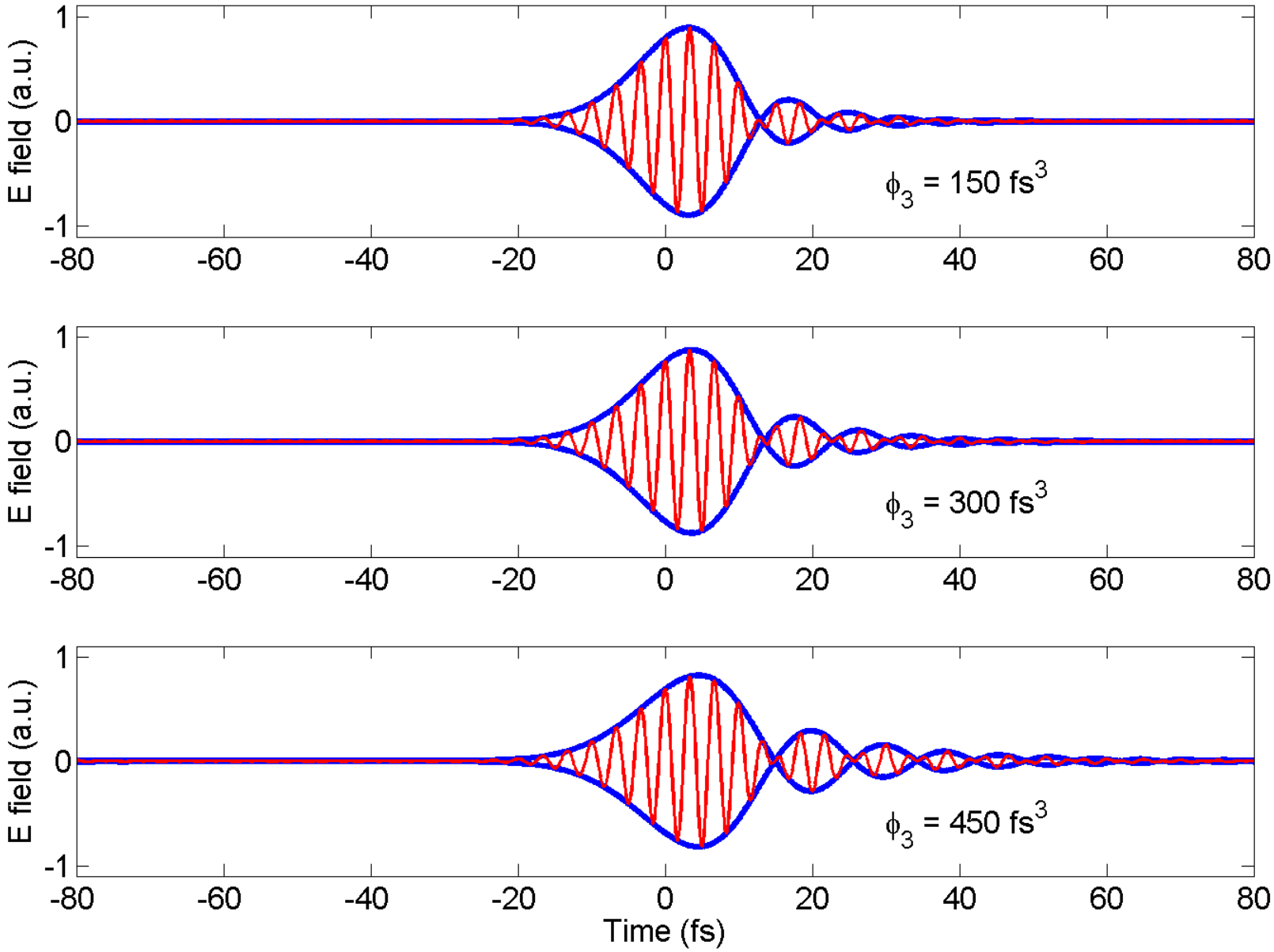
Effect of positive 2nd order dispersion

λ = 1 μm, t_0 = 5 fs.



