# Ultrafast Optical Physics II (SoSe 2020s)
# Lecture 4, May 29, 2020

## Finish Laser dynamics: semi-classical laser theory

(1) Two-level system and Bloch equations

(2) Rabi oscillation: coherent light-matter interaction

(3) Steady-state solution of Bloch equations: linear susceptibility

(4) Adiabatic solution of Bloch equations: laser rate equation

(5) Laser CW operation: stability and relaxation oscillation

(6) Q-switching: active and passive

# Bloch equations

$$\dot{w} \;=\; -\frac{w - w_0}{T_1} + \mathrm{j}\Omega_r^* \, \underline{d} - \mathrm{j}\Omega_r \, \underline{d}^*$$

$$\dot{\underline{d}} \;=\; -\left(\frac{1}{T_2} - \mathrm{j}\,(\omega_{eg} - \omega)\right)\underline{d} + \mathrm{j}\frac{\Omega_r}{2} \, w$$

$$\Omega_r = \frac{\vec{M}_{eg}^* \cdot \vec{e}}{\hbar}\left(\underline{E}_0 + \underline{E}_0^* e^{-\mathrm{j}2\omega t}\right)$$

◯: **Irreversible dynamics**

**Statistical interpretation**

$w_0$: **Nonequilibr. population**

$T_1$: **Energy relaxation**

$T_2$: **Phase relaxation**

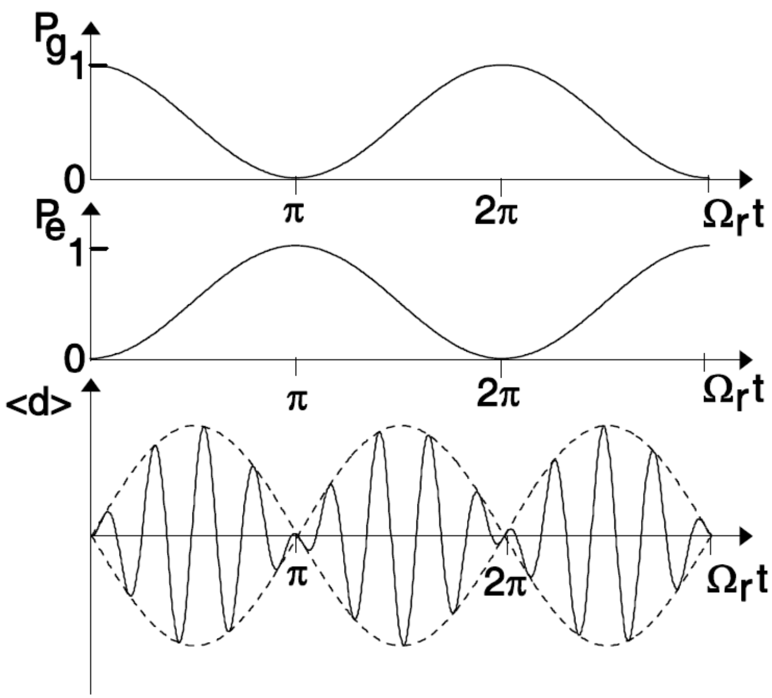
- **Bloch equations describe the dynamics of a statistical ensemble of two-level atoms interacting with a classical electric field.**

- **Polarization of the medium: expectation value of dipole moment of atomic ensemble. Source in Maxwell equations**

→ **Maxwell-Bloch Equations.**

# Steady-state solution

For moderate field strength $E_0$, no dephasing and no energy relaxation, the magnitude of the Rabi-frequency is much smaller than the optical frequency, $|\Omega_r| << \omega$, inversion and dipole moment do not change much within an optical cycle of the field.




If the optical pulse duration is longer than energy relaxation time constant $T_1$, implying that the temporal variation of the EM field is slower than energy decay, we can assume that population inversion and dipole moment are always at the steady-state though the steady state value adjust following the amplitude variation of the EM field.

$$\boxed{\dot{\underline{d}} = 0 \qquad \dot{w} = 0}$$

$$\underline{d}_s = \frac{\mathrm{j}}{2\hbar}\frac{\left(\vec{M}^*_{eg}\cdot\vec{e}\right)w_s}{1/T_2+\mathrm{j}(\omega-\omega_{eg})}\underline{E}_0$$

$$w_s = \frac{w_0}{1+\frac{T_1}{\hbar^2}\frac{1/T_2\,|\vec{M}^*_{eg}\cdot\vec{e}|^2}{(1/T_2)^2+(\omega_{eg}-\omega)^2}|\underline{E}_0|^2}$$

# Inversion Saturation

**We introduce the normalized lineshape function, which is in this case a Lorentzian:**

$$L(\omega) = \frac{(1/T_2)^2}{(1/T_2)^2 + (\omega_{eg} - \omega)^2}$$

**Intensity**: $I = \frac{1}{2Z_F}|\underline{E}_0|^2$

**Steady state inversion:** $w_s = \frac{w_0}{1 + \frac{I}{I_s}L(\omega)}$

Unsaturated inversion

Stationary inversion depends on the intensity of the incident light

Saturated inversion

**Saturation intensity:**

$$I_s = \left[\frac{2T_1T_2Z_F}{\hbar^2}|\vec{M}^*_{eg} \cdot \vec{e}|^2\right]^{-1}$$

# Dielectric Susceptibility

**Expectation value of the dipole moment** $\left\langle \vec{d} \right\rangle = \vec{M}_{eg} \underline{d}\, e^{j\omega t} + c.c.$

**Multiplication with the number of atoms per unit volume, N, relates the dipole moment of the atom to the macroscopic polarization P**

$$\vec{P}(t) = \frac{1}{2}\left(\underline{\vec{P}}_0 e^{j\omega t} + \underline{\vec{P}}_0^* e^{-j\omega t}\right) = N\vec{M}_{eg}\underline{d}_s\, e^{j\omega t} + c.c.$$

$$\underline{\vec{P}}_0 = 2N\vec{M}_{eg}\underline{d}_s$$

**Definition of the complex susceptibility** $\underline{\vec{P}}_0 = \epsilon_0 \chi(\omega) \vec{e}\underline{E}_0$

**Linear susceptibility of the medium**

$$\chi(\omega) = \vec{M}_{eg}\vec{M}_{eg}^{\dagger} \frac{jN}{\hbar\epsilon_0} \frac{w_s}{1/T_2 + j(\omega - \omega_{eg})}$$

# Linear susceptibility of the medium is a 2nd-rank tensor

$$\chi(\omega) = \vec{M}_{eg}\vec{M}_{eg}^{\dagger}\frac{\mathrm{j}N}{\hbar\epsilon_0}\frac{w_s}{1/T_2 + \mathrm{j}(\omega - \omega_{eg})}$$

Assume that the direction of the atom is random, i.e. the alignment of the atomic dipole moment, and the electric field is random. We have to average over the angle enclosed between the electric field of the wave and the atomic dipole moment, which results in

$$\overline{\begin{pmatrix} M_{egx}M_{egx}^* & M_{egx}M_{egy}^* & M_{egx}M_{egz}^* \\ M_{egy}M_{egx}^* & M_{egy}M_{egy}^* & M_{egy}M_{egz}^* \\ M_{egz}M_{egx}^* & M_{egz}M_{egy}^* & M_{egz}M_{egz}^* \end{pmatrix}} = \begin{pmatrix} \overline{M_{egx}^2} & 0 & 0 \\ 0 & \overline{M_{egy}^2} & 0 \\ 0 & 0 & \overline{M_{egz}^2} \end{pmatrix} = \frac{1}{3}|\vec{M}_{eg}|^2\,\mathbf{1}$$

**For homogeneous and isotropic media the susceptibility tensor shrinks to a scalar**

$$\chi(\omega) = \frac{1}{3}|\vec{M}_{eg}|^2\frac{\mathrm{j}N}{\hbar\epsilon_0}\frac{w_s}{1/T_2 + \mathrm{j}(\omega - \omega_{eg})}$$

# Linear susceptibility

**If the incident EM field is weak** $\frac{I}{I_s}L(\omega) \ll 1 \longrightarrow w_s \approx w_0$

**Linear susceptibility derived using semi-classical model**

$$\chi(\omega) = \frac{1}{3}|\vec{M}_{eg}|^2 \frac{\mathrm{j}N}{\hbar\epsilon_0} \frac{w_0}{1/T_2 + \mathrm{j}(\omega - \omega_{eg})}$$

**Linear susceptibility derived using classical harmonic oscillator model**

$$\tilde{\chi}(\omega) = \frac{N\frac{e_0^2}{m}\frac{1}{\epsilon_0}}{(\Omega_0^2 - \omega^2) + 2\mathrm{j}\omega\frac{\Omega_0}{Q}} \xrightarrow{\omega \approx \Omega_0} \tilde{\chi}(\omega) = \frac{-\mathrm{j}N\frac{e_0^2}{m}\frac{1}{\epsilon_0}/(2\Omega_0)}{\mathrm{j}(\omega - \Omega_0) + \frac{\Omega_0}{Q}}$$

**As the EM field has a frequency close to the oscillator's intrinsic frequency and define** $Q = T_2\omega_{eg}$ **, the shape of the susceptibility computed quantum mechanically agrees well with the classical susceptibility derived from the harmonic oscillator model.**

