UFS Lecture 2: Linear Pulse Propagation

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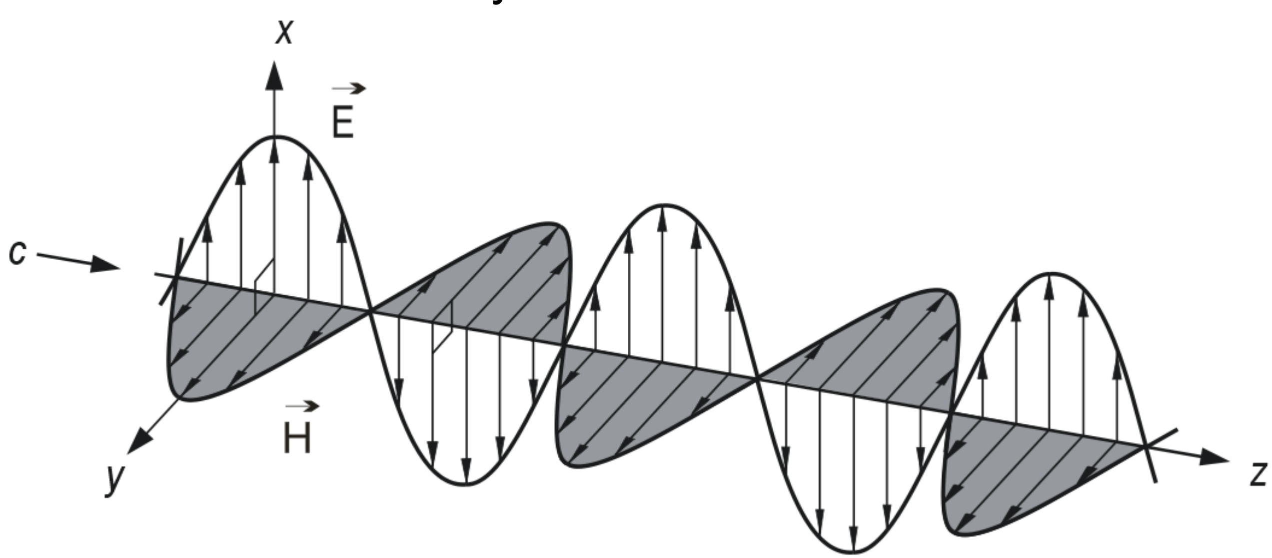


Figure 2.1: Transverse electromagnetic wave (TEM)

2.1 Maxwell's Equations of Isotropic Media

Maxwell's Equations: Differential Form

Current due to free charges

Ampere's Law
$$\vec{\nabla}\times\vec{H} = \frac{\partial\vec{D}}{\partial t}+\vec{J}, \quad (2.1a)$$

Faraday's Law
$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \quad (2.1b)$$

Gauss's Law
$$\vec{\nabla}\cdot\vec{D} = \rho, \quad (2.1c)$$

Free charge density

No magnetic charge
$$\vec{\nabla}\cdot\vec{B} = 0. \quad (2.1d)$$

Material Equations: Bring Life into Maxwell's Equations

$$\vec{D} = \epsilon_0\vec{E}+\vec{P}, \quad (2.2a)$$

Polarization

$$\vec{B} = \mu_0\vec{H}+\vec{M}. \quad (2.2b)$$

Magnetization

Vector Identity: $$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right)=\vec{\nabla}\left(\vec{\nabla}\cdot\vec{E}\right)-\Delta\vec{E},$$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right)=-\vec{\nabla}\times\frac{\partial\vec{B}}{\partial t}=-\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{B}\right)$$

$$=-\frac{\partial}{\partial t}\left(\vec{\nabla}\times\left(\mu_0\vec{H}+\vec{M}\right)\right)=-\frac{\partial}{\partial t}\left(\mu_0\vec{\nabla}\times\vec{H}+\vec{\nabla}\times\vec{M}\right)$$

$$=-\frac{\partial}{\partial t}\left(\mu_0\left(\varepsilon_0\frac{\partial\vec{E}}{\partial t}+\frac{\partial\vec{P}}{\partial t}+\vec{J}\right)+\vec{\nabla}\times\vec{M}\right)$$

$$\Delta\vec{E}-\mu_0\frac{\partial}{\partial t}\left(\vec{j}+\epsilon_0\frac{\partial\vec{E}}{\partial t}+\frac{\partial\vec{P}}{\partial t}\right)=\frac{\partial}{\partial t}\nabla\times\vec{M}+\nabla\left(\nabla\cdot\vec{E}\right) \qquad (2.3)$$

Vacuum speed of light: $$c_0=\sqrt{\frac{1}{\mu_0\epsilon_0}}.$$

$$\left(\Delta-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}=\mu_0\left(\frac{\partial\vec{j}}{\partial t}+\frac{\partial^2}{\partial t^2}\vec{P}\right)+\frac{\partial}{\partial t}\vec{\nabla}\times\vec{M}+\vec{\nabla}\left(\vec{\nabla}\cdot\vec{E}\right). \qquad (2.4)$$

No free charges, No currents from free charges, Non magnetization

$$\left(\Delta - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = \mu_0\left(\cancel{\frac{\partial \vec{j}}{\partial t}} + \frac{\partial^2}{\partial t^2}\vec{P}\right) + \cancel{\frac{\partial}{\partial t}\nabla\times\vec{M}} + \nabla\left(\cancel{\nabla\cdot\vec{E}}\right). \qquad (2.4)$$