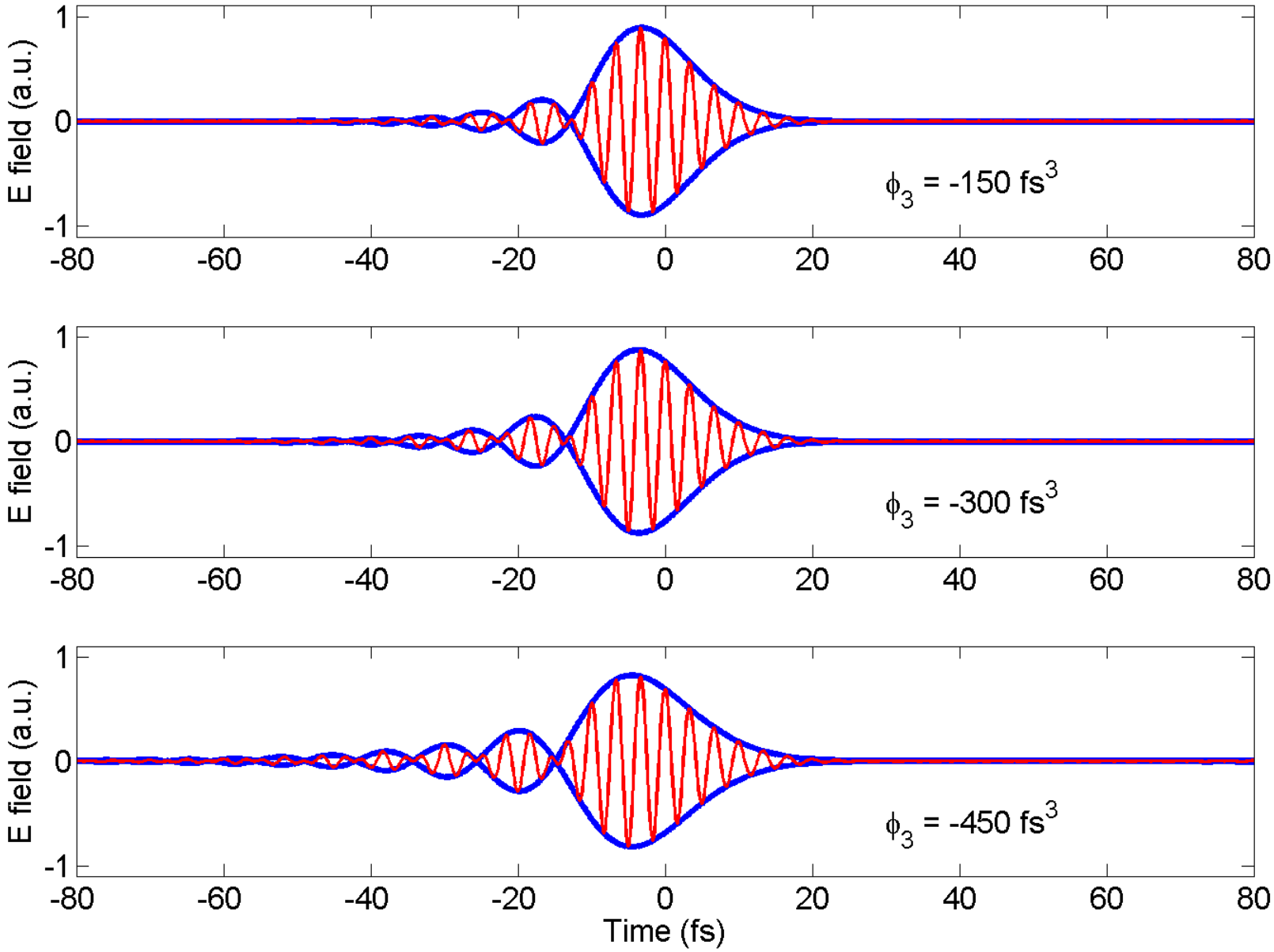
Effect of positive 3rd order dispersion

λ = 1 μm, t_0 = 5 fs.