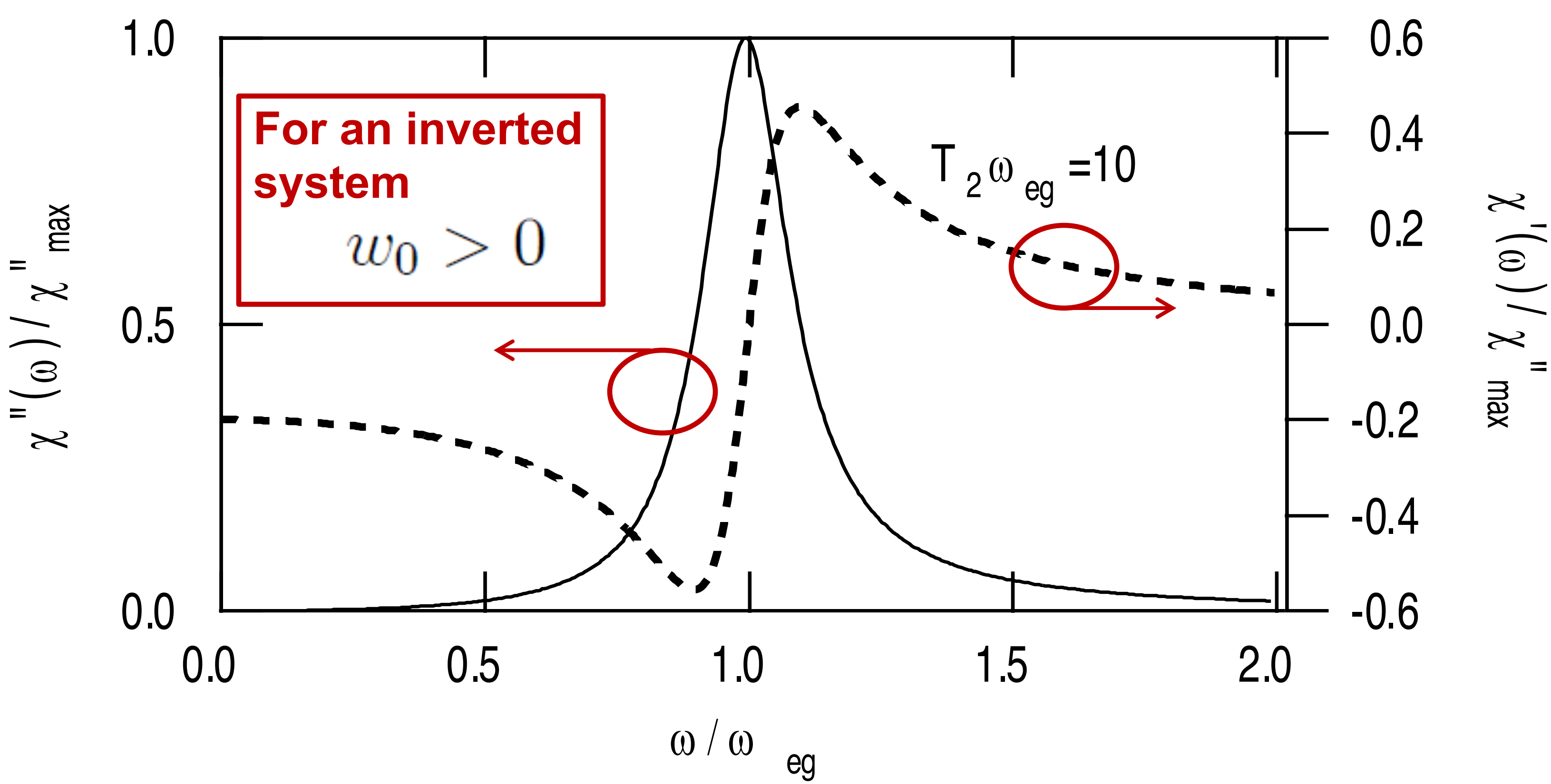
# Linear Susceptibility

**Real and imaginary part of the susceptibility** $\chi(\omega) = \chi'(\omega) + \mathrm{j}\chi''(\omega)$

$$\chi'(\omega) = -\frac{|\vec{M}_{eg}|^2 N w_s T_2^2 (\omega_{eg} - \omega)}{3\hbar\epsilon_0} L(\omega)$$

$$\chi''(\omega) = \frac{|\vec{M}_{eg}|^2 N w_s T_2}{3\hbar\epsilon_0} L(\omega).$$

Positive imaginary susceptibility indicates exponential growth of an EM wave traveling in the medium.



# Linear susceptibility: semi-classical versus classical

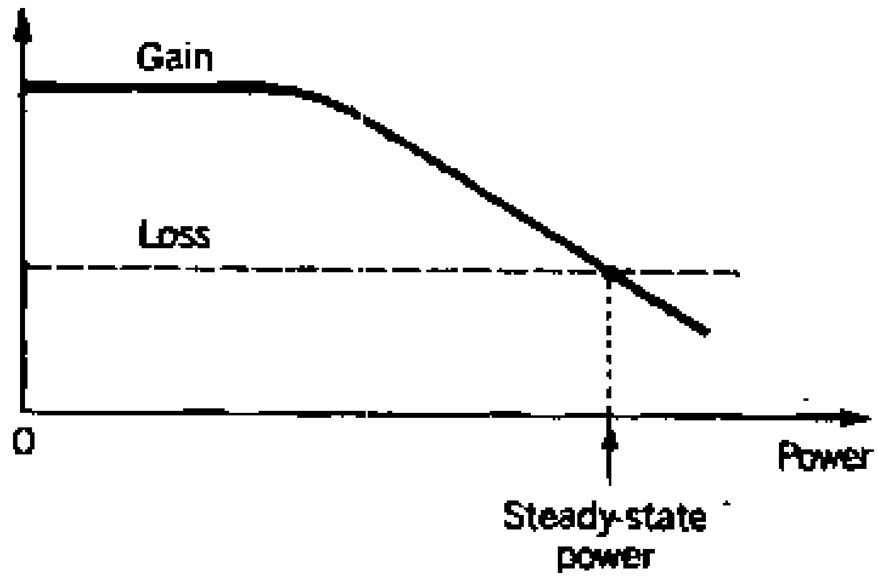
**The phase relaxation rate $1/T_2$ of the dipole moment determines the width of the absorption line or the bandwidth of the amplifier.**

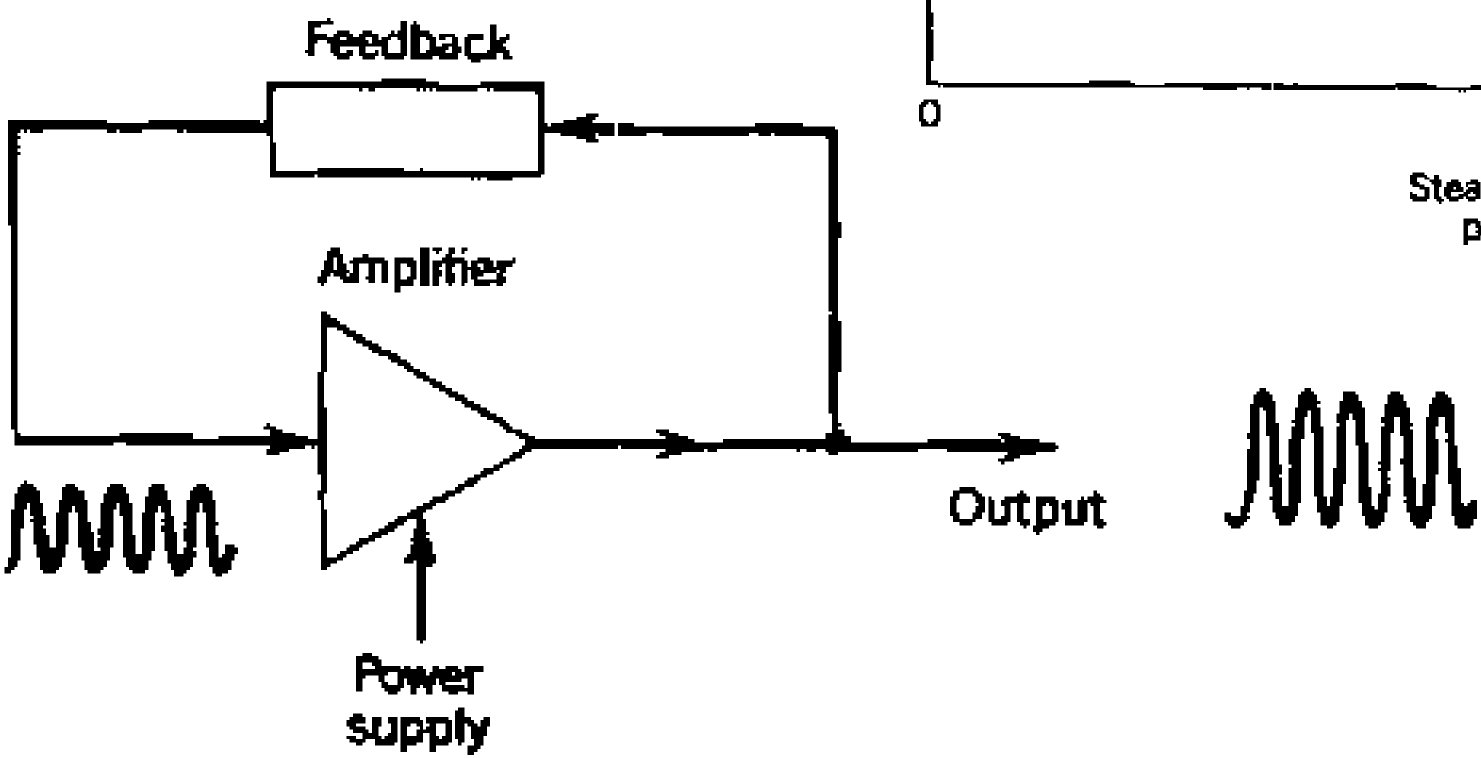
**The amplification can not occur forever, because the amplifier saturates when the intensity reaches the saturation intensity. This is a strong deviation from the linear susceptibility derived from the classical oscillator model.**

- **Light can not extract more energy from the atoms than the energy stored in them, i.e., energy conservation holds.**
- **Induced dipole moment in a two-level atom is limited by the maximum value of the matrix element.**
- **In contrast, the induced dipole moment in a classical oscillator growth proportionally to the applied field without limits.**

# Gain saturation is critical in laser operation

Initially, unstable feedback loop. Oscillation builds up until amplifier saturates such that there is zero net roundtrip gain.