Every field can be written as the sum of tansverse and longitudinal fields:

$$\vec{\nabla}\times\vec{E}_L = 0 \text{ and } \vec{\nabla}\cdot\vec{E}_T = 0$$

Only free charges create a longitudinal electric field:

$$\vec{E} = \vec{E}_T$$ **Pure radiation field**

Simplified wave equation:

$$\left(\Delta - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = \mu_0\frac{\partial^2}{\partial t^2}\vec{P}. \qquad (2.7)$$

Wave in vacuum **Source term**

2.1.1 Helmholtz Equation

Linear medium:

$$\vec{P}(\vec{r},t) = \epsilon_0 \int dt' \, \chi\,(t-t')\, \vec{E}\,(\vec{r},t')\,. \tag{2.8}$$

dielectric susceptibility

$$\tilde{\vec{E}}(\vec{r},\omega) = \int_{-\infty}^{+\infty} \vec{E}(\vec{r},t)e^{-j\omega t}dt, \tag{2.11}$$

$$\tilde{\vec{P}}(\vec{r},\omega) = \epsilon_0 \tilde{\chi}(\omega)\tilde{\vec{E}}(\vec{r},\omega), \tag{2.12}$$

$$\left(\Delta + \frac{\omega^2}{c_0^2}\right)\tilde{\vec{E}}(\omega) = -\omega^2 \mu_0 \epsilon_0 \tilde{\chi}(\omega)\tilde{\vec{E}}(\omega), \tag{2.13}$$

$$\left(\Delta + \frac{\omega^2}{c_0^2}(1+\tilde{\chi}(\omega)\right)\tilde{\vec{E}}(\omega) = 0, \tag{2.14}$$

Medium speed of light: $c(\omega) = c_0/\tilde{n}(\omega)$ with $1+\tilde{\chi}(\omega) = \tilde{n}(\omega)^2$;

Can be complex

Refractive Index

2.1.2 Plane-Wave Solutions (TEM-Waves) and Complex Notation

Real field:

$$\vec{E}_{\vec{k}}(\vec{r},t) = \frac{1}{2}\left[\underline{\vec{E}}_{\vec{k}}(\vec{r},t) + \underline{\vec{E}}_{\vec{k}}(\vec{r},t)^*\right] = \Re e\left\{\underline{\vec{E}}_{\vec{k}}(\vec{r},t)\right\},$$

Artificial, complex

$$\underline{\vec{E}}_{\vec{k}}(\vec{r},t) = \underline{E}_{\vec{k}}\; e^{\mathrm{j}(\omega t - \vec{k}\cdot\vec{r})}\; \vec{e}(\vec{k}).$$

Into wave equation (2.14):

Dispersion relation:

$$|\vec{k}|^2 = \frac{\omega^2}{c(\omega)^2} = k(\omega)^2.$$

$$k = \left|\vec{k}\right| \qquad k(\omega) = \pm\frac{\omega}{c_0} n(\omega). \tag{2.21}$$

$$k = 2\pi/\lambda, \quad \text{Wavelength} \tag{2.22}$$

$$\nabla \cdot \vec{E} = 0 \longrightarrow \vec{k} \perp \vec{e}.$$

What about the magnetic field?

$$\vec{H}_{\vec{k}}(\vec{r},t) = \frac{1}{2}\left[\underline{\vec{H}}_{\vec{k}}(\vec{r},t) + \underline{\vec{H}}_{\vec{k}}(\vec{r},t)^*\right] \tag{2.21}$$

$$\underline{\vec{H}}_{\vec{k}}(\vec{r},t) = \underline{H}_{\vec{k}}\ e^{\mathrm{j}(\omega t - \vec{k}\cdot\vec{r})}\ \vec{h}(\vec{k}). \tag{2.22}$$

Faraday's Law:

$$-\mathrm{j}\vec{k} \times \left(\underline{E}_{\vec{k}}\ e^{\mathrm{j}(\omega t - \vec{k}\cdot\vec{r})}\ \vec{e}(\vec{k})\right) = -\mathrm{j}\mu_0\omega\underline{\vec{H}}_{\vec{k}}(\vec{r},t), \tag{2.23}$$

$$\underline{\vec{H}}_{\vec{k}}(\vec{r},t) = \frac{\underline{E}_{\vec{k}}}{\mu_0\omega} e^{\mathrm{j}(\omega t - \vec{k}\cdot\vec{r})}\vec{k} \times \vec{e} = \underline{H}_{\vec{k}} e^{\mathrm{j}(\omega t - \vec{k}\cdot\vec{r})}\vec{h} \tag{2.24}$$

$$\longrightarrow \qquad \vec{h}(\vec{k}) = \frac{\vec{k}}{|k|} \times \vec{e}(\vec{k})$$

$$\longrightarrow \qquad \underline{H}_{\vec{k}} = \frac{|k|}{\mu_0\omega}\underline{E}_{\vec{k}} = \frac{1}{Z_F}\underline{E}_{\vec{k}}.$$

Characteristic Impedance

$$Z_F = \mu_0 c = \sqrt{\frac{\mu_0}{\epsilon_0 \epsilon_r}} = \frac{1}{n} Z_{F_0}$$

Vacuum Impedance:

$$Z_{F_0} = \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 377\,\Omega.$$

Stealth Airplane?