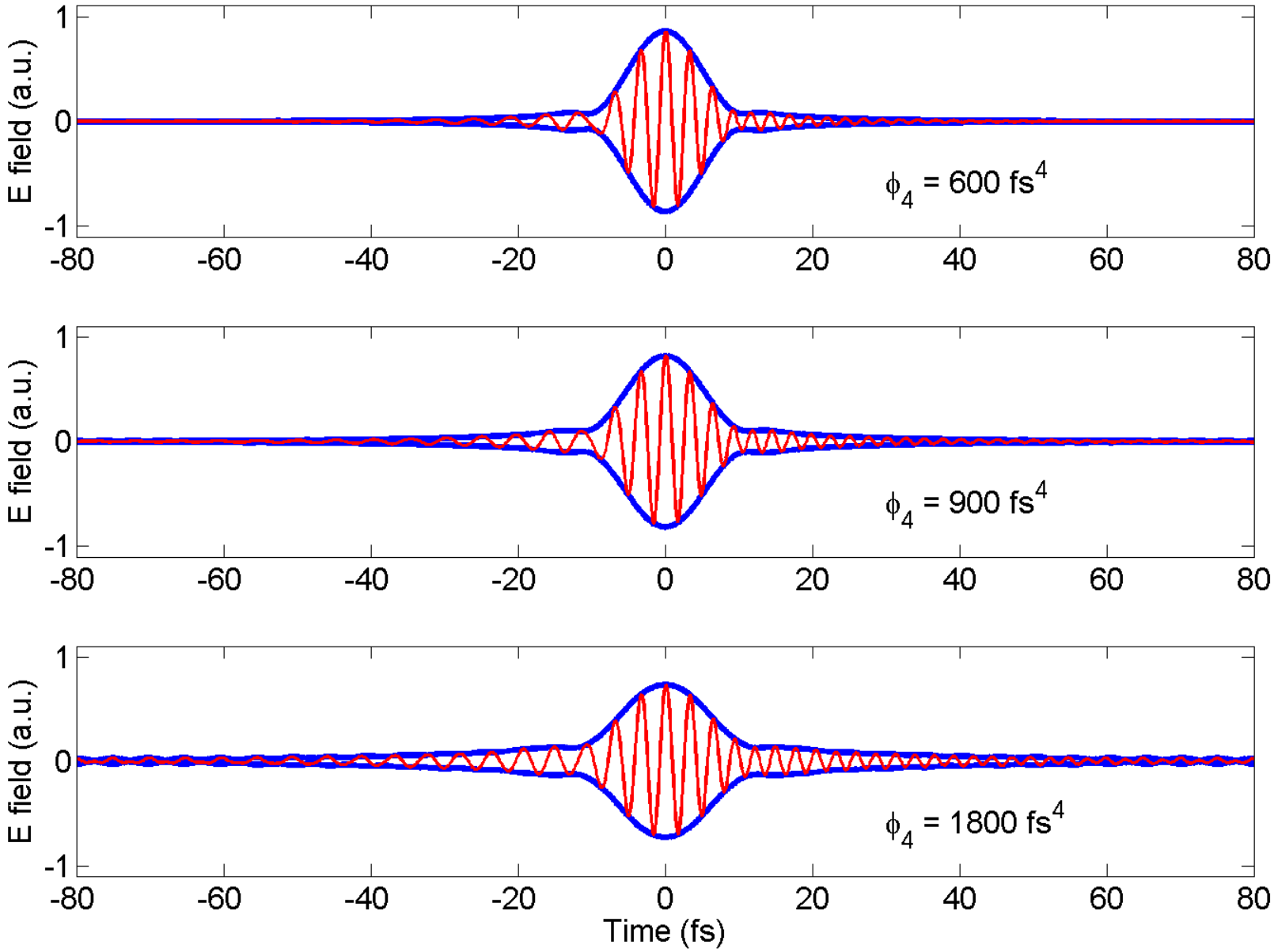
Effect of negative 3rd order dispersion

λ = 1 μm, t_0 = 5 fs.