The Laser (Oscillator) Concept

# Self-consistent in steady state

**Radiation field *E***
classical field, no photons, Maxwell's equations, mainly in the SVE-approximation of the wave equation

**Optical resonator/amplifier**
The field interacts with the electronic system (atoms or molecules) inside an optical resonator or amplifier.

**Atoms/ molecules**
are approximated by a two-level system, interacting with the field in the dipole approximation. Schrödinger's equation delivers a macroscopic polarization ***P***.

**Polarization *P***
is the source of a new coherent electromagnetic field.

**Spontaneous emission**
is introduced phenomenologically and produces a stochastic field which is added to the coherent field.

**self-consistent in steady state**

V.A. Lopota and H. Weber, "fundamentals of the semiclassical laser theory"

# Three regimes of solving Bloch equations

$$\dot{w} = -\frac{w - w_0}{T_1} + \mathrm{j}\Omega_r^* \, \underline{d} - \mathrm{j}\Omega_r \, \underline{d}^*$$

$$\dot{\underline{d}} = -(\frac{1}{T_2} - \mathrm{j}\,(\omega_{eg} - \omega))\underline{d} + \mathrm{j}\frac{\Omega_r}{2} \, w$$

$$\Omega_r = \frac{\vec{M}_{eg}^* \cdot \vec{e}}{\hbar} \left(\underline{E}_0 + \underline{E}_0^* e^{-\mathrm{j}2\omega t}\right)$$

**Coherent equations:**
Rabi oscillation

$$\boxed{\begin{aligned} \dot{w} &= \mathrm{j}\Omega_r^* \, \underline{d} - \mathrm{j}\Omega_r \, \underline{d}^* \\ \dot{\underline{d}} &= \mathrm{j}\,(\omega_{eg} - \omega)\,\underline{d} + \mathrm{j}\frac{\Omega_r}{2} \, w \end{aligned}}$$

**Steady state equations:**
Optical pulse duration >> $T_1$, $T_2$

$$\boxed{\dot{\underline{d}} = 0 \qquad \dot{w} = 0}$$

**Adiabatic equations:**
$T_2$ << $T_1$, polarization is in equilibrium with the applied field. No transient oscillations of the electronic system.

$$\boxed{\dot{\underline{d}} = 0 \qquad \dot{w} \neq 0}$$

e.g. semiconductors: $T_2$~50 fs

# Adiabatic equations: induced transitions

$$\dot{w} = -\frac{w - w_0}{T_1} + \mathrm{j}\Omega_r^* \, \underline{d} - \mathrm{j}\Omega_r \, \underline{d}^*$$

$$\underline{\dot{d}} = -(\frac{1}{T_2} - \mathrm{j}\,(\omega_{eg} - \omega))\underline{d} + \mathrm{j}\frac{\Omega_r}{2} \, w$$

**Adiabatic equations:** $T_2 << T_1$

$$\boxed{\underline{\dot{d}} = 0 \qquad \dot{w} \neq 0}$$

$$\dot{w} = -\underbrace{\frac{w(t) - w_0}{T_1}} - \underbrace{\frac{w(t)}{T_1 I_s} L(\omega) I(t)}$$

energy relaxation (e.g., spontaneous emission)

Induced transitions (absorption, stimulated emission)

**Light intensity:**

$$I(t) = |E_0(t)|^2 / (2Z_F)$$

**Resonant interaction between atom and EM field:** $\omega = \omega_{eg}$ $L(\omega) = 1$

$$\dot{w}|_{induced} = -\frac{w}{T_1 I_s} I = -\sigma w I_{ph}$$

Interaction cross section

**Photon flux density** $I_{ph} = I / \hbar\omega_{eg}$

# Laser rate equations

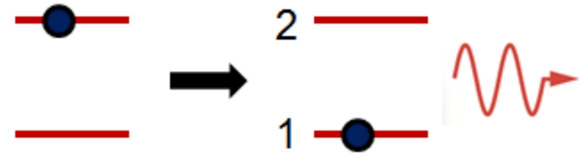
**Interaction cross section:** [Unit: $cm^2$]

$$\sigma = \frac{\hbar\omega_{eg}}{T_1 I_s} = \frac{2\omega_{eg} T_2 Z_F}{\hbar} |\vec{M}_{eg}^* \cdot \vec{e}|^2$$

$$\dot{w}\big|_{induced} = -\frac{w}{T_1 I_s} I = -\sigma w I_{ph}$$

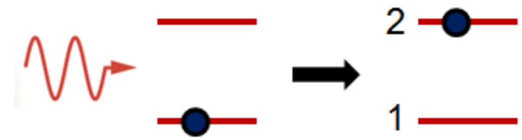
- Interaction cross section is the probability that an interaction will occur between EM field and the atomic system.
- Interaction cross  section only depends on the dipole matrix element and the linewidth of the transition