$\vec{e}$, $\vec{h}$ and $\vec{k}$ form an orthogonal trihedral

$$\vec{e} \perp \vec{h}, \quad \vec{k} \perp \vec{e}, \quad \vec{k} \perp \vec{h}.$$

Example: EM-Wave polarized along x-axis and propagation along z-direction:

$$\vec{e}(\vec{k}) = \vec{e}_x. \qquad \vec{E}(\vec{r}, t) = E_0 \cos(\omega t - kz)\ \vec{e}_x$$

$$\frac{\vec{k}}{|k|} = \vec{e}_z, \quad \longrightarrow \quad \vec{H}(\vec{r}, t) = \frac{E_0}{Z_{F_0}} \cos(\omega t - kz)\ \vec{e}_y$$

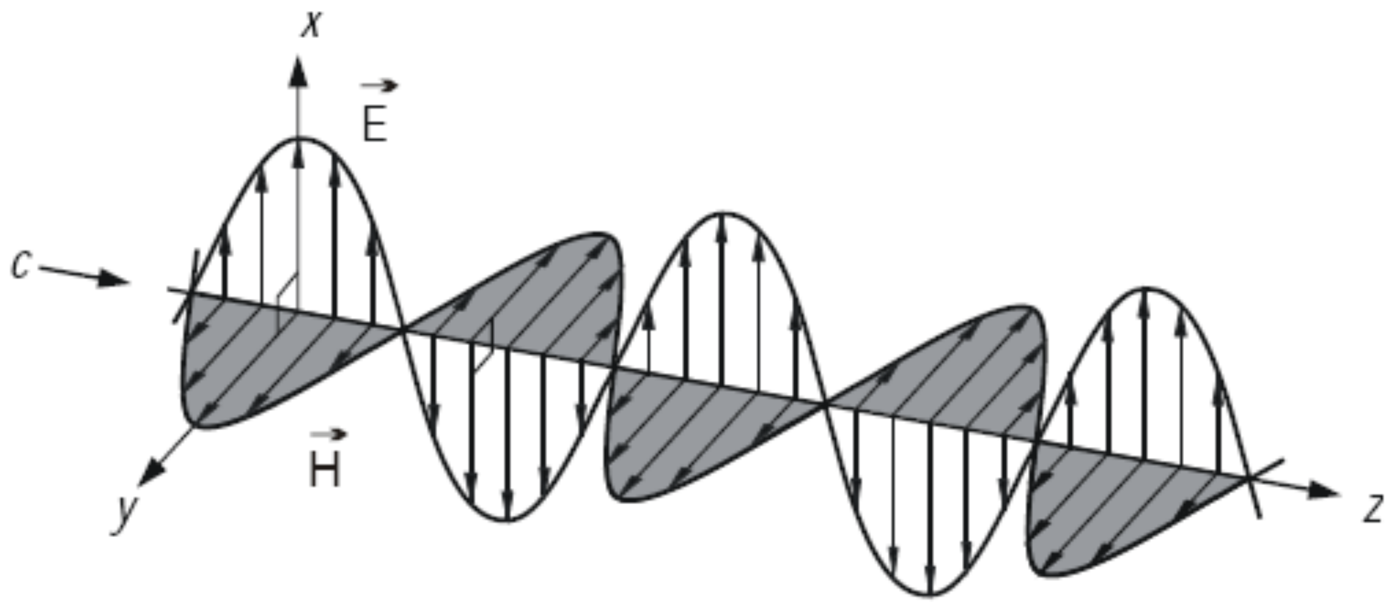


Figure 2.1: Transverse electromagnetic wave (TEM) [6]

Backwards Traveling Wave

Backwards

$$\underline{\vec{E}}(\vec{r},t) = \underline{E}\; e^{\mathrm{j}\omega t+\mathrm{j}\vec{k}\cdot\vec{r}}\; \vec{e}_x,$$

$$\underline{\vec{H}}(\vec{r},t) = \underline{H}\; e^{\mathrm{j}(\omega t+\vec{k}\vec{r})}\; \vec{e}_y,$$

$$\underline{H} = -\frac{|k|}{\mu_0\omega}\underline{E},$$

2.1.3 Poynting Vector, Energy Density and Intensity

relative permittivity: $\varepsilon_r = 1 + \chi$

Quantity	Real fields	Complex fields
Electric and magnetic energy density	$w_e = \frac{1}{2}\vec{E}\cdot\vec{D} = \frac{1}{2}\epsilon_0\epsilon_r\vec{E}^2$ $w_m = \frac{1}{2}\vec{H}\cdot\vec{B} = \frac{1}{2}\mu_0\mu_r\vec{H}^2$ $w = w_e + w_m$	$\langle w_e\rangle = \frac{1}{4}\epsilon_0\epsilon_r\left\|\underline{\vec{E}}\right\|^2$ $\langle w_m\rangle = \frac{1}{4}\mu_0\mu_r\left\|\underline{\vec{H}}\right\|^2$ $\langle w\rangle = \langle w_e\rangle + \langle w_m\rangle$
Poynting vector	$\vec{S} = \vec{E}\times\vec{H}$	$\underline{\vec{T}} = \frac{1}{2}\underline{\vec{E}}\times\underline{\vec{H}}^*$
Poynting theorem	$\mathrm{div}\vec{S} + \vec{E}\cdot\vec{j} + \frac{\partial w}{\partial t} = 0$	$\mathrm{div}\underline{\vec{T}} + \frac{1}{2}\underline{\vec{E}}\cdot\underline{\vec{j}}^* +$ $+2j\omega(\langle w_m\rangle - \langle w_e\rangle) = 0$
Intensity	$I = \left\|\vec{S}\right\| = cw$	$I = \mathrm{Re}\{\underline{\vec{T}}\} = c\langle w\rangle$

Table 2.1: Poynting vector and energy density in EM-fields

Example: Plane Wave:

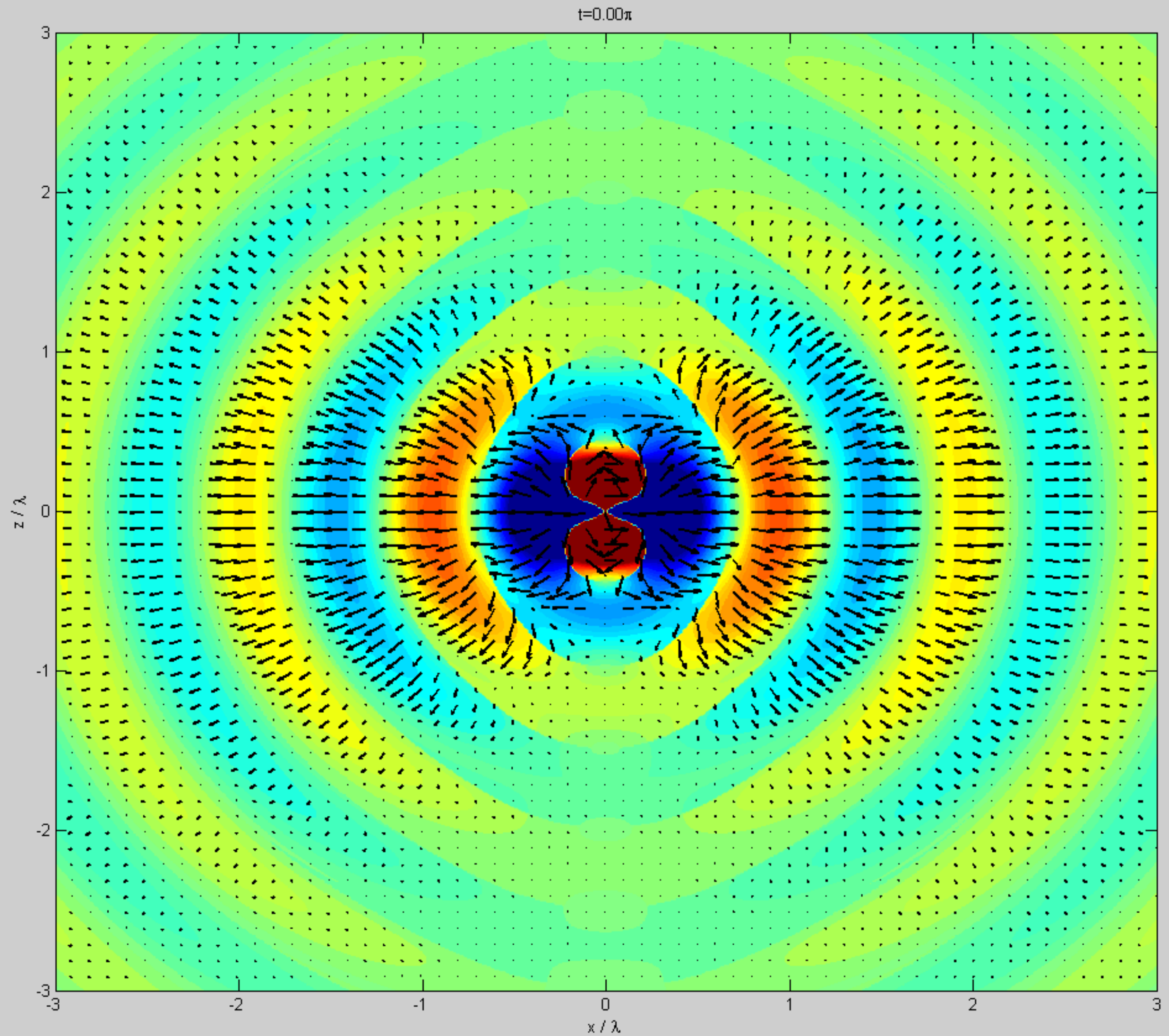
$$\langle w\rangle = \frac{1}{2}\epsilon_r\epsilon_0|\underline{E}|^2,$$

$$\underline{\vec{E}}(\vec{r},t) = \underline{E}e^{\mathrm{j}(\omega t - kz)}\ \vec{e}_x$$

$$\boxed{I = \frac{1}{2Z_F}|\underline{E}|^2 = \frac{1}{2}Z_F|\underline{H}|^2.}$$

$$\underline{\vec{T}} = \frac{1}{2Z_F}|\underline{E}|^2\ \vec{e}_z,$$

Oscillating dipole moment emits new EM wave at the oscillating frequency

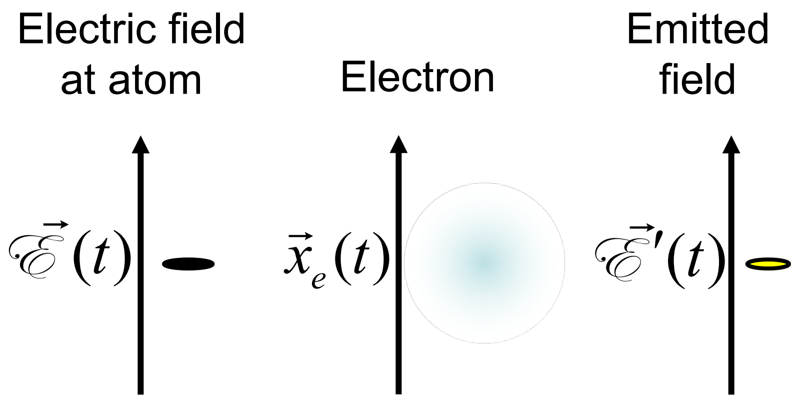


It does not emit in the oscillating direction!

