Solution to PSet 2 UPO - UHH - 2015

Problem 2.1

(a.) $$\frac{\partial A}{\partial z} = j\cdot D_2 \frac{\partial^2 A}{\partial t^2} - j\cdot\delta |A|^2\cdot A \quad \text{substitute } \xi = \frac{z}{L_D} \quad \tau = \frac{t}{\tau_0}$$

$$\tau_0^2\cdot\frac{\partial A}{\partial \xi} = j\cdot D_2\cdot L_D\cdot\frac{\partial^2 A}{\partial \tau^2} - j\cdot\delta\cdot L_D\cdot\tau_0^2\cdot|A|^2\cdot A, \quad \text{note: } D_2\cdot L_D = -\tau_0^2$$

$$\frac{\partial A}{\partial \xi} = -j\cdot\frac{\partial^2 A}{\partial \tau^2} - j\,\delta L_D\,|A|^2\cdot A \quad \text{substitute } A = u\cdot\sqrt{\frac{|D_2|}{\delta}}\cdot\frac{1}{\tau_0}$$

$$\frac{\partial u}{\partial \xi} = -j\cdot\frac{\partial^2 u}{\partial \tau^2} - j\cdot\delta\cdot L_D\cdot\frac{D_2}{\delta}\cdot\frac{1}{\tau_0^2}\cdot|u|^2\cdot u$$

$$\Rightarrow \frac{\partial u}{\partial \xi} = -j\cdot\frac{\partial^2 u}{\partial \tau^2} - j\cdot|u|^2\cdot u \qquad \text{Q.E.D}$$

(b.) $$u = \operatorname{sech}\left(\frac{\tau}{\sqrt{2}}\right)\cdot\exp\left(-j\cdot\frac{\xi}{2}\right)$$

for normalized NSE

$$(\text{left}) = -\frac{j}{2}\cdot\operatorname{sech}\left(\frac{\tau}{\sqrt{2}}\right)\cdot\exp\left(-j\cdot\frac{\xi}{2}\right)$$

$$(\text{right}) = -j\cdot\frac{1}{4}\left(\cosh(\sqrt{2}\,\tau)-3\right)\cdot\operatorname{sech}^3\left(\frac{\tau}{\sqrt{2}}\right)\cdot\exp\left(-j\frac{\xi}{2}\right) - j\cdot\operatorname{sech}^3\left(\frac{\tau}{\sqrt{2}}\right)\cdot\exp\left(-j\cdot\frac{\xi}{2}\right)$$

$$= -j\cdot\frac{\cosh(\sqrt{2}\,\tau)+1}{4}\cdot\operatorname{sech}^3\left(\frac{\tau}{\sqrt{2}}\right)\cdot\exp\left(-j\cdot\frac{\xi}{2}\right)$$

$$= -\frac{j}{2}\cdot\operatorname{sech}\left(\frac{\tau}{\sqrt{2}}\right)\cdot\exp\left(-j\cdot\frac{\xi}{2}\right) = (\text{left}). \qquad \text{Q.E.D}$$

(c.) $$D = \frac{S_0}{4}\left[\lambda - \frac{\lambda_0^4}{\lambda}\right] = \frac{0.092}{4}\left(1.55 - \frac{1.3115^4}{1.55^3}\right)\times 10^3 = 17.38\ \mathrm{ps/nm\cdot km}$$

$$\beta_2 = -\frac{\lambda^2}{2\pi c}\cdot D = -22.2\ \mathrm{fs^2/mm}. \qquad D_2 = \frac{\beta_2}{2} = -11.1\ \mathrm{fs^2/mm}$$

$$\delta = k_0\cdot n_2 = \frac{2\pi}{\lambda}\,n_2$$

$$A = \sqrt{\frac{|D_2|}{\delta}}\cdot\frac{1}{\tau_0}\cdot u = 7.553\times10^4\cdot u\ \left[\sqrt{\frac{W}{m^2}}\right]$$

$$I_p = 5.705\times10^9\ \mathrm{W/m^2} \qquad P_p = I_p\cdot A_{eff} = I_p\cdot\frac{d_{MFD}^2}{4}\cdot\pi = 0.4846\ W$$

Problem 2.3

(a)

$$A(t) = e^{-(\Gamma_1 + i\Gamma_2)t^2} \cdot e^{i\omega_0 t}$$

$$\tilde{A}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} A(t) e^{-i\omega t} dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-(\Gamma_1 + i\Gamma_2)t^2} \cdot e^{-i(\omega-\omega_0)t} dt$$

$$= \frac{1}{2\pi} \cdot \sqrt{\frac{\pi}{\Gamma}} \cdot e^{-(\omega-\omega_0)^2/4\Gamma}, \text{ where } \Gamma = \Gamma_1 + i\Gamma_2$$

$$\tilde{A}'(\omega) = \frac{1}{2\pi} \cdot \sqrt{\frac{\pi}{\Gamma}} \cdot e^{-\frac{(\omega-\omega_0)^2}{4\Gamma}} \cdot e^{iD_2(\omega-\omega_0)^2}$$

$$= \frac{1}{2\pi} \cdot \sqrt{\frac{\pi}{\Gamma}} \cdot e^{-\left(\frac{1}{4\Gamma} - iD_2\right)(\omega-\omega_0)^2}$$

$$= C \cdot e^{-(\omega-\omega_0)^2/4\Gamma'}$$

$$\Rightarrow \quad \frac{1}{4\Gamma'} = \frac{1}{4\Gamma} - iD_2$$

$$\Gamma' = \frac{\Gamma}{1 - 4iD_2\Gamma} = \frac{\Gamma_1 + i\Gamma_2}{1 + 4D_2\Gamma_2 - i4D_2\Gamma_1}$$

(b) The FWHM pulse duration is proportional to $\Re\{\frac{1}{\Gamma}\}$ after the nonlinear phase is completely compensated by negative dispersion.

Assume the width of the chirp pulse is τ_c.

$$\tau_c^2 \propto \Re\{\frac{1}{\Gamma}\} = \Re\{\frac{1}{\Gamma_1 + i\Gamma_2}\} = \frac{\Gamma_1}{\Gamma_1^2 + \Gamma_2^2}$$

$$\tau^2 \propto \Re\{\frac{1}{\Gamma}\} = \Re\{\frac{1}{\Gamma_1}\} \qquad \text{(unchirped initial pulse)}$$

Given $\tau_c = \frac{\tau}{2}$. $\Rightarrow \Gamma_2 = \sqrt{3} \cdot \Gamma_1$

$$\phi_0 = -\Gamma_2 \cdot \tau^2 = -\sqrt{3} \cdot \frac{1}{2\tau^2} \cdot \tau^2 = -\frac{\sqrt{3}}{2} \quad \text{(rad)}$$

(c).   to compensate the dispersion, we need to make the imaginary part of Γ zero.

$$\mathrm{Im}\,\Gamma' = 4D_2(\Gamma_1^2+\Gamma_2^2)+\Gamma_2 = 0$$

$$D_2 = -\frac{\Gamma_2}{4(\Gamma_1^2+\Gamma_2^2)} \quad \text{having} \quad \Gamma_2 = \sqrt{3}\,\Gamma_1$$

$$D_2 = -\frac{\sqrt{3}}{16\Gamma_1} = -\frac{\sqrt{3}}{8}\cdot\tau^2$$

assume Γ_2 is positive (which is true in many cases) we need negative D_2 to compress the spectral-broadened pulse. Since most glass have positive dispersion in the ultrafast laser wavelengths. it is necessary to have prism/grating pair or chirped mirror to introduce negative dispersion.