$$\dot{w} = -\frac{w(t) - w_0}{T_1} - \sigma w I_{ph}$$



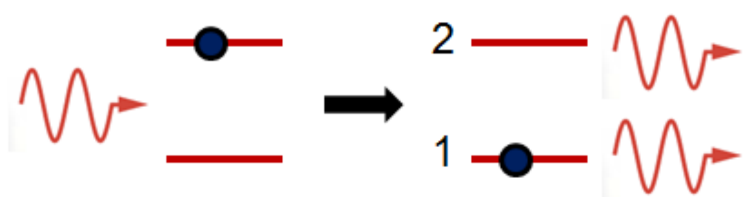

$$\frac{dN_2}{dt} = -AN_2 = -\frac{N_2}{\tau_{21}}$$

**Spontaneous emission**




$$\frac{dN_1}{dt} = -BN_1 I$$

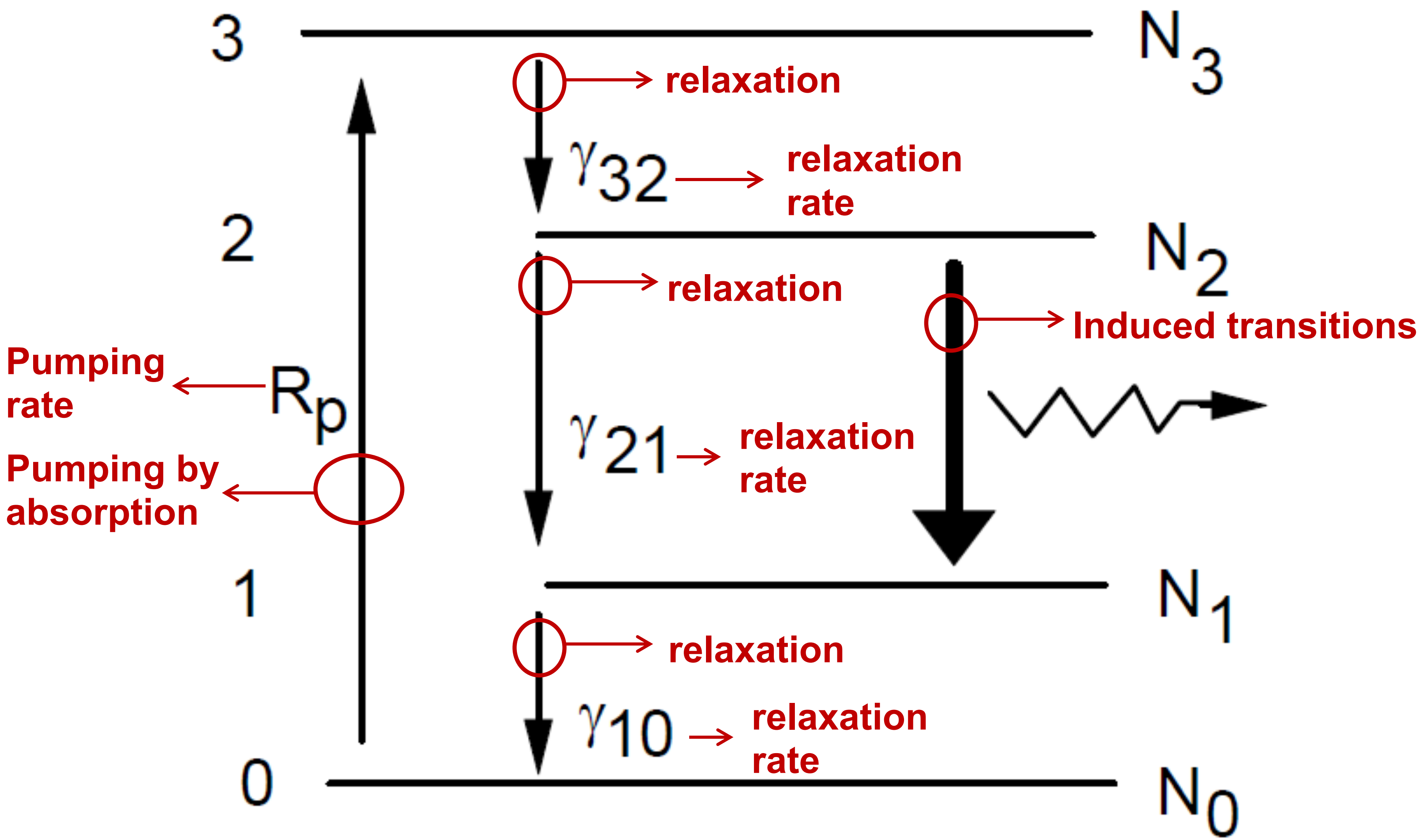
**Absorption**




$$\frac{dN_2}{dt} = -BN_2 I$$

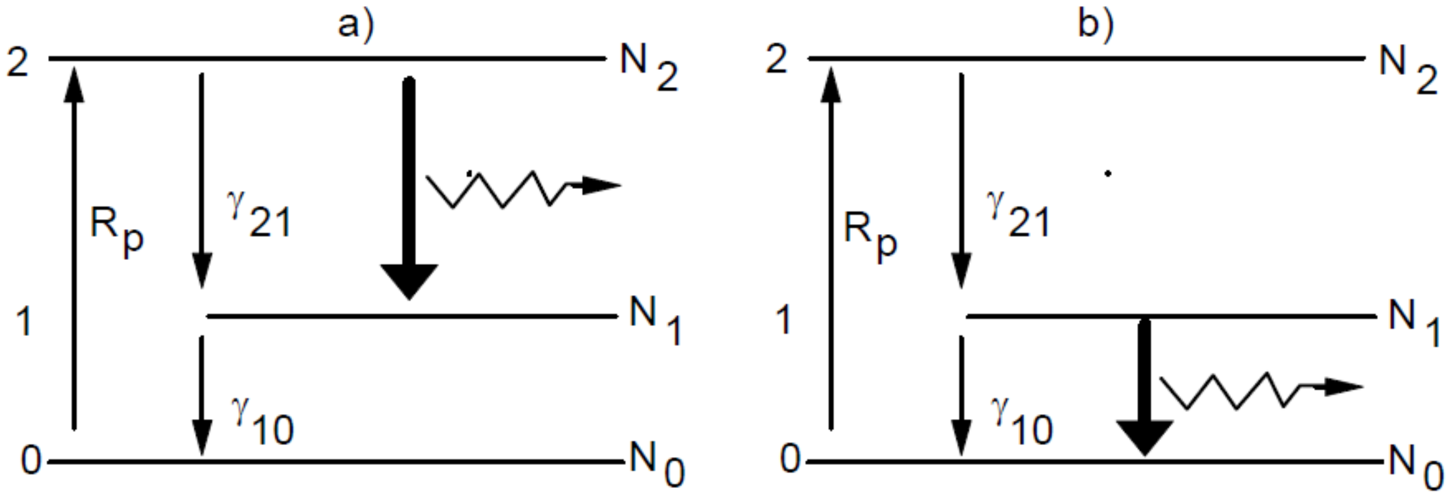
**Stimulated emission**

# How to achieve population inversion?




Four-level gain medium

# Laser rate equations for three-level laser medium




**For (a):**

$$\frac{d}{dt}N_2 = -\gamma_{21}N_2 - \sigma_{21}\left(N_2 - N_1\right)I_{ph} + R_p$$

$$\frac{d}{dt}N_1 = -\gamma_{10}N_1 + \gamma_{21}N_2 + \sigma_{21}\left(N_2 - N_1\right)I_{ph}$$

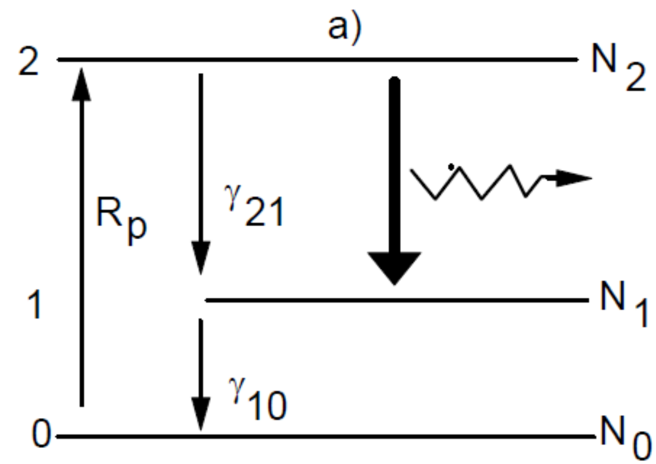
$$\frac{d}{dt}N_0 = \gamma_{10}N_1 - R_p$$

Many atoms are available in the ground state such that optical pumping can never deplete $N_0$. That is why we can assume a constant pump rate $R_p$.

**$\sigma_{21}$ is the cross section for stimulated emission between the levels 2 and 1. $I_{ph}$ is the photon flux.**

# Laser rate equations for three-level laser medium

If the relaxation rate $\gamma_{10}$ is much faster than $\gamma_{21}$ and the number of possible stimulated emission events that can occur $\sigma_{21}\left(N_2 - N_1\right) I_{ph}$ , we can set $N_1 = 0$ and obtain only a rate equation for the upper laser level:




$$\frac{d}{dt} N_2 = -\gamma_{21}\left(N_2 - \frac{R_p}{\gamma_{21}}\right) - \sigma_{21} N_2 \cdot I_{ph}$$

**This equation is identical to the equation for the inversion of the two-level system:**

$$\dot{w} = -\frac{w(t) - w_0}{T_1} - \sigma w I_{ph}$$

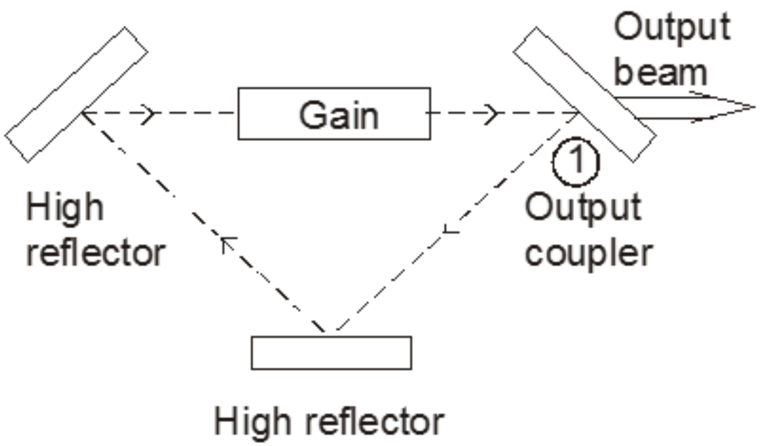
$\frac{R_p}{\gamma_{21}} \longrightarrow$ **equilibrium upper state population w/o photons present**

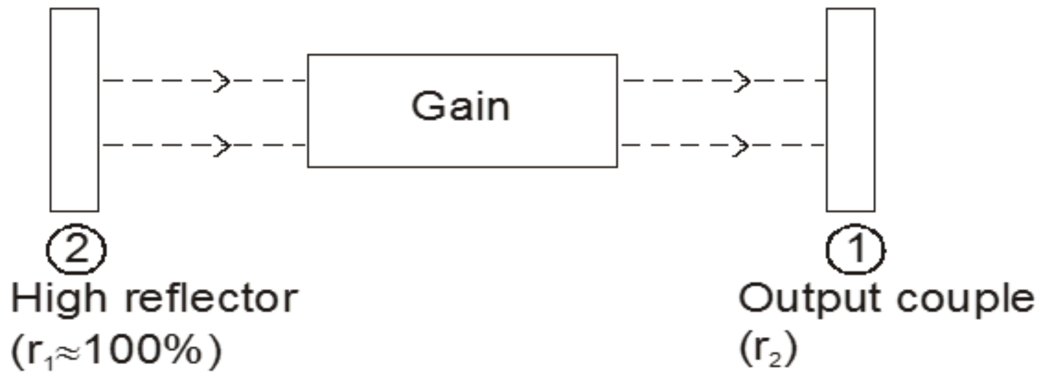
$\gamma_{21} = \frac{1}{\tau_L} \longrightarrow$ **upper level lifetime due to radiative and non-radiative processes**