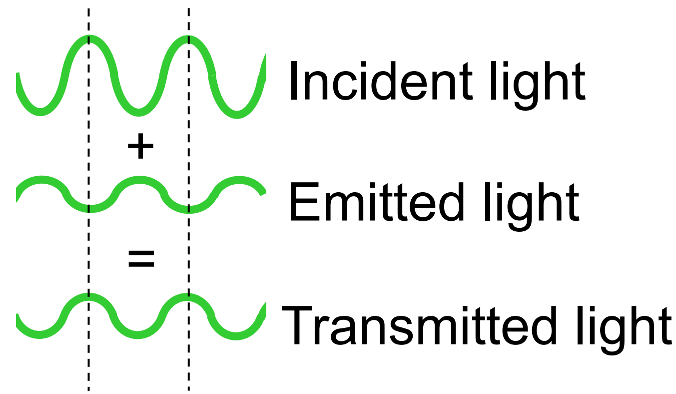
From wiki

Lorentz model of light-atom interaction

When light of frequency ω excites an atom with resonant frequency ω_0 :



On resonance ($\omega = \omega_0$)



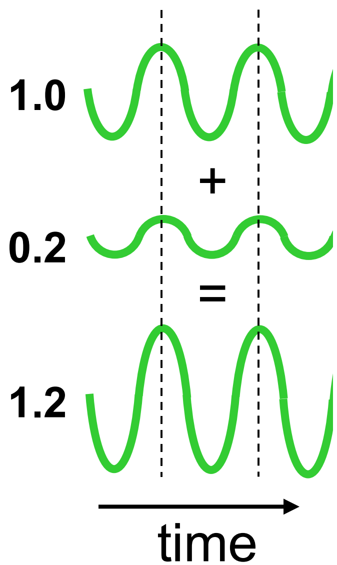
Incident Light excites electron oscillation → electron oscillation emits new light at the same frequency → incident light interferes with the new light leading to the transmitted light.

The crucial issue is the **relative phase** of the incident light and this emitted light. For example, if these two waves are ~180° out of phase, the beam will be attenuated. We call this absorption.

Interference depends on relative phase

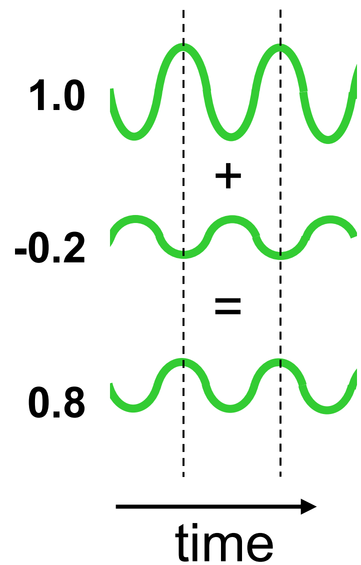
When two waves add together with the same complex exponentials, we add the complex amplitudes, $\underset{\sim}{E}_0 + \underset{\sim}{E}_0'$.

Constructive interference:



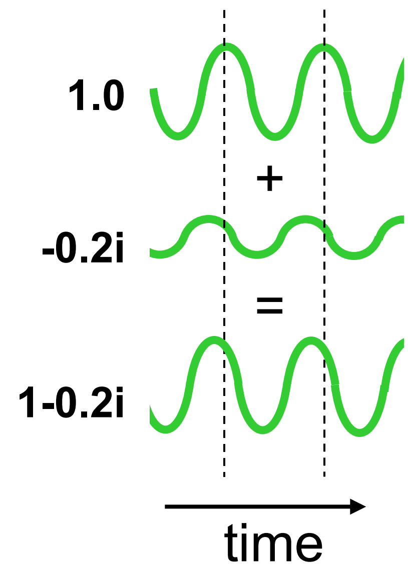
Laser

Destructive interference:



Absorption

Quadrature phase: $\pm 90^\circ$ interference:



Slower phase velocity (when accumulated over distance)

Adapted from Rick Trebino's course slides

2.2 Classical (dielectric) Permittivity

Polarization: $\vec{P}(t) = \dfrac{\text{dipole moment}}{\text{volume}} = N \cdot \vec{p}(t)$

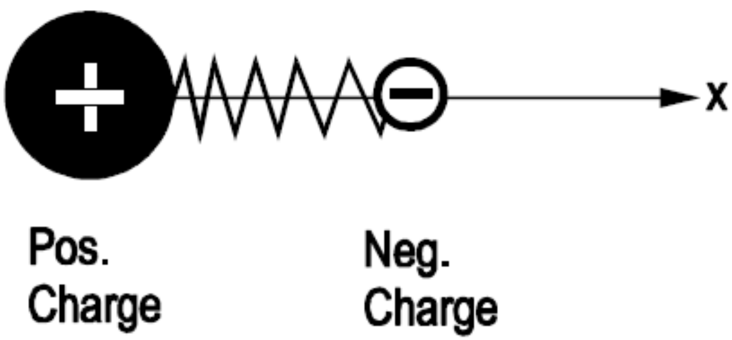
Density

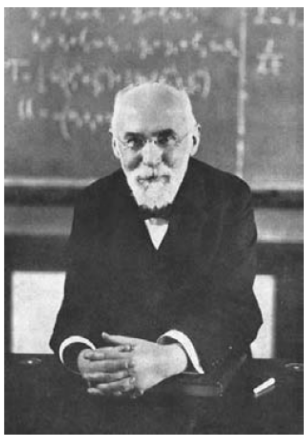
Elementary Dipole

$$\underline{\tilde{\vec{P}}}(\omega) = \frac{\text{dipole moment}}{\text{volume}} = N \cdot \underline{\tilde{\vec{p}}}(\omega) = \epsilon_0 \underline{\tilde{\chi}}(\omega) \underline{\tilde{\vec{E}}}(\omega)$$