Effect of positive 4th order dispersion

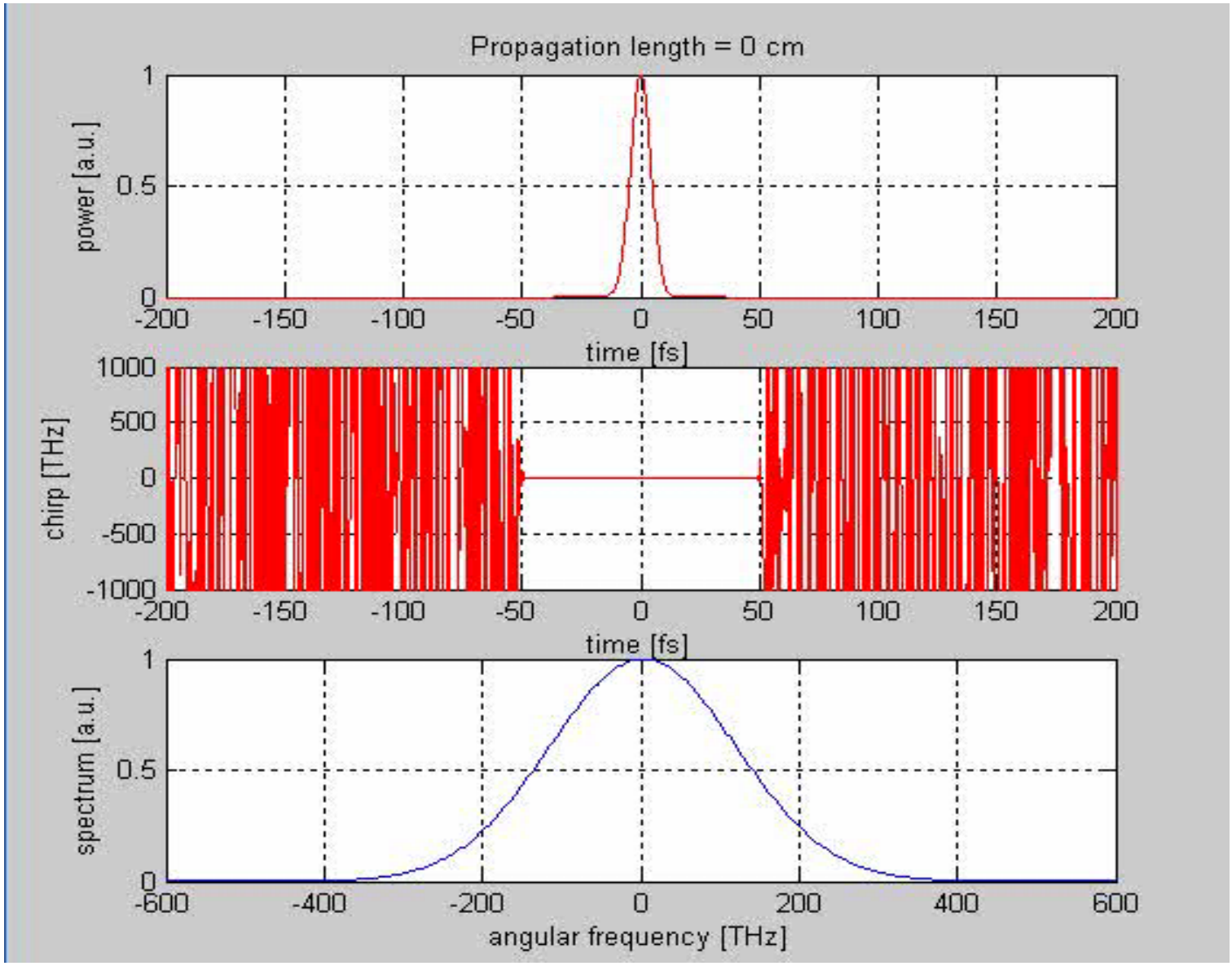
λ = 1 μm, t_0 = 5 fs.



Dispersion parameters for various materials

material	λ [nm]	$n(\lambda)$	$\frac{dn}{d\lambda}\cdot 10^{-2}\left[\frac{1}{\mu m}\right]$	$\frac{d^2n}{d\lambda^2}\cdot 10^{-1}\left[\frac{1}{\mu m^2}\right]$	$\frac{dn^3}{d\lambda^3}\left[\frac{1}{\mu m^3}\right]$	$T_g\left[\frac{fs}{mm}\right]$	$GDD\left[\frac{fs^2}{mm}\right]$	$TOD\left[\frac{fs^3}{mm}\right]$
BK7	400	1,5308	-13,17	10,66	-12,21	5282	120,79	40,57
	500	1,5214	-6,58	3,92	-3,46	5185	86,87	32,34
	600	1,5163	-3,91	1,77	-1,29	5136	67,52	29,70
	800	1,5108	-1,97	0,48	-0,29	5092	43,96	31,90
	1000	1,5075	-1,40	0,15	-0,09	5075	26,93	42,88
	1200	1,5049	-1,23	0,03	-0,04	5069	10,43	66,12
SF10	400	1,7783	-52,02	59,44	-101,56	6626	673,68	548,50
	500	1,7432	-20,89	15,55	-16,81	6163	344,19	219,81
	600	1,7267	-11,00	6,12	-4,98	5980	233,91	140,82
	800	1,7112	-4,55	1,58	-0,91	5830	143,38	97,26
	1000	1,7038	-2,62	0,56	-0,27	5771	99,42	92,79
	1200	1,6992	-1,88	0,22	-0,10	5743	68,59	107,51
Sapphire	400	1,7866	-17,20	13,55	-15,05	6189	153,62	47,03
	500	1,7743	-8,72	5,10	-4,42	6064	112,98	39,98
	600	1,7676	-5,23	2,32	-1,68	6001	88,65	37,97
	800	1,7602	-2,68	0,64	-0,38	5943	58,00	42,19
	1000	1,7557	-1,92	0,20	-0,12	5921	35,33	57,22
	1200	1,7522	-1,70	0,04	-0,05	5913	13,40	87,30
Quartz	300	1,4878	-30,04	34,31	-54,66	5263	164,06	46,49
	400	1,4701	-11,70	9,20	-10,17	5060	104,31	31,49
	500	1,4623	-5,93	3,48	-3,00	4977	77,04	26,88
	600	1,4580	-3,55	1,59	-1,14	4934	60,66	25,59
	800	1,4533	-1,80	0,44	-0,26	4896	40,00	28,43
	1000	1,4504	-1,27	0,14	-0,08	4880	24,71	38,73
	1200	1,4481	-1,12	0,03	-0,03	4875	9,76	60,05