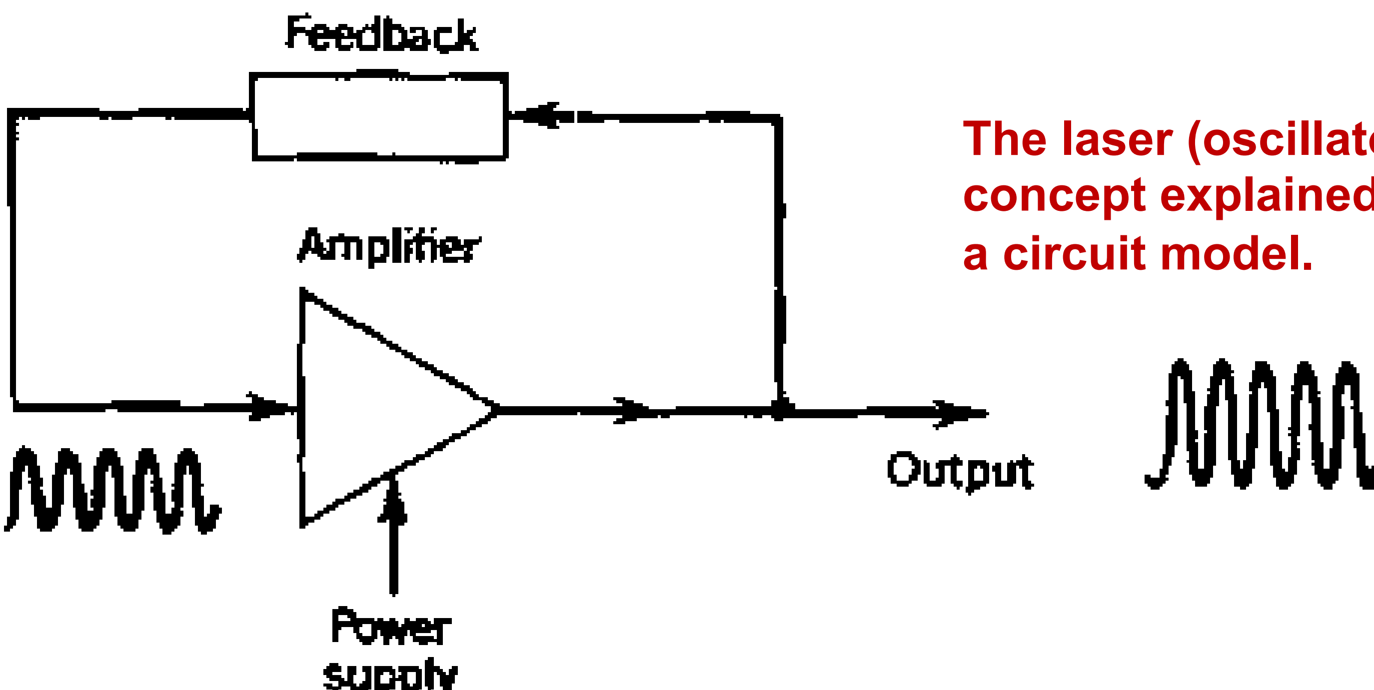
# Spectroscopic parameters of selected laser materials

| Laser Medium | Wave-length $\lambda_0(nm)$ | Cross Section $\sigma$ (cm$^2$) | Upper-St. Lifetime $\tau_L$ ($\mu s$) | Linewidth $\Delta f_{FWHM}$ $\frac{2}{T_2}$(THz) | Typ | Refr. index $n$ |
|---|---|---|---|---|---|---|
| $Nd^{3+}$:YAG | 1,064 | $4.1 \cdot 10^{-19}$ | 1,200 | 0.210 | H | 1.82 |
| $Nd^{3+}$:LSB | 1,062 | $1.3 \cdot 10^{-19}$ | 87 | 1.2 | H | 1.47 |
| $Nd^{3+}$:YLF | 1,047 | $1.8 \cdot 10^{-19}$ | 450 | 0.390 | H | 1.82 |
| $Nd^{3+}$:$YVO_4$ | 1,064 | $2.5 \cdot 10^{-19}$ | 50 | 0.300 | H | 2.19 |
| $Nd^{3+}$:glass | 1,054 | $4 \cdot 10^{-20}$ | 350 | 3 | H/I | 1.5 |
| $Er^{3+}$:glass | 1,55 | $6 \cdot 10^{-21}$ | 10,000 | 4 | H/I | 1.46 |
| Ruby | 694.3 | $2 \cdot 10^{-20}$ | 1,000 | 0.06 | H | 1.76 |
| $Ti^{3+}$:$Al_2O_3$ | 660-1180 | $3 \cdot 10^{-19}$ | 3 | 100 | H | 1.76 |
| $Cr^{3+}$:LiSAF | 760-960 | $4.8 \cdot 10^{-20}$ | 67 | 80 | H | 1.4 |
| $Cr^{3+}$:LiCAF | 710-840 | $1.3 \cdot 10^{-20}$ | 170 | 65 | H | 1.4 |
| $Cr^{3+}$:LiSGAF | 740-930 | $3.3 \cdot 10^{-20}$ | 88 | 80 | H | 1.4 |
| He-Ne | 632.8 | $1 \cdot 10^{-13}$ | 0.7 | 0.0015 | I | $\sim$1 |
| $Ar^+$ | 515 | $3 \cdot 10^{-12}$ | 0.07 | 0.0035 | I | $\sim$1 |
| $CO_2$ | 10,600 | $3 \cdot 10^{-18}$ | 2,900,000 | 0.000060 | H | $\sim$1 |
| Rhodamin-6G | 560-640 | $3 \cdot 10^{-16}$ | 0.0033 | 5 | H | 1.33 |
| semiconductors | 450-30,000 | $\sim 10^{-14}$ | $\sim$ 0.002 | 25 | H/I | 3 - 4 |

# Possible laser cavity configurations

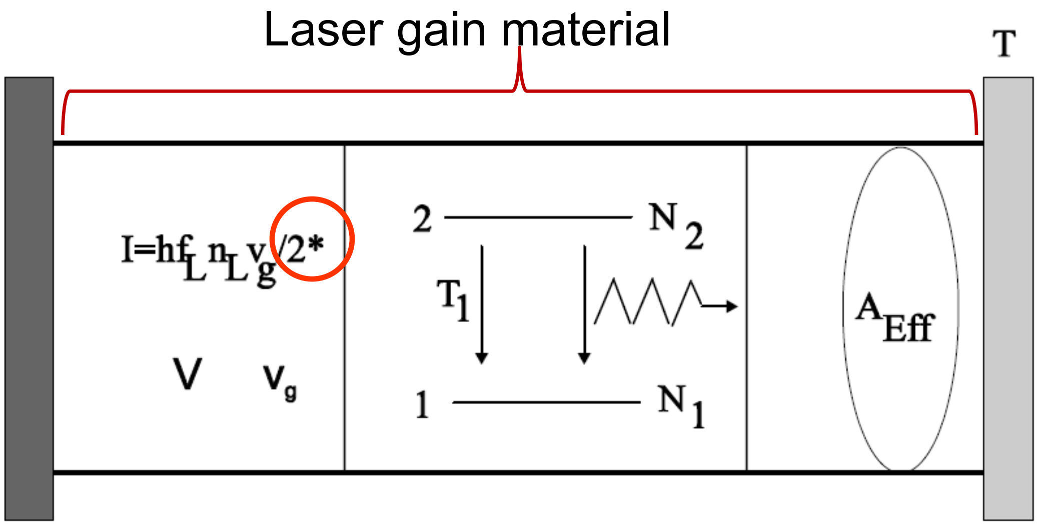



**The laser (oscillator) concept explained using a circuit model.**

# More on laser rate equations




***V*:= $A_{eff}$ *L* Mode volume**
**$f_L$: laser frequency**
***I*: Intensity**
**$v_g$: group velocity at laser frequency**
**$N_L$: number of photons in mode**
**$n_L$: photon density in mode**

Intensity I in a mode propagating at group velocity $v_g$ in one direction with a mode volume V is related to the number of photons $N_L$ stored in the mode with volume V by

$$I = hf_L \frac{N_L}{2^* V} v_g = \frac{1}{2^*} hf_L n_L v_g$$

$$I_{ph} = I/\hbar\omega_{eg} = \frac{N_L}{2^* V} v_g$$

$2^*$ = 2 for a linear laser resonator (then only half of the photons are going in one direction)

$2^*$ = 1 for a ring laser

**$\sigma$: interaction cross section** $\sigma = hf_L/(I_s \tau_L)$

# More on laser rate equations

**Number of atoms in upper level:**

$$\frac{d}{dt}N_2 = -\frac{N_2}{T_1} - \frac{\sigma v_g}{V}N_2 N_L + R_p$$

upper level lifetime

**Number of photons in mode:**

$$\frac{d}{dt}N_L = -\frac{N_L}{\tau_p} + \frac{\sigma v_g}{V}N_2\left(N_L + 1\right)$$

Photon lifetime in the cavity or cavity decay time

Number of photons spontaneously emitted into laser mode

Laser cavity with a semi-transparent mirror with transmission T produces a small power loss 2l = − ln(1−T ) ≈ T (for small T ) per roundtrip in the cavity.

**Cavity round trip time:** $T_R = 2^* L / v_g$

**Photon lifetime:** $\tau_p = T_R / 2l$

***l*: amplitude loss per roundtrip**

***2l*: power loss per roundtrip**

# Rewrite rate equations using measurable quantities

$$\frac{d}{dt}N_2 = -\frac{N_2}{\tau_L} - \sigma v_g N_2 n_L + R_p$$

$$\frac{d}{dt}N_L = -\frac{N_L}{\tau_p} + \frac{\sigma v_g}{V} N_2 (N_L + 1)$$

**Circulating intracavity power**

$$P = I \cdot A_{eff} = h f_L \frac{N_L}{T_R}$$

**Round trip amplitude gain**

$$g = \frac{\sigma v_g}{2V} N_2 T_R$$

$$\frac{d}{dt}g = -\frac{g - g_0}{\tau_L} - \frac{gP}{E_{sat}}$$

$$\frac{d}{dt}P = -\frac{1}{\tau_p}P + \frac{2g}{T_R}(P + P_{vac})$$

$$E_{sat} = \frac{h f_L V}{\sigma v_g T_R} = \frac{1}{2^*} I_s A_{eff} \tau_L$$

$$P_{sat} = E_{sat}/\tau_L$$

$$P_{vac} = h f_L / T_R$$

$$g_0 = 2^* \frac{R_p}{2A_{eff}} \sigma \tau_L,$$

**Output power:** $P_{out} = T \cdot P.$

**small signal gain ~ $\sigma\tau_L$ - product**

# Buildup of laser oscillation

$$\frac{d}{dt}g = -\frac{g-g_0}{\tau_L} - \frac{gP}{E_{sat}} \qquad \frac{d}{dt}P = -\frac{1}{\tau_p}P + \frac{2g}{T_R}(P+P_{vac})$$