Dielectric Suszeptibility: $\underline{\tilde{\chi}}(\omega) = \dfrac{N \cdot \underline{\tilde{\vec{p}}}(\omega)}{\epsilon_0 \underline{\tilde{\vec{E}}}(\omega)}$

Lorentz Model:





H. A. Lorentz (1853-1928)

Figure 2.2: Classical harmonic oscillator model for radiation matter interaction

Response to a Monochromatic Field

$$\vec{E}(\vec{r},t) = \vec{E}(\vec{r}_A,t) = E(t)\vec{e}_x$$

$$\underline{E}(t) = \underline{\tilde{E}}e^{\mathrm{j}\omega t} \rightarrow \underline{x}(t) = \underline{\tilde{x}}e^{\mathrm{j}\omega t} \rightarrow \underline{p}(t) = e_0\underline{x}(t) = \underline{\tilde{p}}e^{\mathrm{j}\omega t}$$

$$m\frac{d^2x}{dt^2} + 2\frac{\Omega_0}{Q}m\frac{dx}{dt} + m\Omega_0^2 x = e_0 E(t) \longleftarrow$$

force

mass

quality factor

frequency of undamped oscillator

$$\underline{\tilde{p}} = \frac{\frac{e_0^2}{m}}{(\Omega_0^2 - \omega^2) + 2\mathrm{j}\frac{\Omega_0}{Q}\omega}\underline{\tilde{E}}.$$

$$\boxed{\underline{\chi}(\omega) = \frac{N\frac{e_0^2}{m}\frac{1}{\epsilon_0}}{(\Omega_0^2 - \omega^2) + 2\mathrm{j}\omega\frac{\Omega_0}{Q}}}$$

Real and Imaginary Part of the Susceptibility

$$\underline{\tilde{\chi}}(\omega) = \tilde{\chi}_r(\omega) + j\tilde{\chi}_i(\omega)$$

$$\tilde{\chi}_r(\omega) = \omega_p^2 \cdot \frac{\left(\Omega_0^2 - \omega^2\right)}{\left(\Omega_0^2 - \omega^2\right)^2 + \left(2\omega\frac{\Omega_0}{Q}\right)^2}$$

$$\tilde{\chi}_i(\omega) = -\omega_p^2 \cdot \frac{2\omega\frac{\Omega_0}{Q}}{\left(\Omega_0^2 - \omega^2\right)^2 + \left(2\omega\frac{\Omega_0}{Q}\right)^2}$$

Real and Imaginary Part of the Susceptibility

$$\underset{\sim}{\widetilde{\chi}}(\omega) = \widetilde{\chi}_r(\omega) + j\widetilde{\chi}_i(\omega)$$