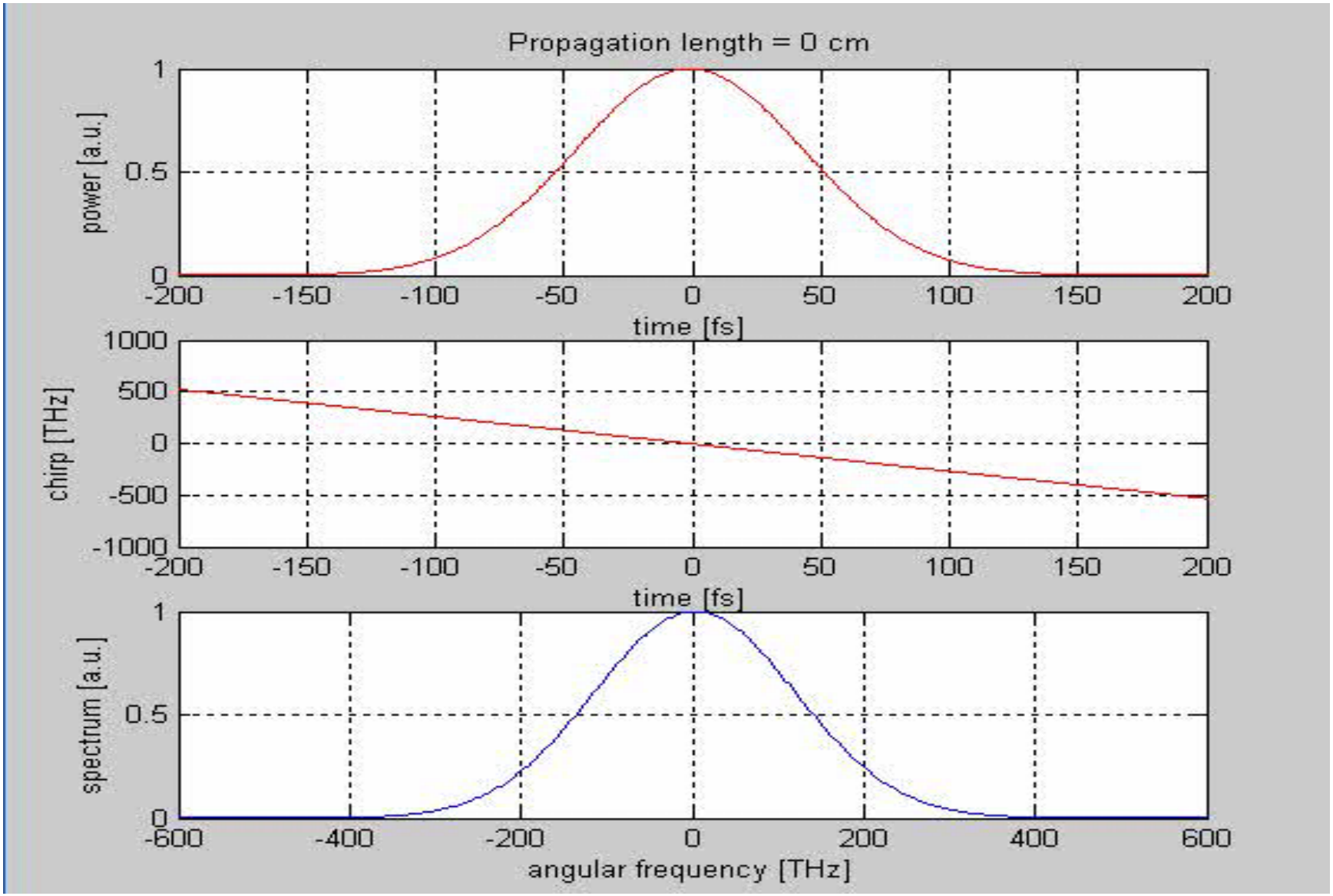
Effect of negative GVD



Courtesy
Noah Chang

GVD $\beta_2 = -25\, ps^2 / km$ **Input pulse duration:10fs**

Effect of positive GVD



Courtesy
Noah Chang

GVD $\beta_2 = 25\,ps^2 / km$ The output of last slide is taken as the input here.