During laser buildup,

$$P_{vac} \ll P \ll P_{sat} = E_{sat}/\tau_L$$

we can neglect the spontaneous emission $P_{vac}$, and the gain is unsaturated: $g = g_0$

$$\frac{d}{dt}P = -\frac{1}{\tau_p}P + \frac{2g}{T_R}(P+P_{vac})$$

$$\tau_p = T_R/2l$$

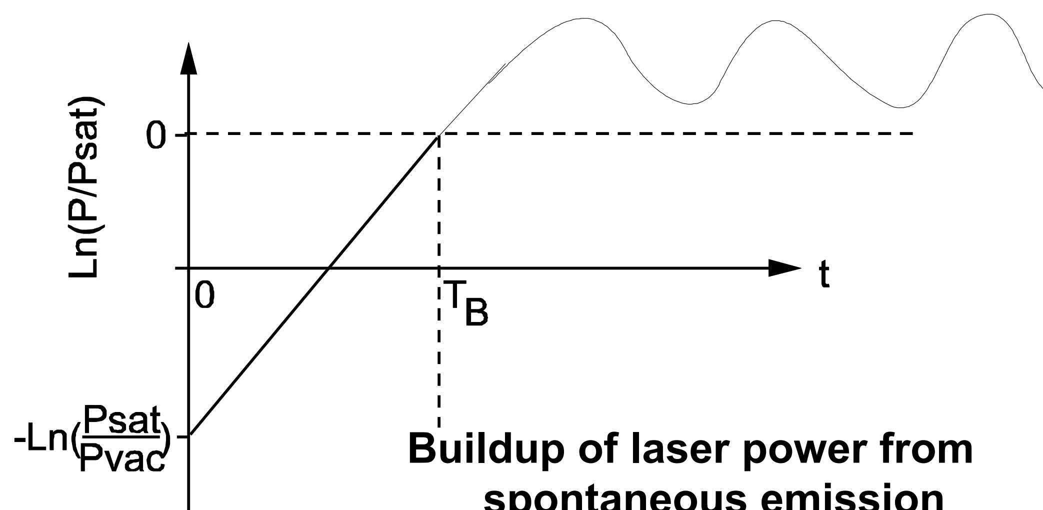
$$\frac{dP}{P} = 2(g_0 - l)\frac{dt}{T_R}$$

$$P(t) = P(0)e^{2(g_0-l)\frac{t}{T_R}}$$

The laser power builds up from vacuum fluctuations once the small signal gain surpasses the laser losses: $g_0 > g_{th} = l$

Saturation sets in within the built-up time

$$T_B = \frac{T_R}{2(g_0-l)}\ln\frac{P_{sat}}{P_{vac}} = \frac{T_R}{2(g_0-l)}\ln\frac{A_{eff}T_R}{\sigma\tau_L}$$

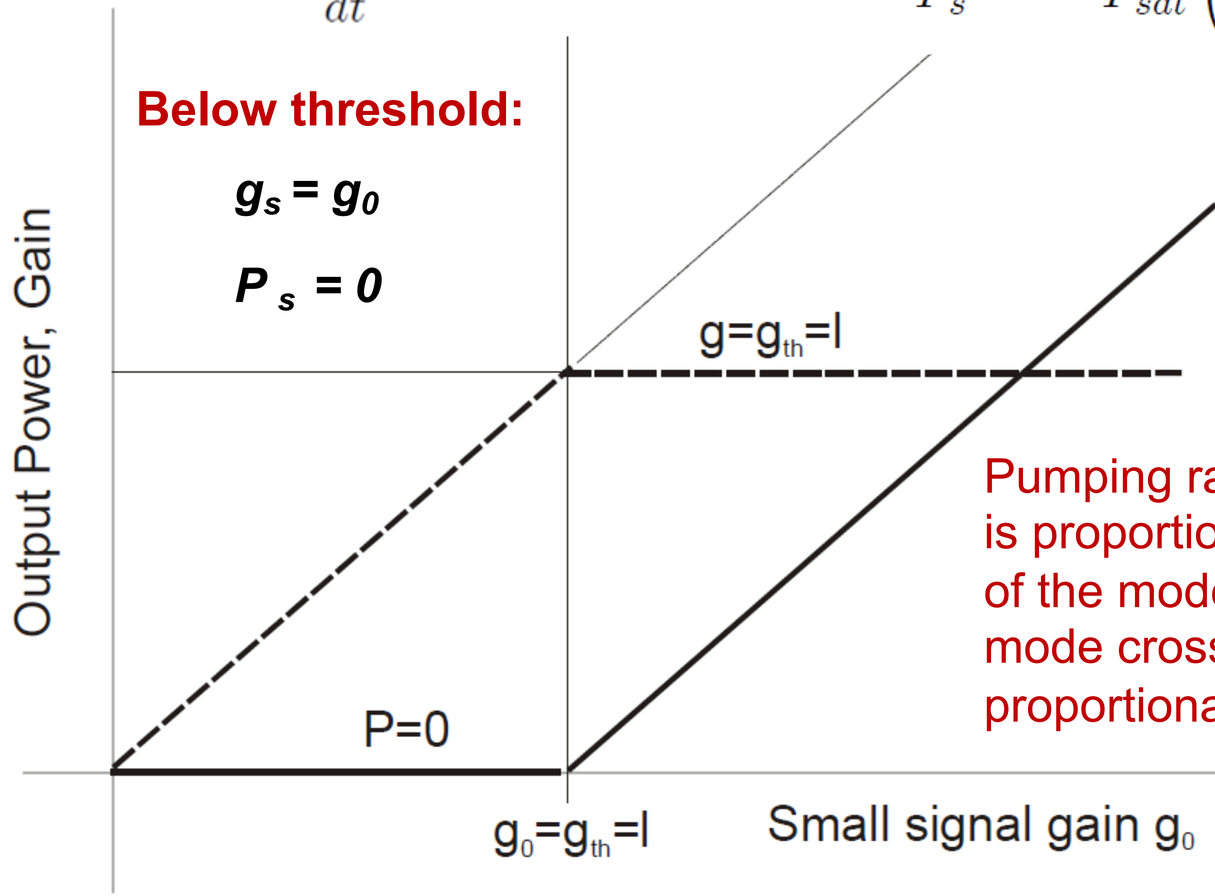



**Buildup of laser power from spontaneous emission**

# Steady state operation: output power vs small signal gain

Beyond the gain threshold, some time after the buildup phase, the laser reaches steady state. Neglecting the spontaneous emission, saturated gain and steady state power can be calculated:

$$\frac{d}{dt}g = 0 \qquad \frac{d}{dt}P = 0 \quad \longrightarrow \quad g_s = \frac{g_0}{1+\frac{P_s}{P_{sat}}} = l \qquad P_s = P_{sat}\left(\frac{g_0}{l}-1\right)$$




$$g_{th} = l, \qquad R_{p,th} = \frac{2l A_{eff}}{2^* \sigma \tau_L}$$

Pumping rate to reach threshold is proportional to the optical loss of the mode per roundtrip, the mode cross section, and inversely proportional to the $\sigma \cdot \tau_L$.

# General description of laser operation

- Pumping begins when a laser is turned on. Then population inversion growth until it eventually reaches a steady-state value.
- This steady-state population inversion is determined by the pumping rate and the $\sigma\tau_L$ - product
- This steady-state population inversion corresponds to the small signal gain $g_0$.
- As the gain exceeds the cavity losses, the laser intra-cavity power begins to grow until it eventually reaches the saturation power and begins to extract energy from the medium.
- As the intra-cavity power grows, stimulated emission reduces the population inversion, and consequently the inversion reaches a new, lower steady-state value such that the reduced gain equals the losses in the cavity: $$g_s = \frac{g_0}{1+\frac{P_s}{P_{sat}}} = l$$

# Stability and relaxation oscillations

**How does the laser reach steady state, once a perturbation occurs?**

$$\begin{aligned} P &= P_s + \Delta P \\ g &= g_s + \Delta g \end{aligned} \Rightarrow \begin{aligned} \frac{d\Delta P}{dt} &= +2\frac{P_s}{T_R}\Delta g \\ \frac{d\Delta g}{dt} &= -\frac{g_s}{E_{sat}}\Delta P - \frac{1}{\tau_{stim}}\Delta g \end{aligned}$$

**Stimulated lifetime**

$$\frac{1}{\tau_{stim}} = \frac{1}{\tau_L}\left(1 + \frac{P_s}{Psat}\right)$$

The perturbations decay or grow like

$$\begin{pmatrix} \Delta P \\ \Delta g \end{pmatrix} = \begin{pmatrix} \Delta P_0 \\ \Delta g_0 \end{pmatrix} e^{st} \Rightarrow A \begin{pmatrix} \Delta P_0 \\ \Delta g_0 \end{pmatrix} = \begin{pmatrix} -s & 2\frac{P_s}{T_R} \\ -\frac{T_R}{E_{sat}2\tau_p} & -\frac{1}{\tau_{stim}} - s \end{pmatrix} \begin{pmatrix} \Delta P_0 \\ \Delta g_0 \end{pmatrix} = 0$$

Non-zero solution exists only if the determinant of the coefficient matrix is 0:

$$s\left(\frac{1}{\tau_{stim}} + s\right) + \frac{P_s}{E_{sat}\tau_p} = 0$$

$$\boxed{s_{1/2} = -\frac{1}{2\tau_{stim}} \pm \sqrt{\left(\frac{1}{2\tau_{stim}}\right)^2 - \frac{P_s}{E_{sat}\tau_p}}}$$

# Stability and relaxation oscillations

Introducing the pump parameter $r = 1 + \frac{P_s}{P_{sat}} = \frac{g_0}{l}$, which tells us how much we pump the laser over threshold, the eigen frequencies can be rewritten as

$$\boxed{s_{1/2} = -\frac{1}{2\tau_{stim}}\left(1 \pm j\sqrt{\frac{4(r-1)}{r}\frac{\tau_{stim}}{\tau_p} - 1}\right) = -\frac{r}{2\tau_L} \pm j\sqrt{\frac{(r-1)}{\tau_L\tau_p} - \left(\frac{r}{2\tau_L}\right)^2}}$$

(i): The stationary state $(0, g_0)$ for $g_0 < l$ and $(P_s, g_s)$ for $g_0 > l$ are always stable, i.e. $\mathrm{Re}\{s_i\} < 0$.