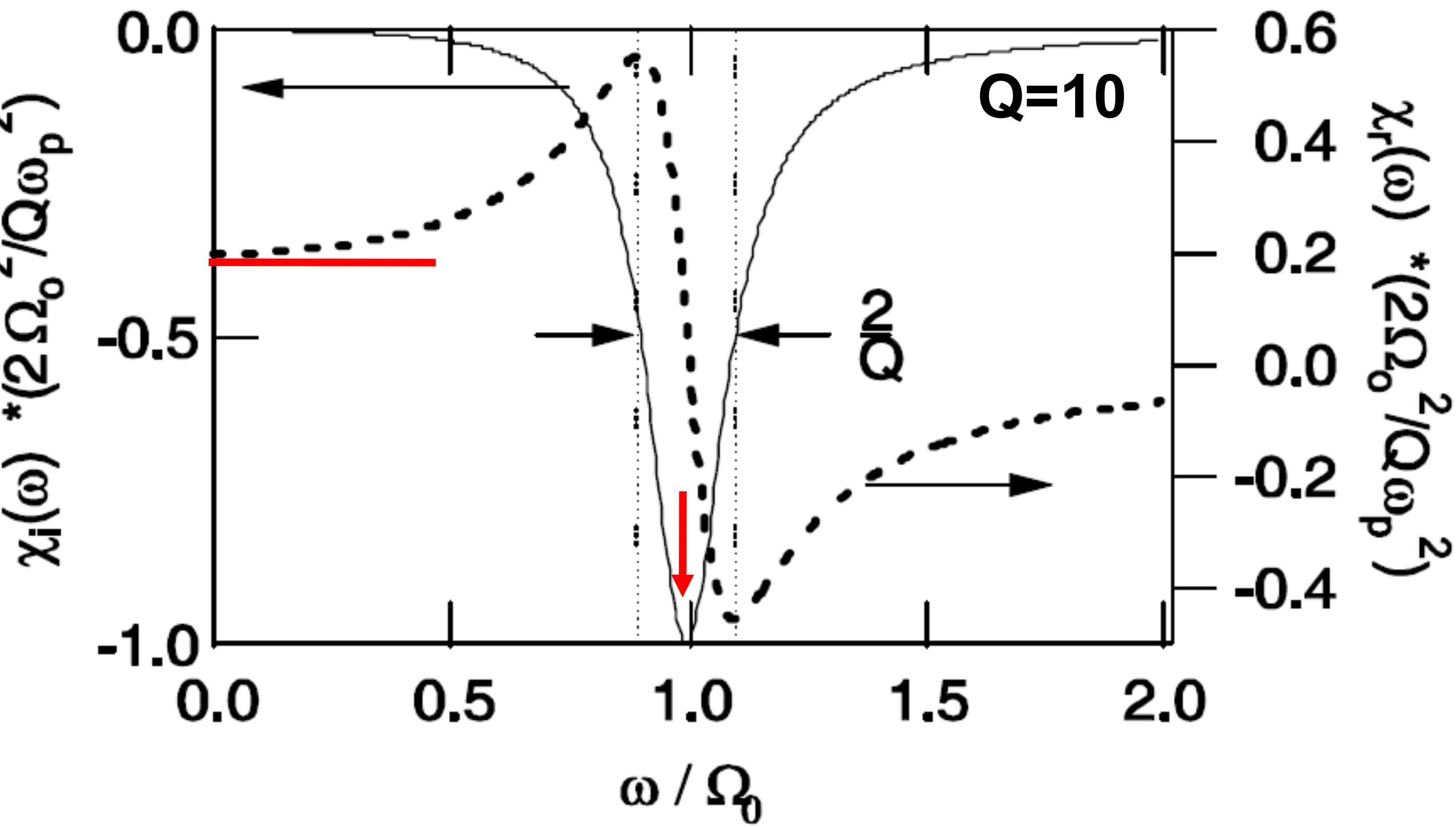


Figure 2.3: Real part (dashed line) and imaginary part (solid line) of the susceptibility of the classical oscillator model for the dielectric polarizability

Real and Imaginary Part of the Susceptibility

$$\underline{\tilde{\chi}}(\omega) = \tilde{\chi}_r(\omega) + j\tilde{\chi}_i(\omega)$$

Example: EM-Wave polarized along x-axis and propagation along z-direction:

$$\underline{\vec{E}}(z,t) = \underline{E}_0 e^{j(\omega t - kz)} \; \vec{e}_x$$

In general:

$$k(\omega) = \frac{\omega}{c_0}\tilde{n}(\omega) = \frac{\omega}{c_0}\sqrt{1+\tilde{\chi}(\omega)} = \frac{\omega}{c_0}\left(\tilde{n}_r(\omega) + j\tilde{n}_i(\omega)\right) = k_r(\omega) - j\alpha(\omega)$$

$$\boxed{\underline{\vec{E}}(z,t) = \underline{E}_0 e^{-\alpha \cdot z} \; e^{j(\omega t - k_r z)} \; \vec{e}_x}$$

damping

$$k(\omega) = \frac{\omega}{c_0}\sqrt{1+\tilde{\chi}(\omega)}$$

for: $\tilde{\chi}(\omega) \ll 1$

$$= \frac{\omega}{c_0}\left(1 + \frac{1}{2}\tilde{\chi}(\omega)\right) = \frac{\omega}{c_0}\left(1 + \frac{1}{2}\tilde{\chi}_r(\omega) + \frac{1}{2}j\tilde{\chi}_i(\omega)\right)$$

In a Metal

Free electrons between background ions

$$m\frac{d^2x}{dt^2} + \cancel{2\frac{\Omega_0}{Q}m\frac{dx}{dt}} + \cancel{m\Omega_0^2 x} = e_0 E(t), \qquad (2.41)$$

In general:

$$\tilde{\chi}_r(\omega) = \omega_p^2 \cdot \frac{\left(\Omega_0^2 - \omega^2\right)}{\left(\Omega_0^2 - \omega^2\right)^2 + \left(2\omega\frac{\Omega_0}{Q}\right)^2} \rightarrow -\frac{\omega_p^2}{\omega^2}$$

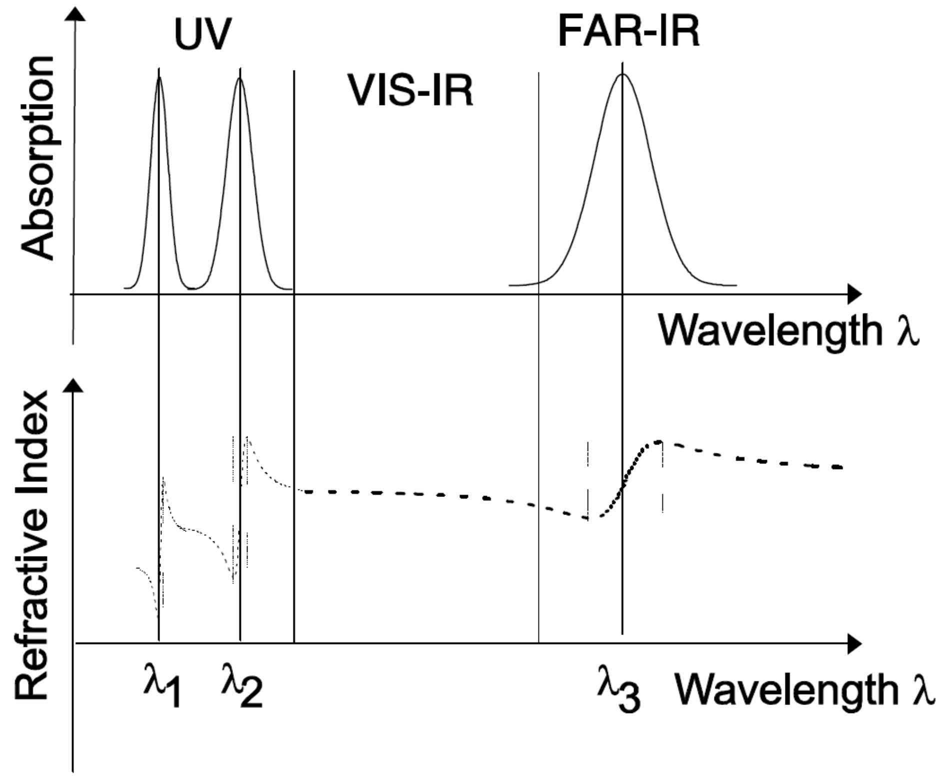
$$\tilde{\chi}_i(\omega) = 0$$

$$k(\omega) = \frac{\omega}{c_0}\tilde{n}(\omega) = \frac{\omega}{c_0}\sqrt{1+\tilde{\chi}(\omega)} = \frac{\omega}{c_0}\sqrt{1-\frac{\omega_p^2}{\omega^2}}$$