(ii): For lasers pumped above threshold, $r > 1$, and long upper state lifetimes, i.e. $\frac{r}{4\tau_L} < \frac{1}{\tau_p}$, the relaxation rate becomes complex, i.e. there are relaxation oscillations

$$s_{1/2} = -\frac{1}{2\tau_{stim}} \pm j\omega_R \qquad \omega_R = \sqrt{\frac{1}{\tau_{stim}\tau_p}} \qquad \tau_{stim} = \frac{\tau_L}{r}$$

If the laser can be pumped strong enough, i.e. $r$ can be made large enough so that the stimulated lifetime becomes as short as the cavity decay time, relaxation oscillations vanish.

# Relaxation oscillations: a case study

**Diode-pumped Nd:YAG-Laser:**

$$\lambda_0 = 1064 \text{ nm}, \sigma = 4 \cdot 10^{-20} cm^2, \; A_{eff} = \pi \left(100\mu m \times 150\mu m\right)$$

$$r = 50 \quad \tau_L = 1.2 \; ms, \; l = 1\%, \; T_R = 10ns$$

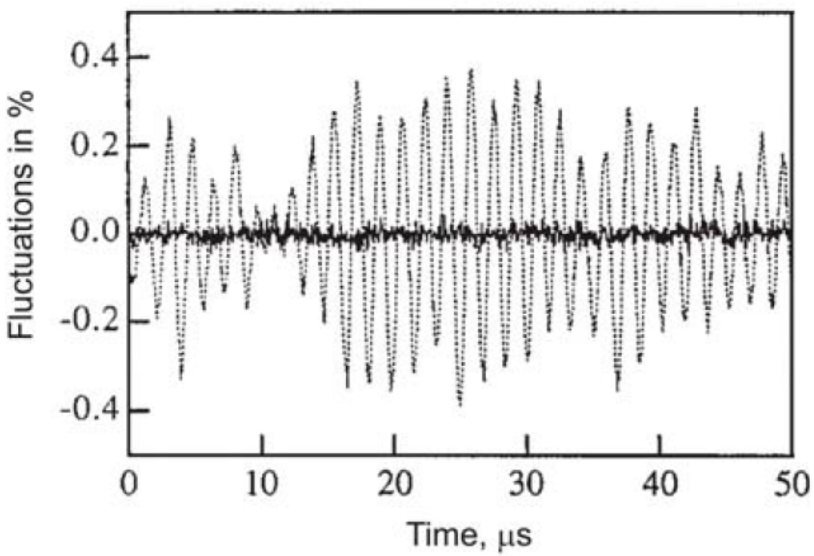
$$I_{sat} = \frac{hf_L}{\sigma \tau_L} = 3.9 \frac{kW}{cm^2}, \; P_s = 91.5W$$

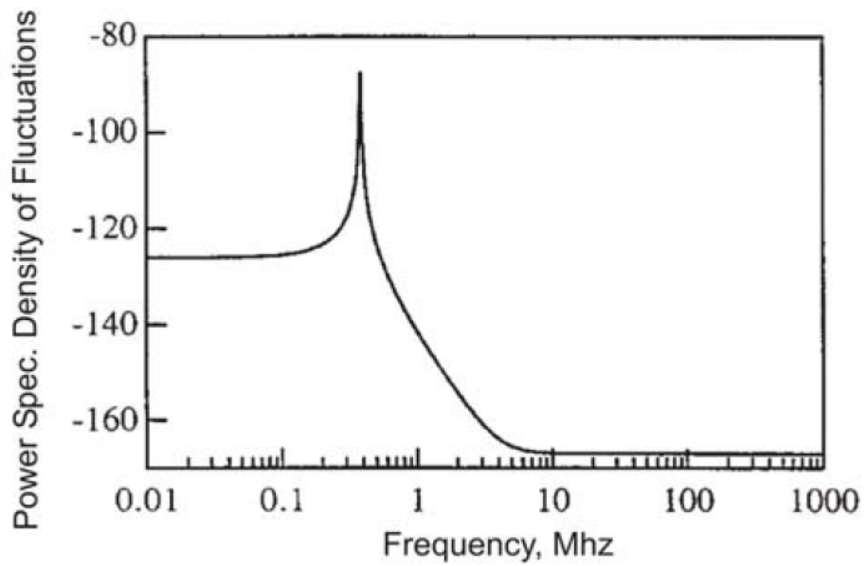
$$P_{sat} = I_{sat} A_{eff} = 1.8 \; W,$$

$$\tau_{stim} = \frac{\tau_L}{r} = 24\mu s, \; \tau_p = 1\mu s$$

$$\omega_R = \sqrt{\frac{1}{\tau_{stim} \tau_p}} = 2 \cdot 10^5 s^{-1}$$

The physical reason for relaxation oscillations and later instabilities is, that the gain reacts too slow on the light field, i.e. the stimulated lifetime is long in comparison with the cavity decay time.




Typically observed relaxation oscillations in time and frequency domain.

# Laser efficiency: how much pump power converted to laser output power

$$2g_0 = 2^* \frac{R_p}{A_{eff}} \sigma \tau_L,$$
$$P_{sat} = \frac{h f_L}{2^* \sigma \tau_L} A_{eff}$$

**Steady-state intracavity power:**

$$P_s = P_{sat} \left( \frac{2g_0}{2l} - 1 \right)$$

Laser power losses include the internal losses $2l_{int}$ and the transmission T through the output coupling mirror:

$$2l = 2l_{int} + T$$

**Pump photon energy**

Laser output power:

$$P_{out} = T \cdot P_{sat} \left( \frac{2g_0}{2l_{int} + T} - 1 \right)$$

Pump power:

$$P_p = R_p h f_P$$

**Efficiency:**

$$\eta = \frac{P_{out}}{P_p}$$

**Differential Efficiency:**

$$\eta_D = \frac{\partial P_{out}}{\partial P_p}$$

If the laser is pumped many times over threshold: $r = 2g_0/2l \to \infty$

$$\eta_D = \eta = \frac{T}{2l_{int} + T} P_{sat} \frac{2^*}{A_{eff} h f_P} \sigma \tau_L = \frac{T}{2l_{int} + T} \cdot \frac{h f_L}{h f_P}$$

Laser efficiency is fundamentally limited by the ratio of output coupling to total losses and the quantum defect in pumping.

# How to efficiently use the energy storage capability?

- Typical cavity length of a solid-state laser: 0.1-10 m
- Cavity round-trip time: ~1-100 ns
- Photon lifetime in the cavity $\tau_p = T_R / 2l$ : ~0.1-1 us
- Upper level lifetime of solid-state gain materials: 10-1000 us

$$\tau_L >> \tau_p >> T_R$$

- Given pumping rate, $\tau_L$ time is needed such that the full energy storage capability is reached.
- However, the intra-cavity power starts to build up as the gain exceeds loss and it grows much faster such that it reaches the saturation power and starts to saturate population inversion. So the full energy storage capability cannot be reached.