$\omega < \omega_p$: Metal reflects and for $\omega > \omega_p$: "transparent"

Absorption and refractive index Vs. wavelength

$$k_r(\omega) = \frac{\omega}{c_0} n_r(\omega)$$



Classical Optics $\begin{cases} \frac{dn}{d\lambda} < 0 : \text{normal dispersion (blue refracts more than red)} \\ \frac{dn}{d\lambda} > 0 : \text{anomalous dispersion} \end{cases}$

Ultrafast Optics $\begin{cases} \frac{d^2n}{d\lambda^2} > 0 : \text{normal dispersion} \\ \text{short wavelengths slower than long wavelengths} \\ \frac{d^2n}{d\lambda^2} < 0 : \text{anomalous dispersion} \\ \text{short wavelengths faster than long wavelengths} \end{cases}$

Sellmeier equations to model refractive index

If the frequency is far away from the absorption resonance $\left|\Omega_0^2 - \omega^2\right| >> 2\omega \frac{\Omega_0}{Q}$

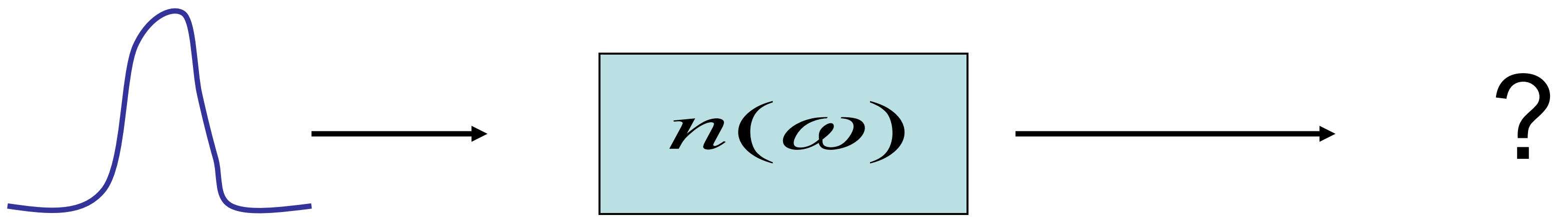
$$\tilde{\chi}_r(\omega) = \omega_p^2 \cdot \frac{\left(\Omega_0^2 - \omega^2\right)}{\left(\Omega_0^2 - \omega^2\right)^2 + \left(2\omega \frac{\Omega_0}{Q}\right)^2} \Longrightarrow \tilde{\chi}_r(\omega) = \frac{\omega_p^2}{\Omega_0^2 - \omega^2}$$

Normally there are multiple resonant frequencies for the electronic oscillators. It means in general the refractive index will have the form

$$\boxed{n^2(\omega) = 1 + \sum_i A_i \frac{\omega_p^2}{\Omega_i^2 - \omega^2} = 1 + \sum_i a_i \frac{\lambda^2}{\lambda^2 - \lambda_i^2}}$$

	Fused Quartz	Sapphire
a_1	0.6961663	1.023798
a_2	0.4079426	1.058364
a_3	0.8974794	5.280792
λ_1^2	$4.679148 \cdot 10^{-3}$	$3.77588 \cdot 10^{-3}$
λ_2^2	$1.3512063 \cdot 10^{-2}$	$1.22544 \cdot 10^{-2}$
λ_3^2	$0.9793400 \cdot 10^{2}$	$3.213616 \cdot 10^{2}$

Linear propagation of a pulse



$$1+\chi(\omega)=n^2(\omega)$$

$$\left(\Delta+\frac{\omega^2}{c_0^2}(1+\tilde{\chi}(\omega))\right)\tilde{\vec{E}}(\omega)=0 \longrightarrow (\nabla^2+\frac{n^2(\omega)\omega^2}{c_0^2})E(\omega)=0$$

Neglecting diffraction (e.g. inside an optical waveguide)

$$\Delta=\frac{\partial^2}{\partial z^2}$$

Fourier transform

$$E(z,t)=A(z,t)e^{j(\omega_0 t-k_0 z)}$$

$$E(z,\omega)=A(z,\omega-\omega_0)e^{-jk_0 z}$$

$$\frac{d^2A}{dz^2}-2jk_0\frac{dA}{dz}+[k^2(\omega)-k_0^2]A=0 \longleftarrow [\frac{d^2}{dz^2}+k^2(\omega)]E(z,\omega)=0$$

$$k(\omega)=\frac{n(\omega)\omega}{c_0}$$

$$\frac{d^2A}{dz^2}-2jk_0\frac{dA}{dz}+\underbrace{[k(\omega)+k_0]}_{\approx 2k_0}[k(\omega)-k_0]A=0$$

Slowly varying amplitude approximation

$$\left|\frac{d^2A}{dz^2}\right|<<\left|2k_0\frac{dA}{dz}\right|$$

$$\boxed{\frac{dA}{dz}=-j[k(\omega)-k_0]A}$$