## How to solve this dilemma?

## Q-Switching: manage cavity loss to obtain giant pulses

The idea: “Charging up” a cavity to “dump” the energy into one big pulse

→Controlling the loss in a cavity to suppress circulating power
→On purpose switching cavity losses to a minimum
→Cavity Power rapidly growth

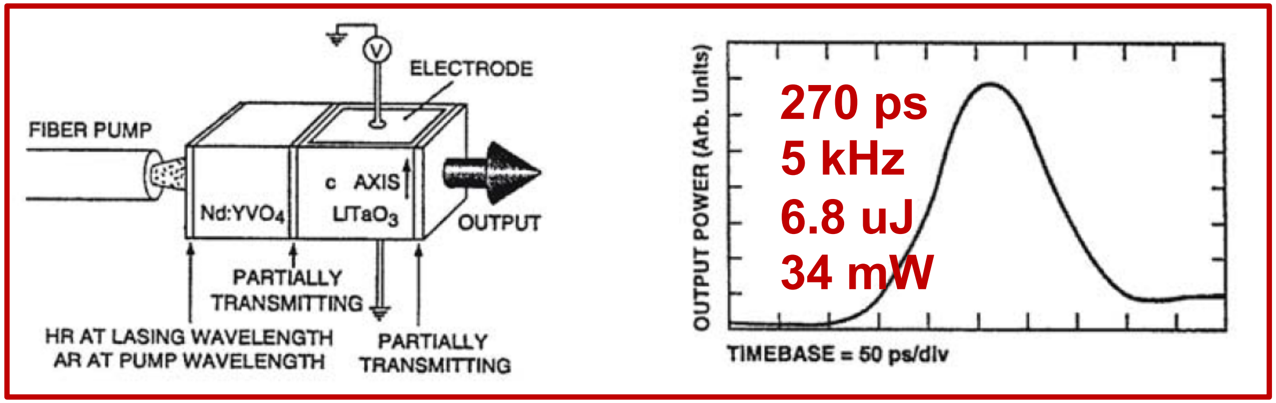
**Rate equations:**

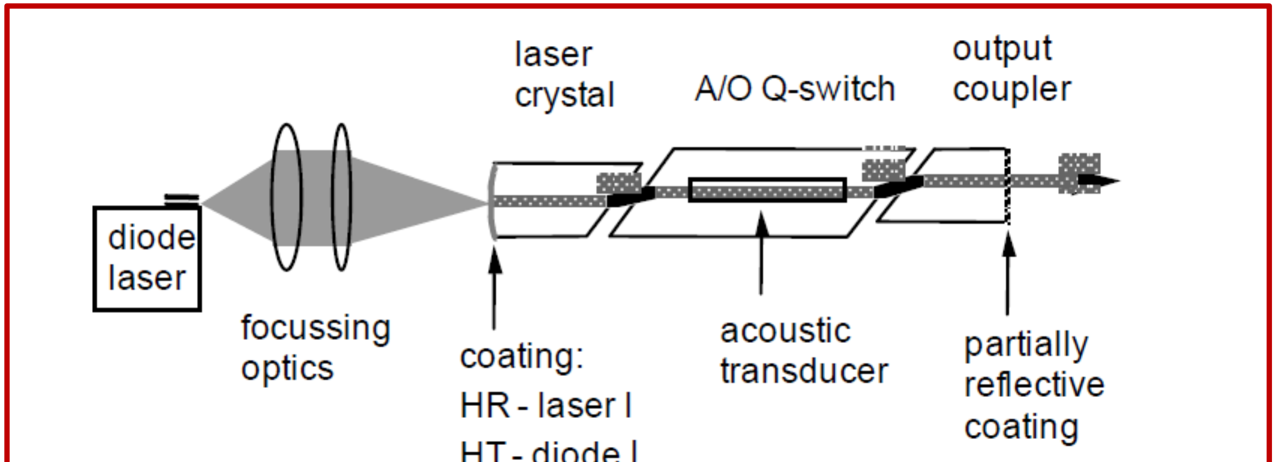
$$\frac{d}{dt}g = -\frac{g-g_0}{\tau_L} - \frac{gP}{E_{sat}} \qquad \frac{d}{dt}P = -\frac{1}{\tau_p}P + \frac{2g}{T_R}(P + P_{vac})$$

Applications for such lasers are wherever high pulse energies in nanosecond p

- Microfabrication – Metal Cutting
- Range Finding
- Tattoo Removal

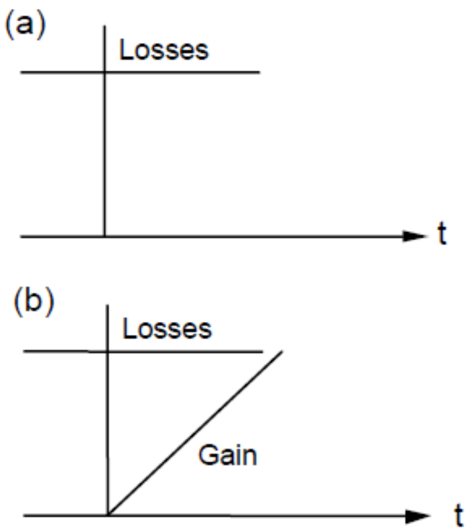
# Active Q-switching lasers: EO switch and AO switch



## Q-Switching: Step by Step – Storing Energy

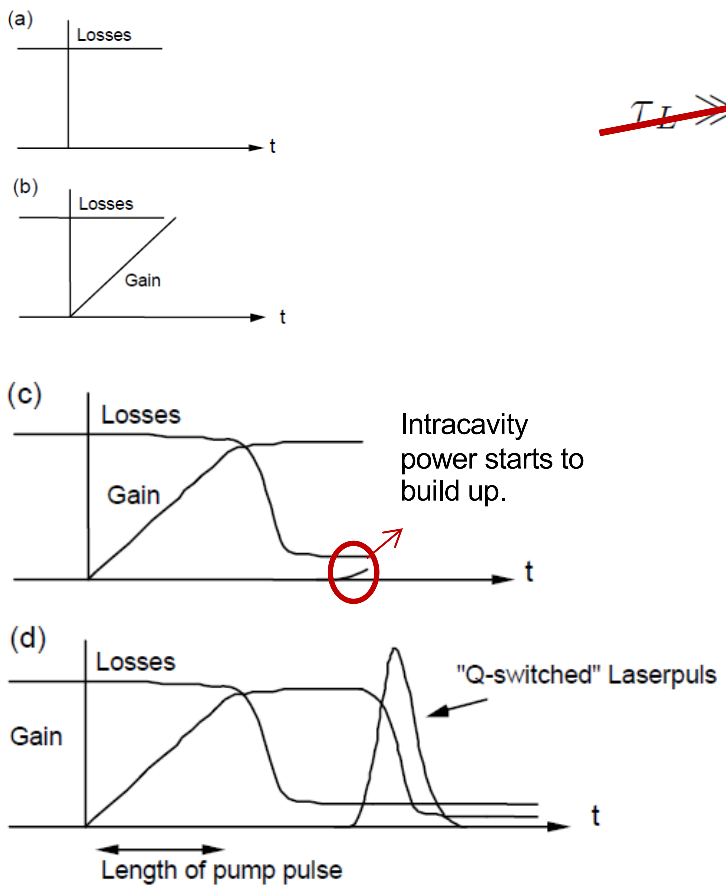
$$\tau_L \gg T_R \gg \tau_p \qquad (4.137)$$




**High losses, laser is below threshold.**

**Build-up of inversion by pumping. Pump population to upper laser level until gain approaches loss.**

# Q-Switching: Step by Step – Pulse Forming



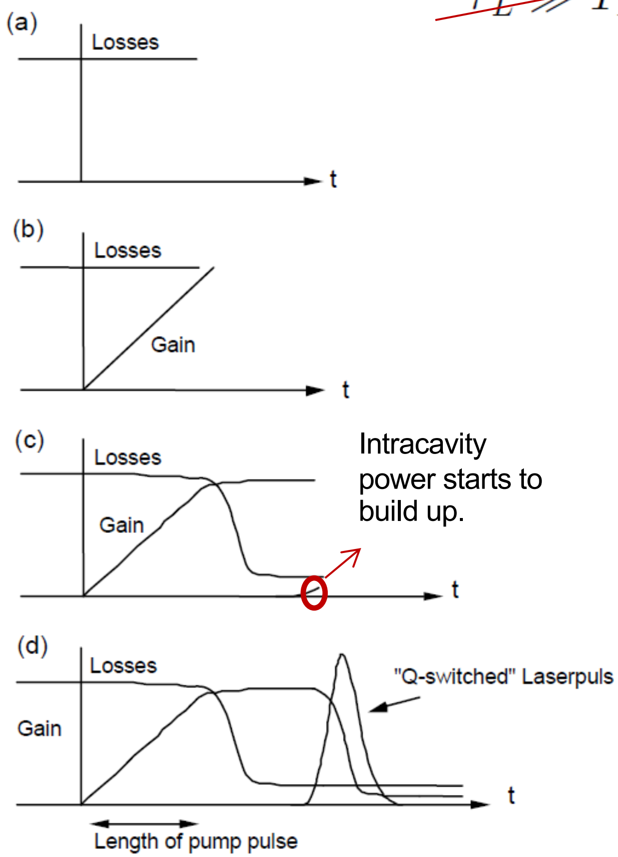

$\cancel{\tau_L \gg T_R \gg \tau_p}$ $\tau_L >> \tau_p >> T_R$ (4.137)

**Active Q-Switching starts, the losses are rapidly reduced. Pulse build up from vacuum noise/sponteneous emission starts**

**Laser emission stops after the energy stored in the gain medium is extracted.**

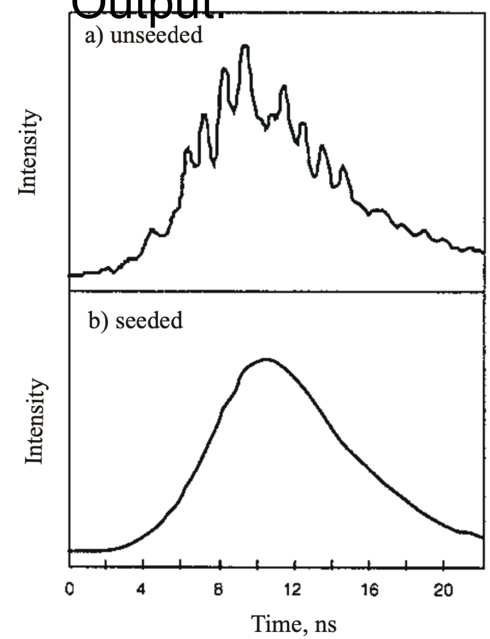
# Q-Switching: Seeded vs Unseeded

$$\cancel{\tau_L \gg T_R \gg \tau_p} \qquad \tau_L \gg \tau_p \gg T_R \qquad (4.137)$$



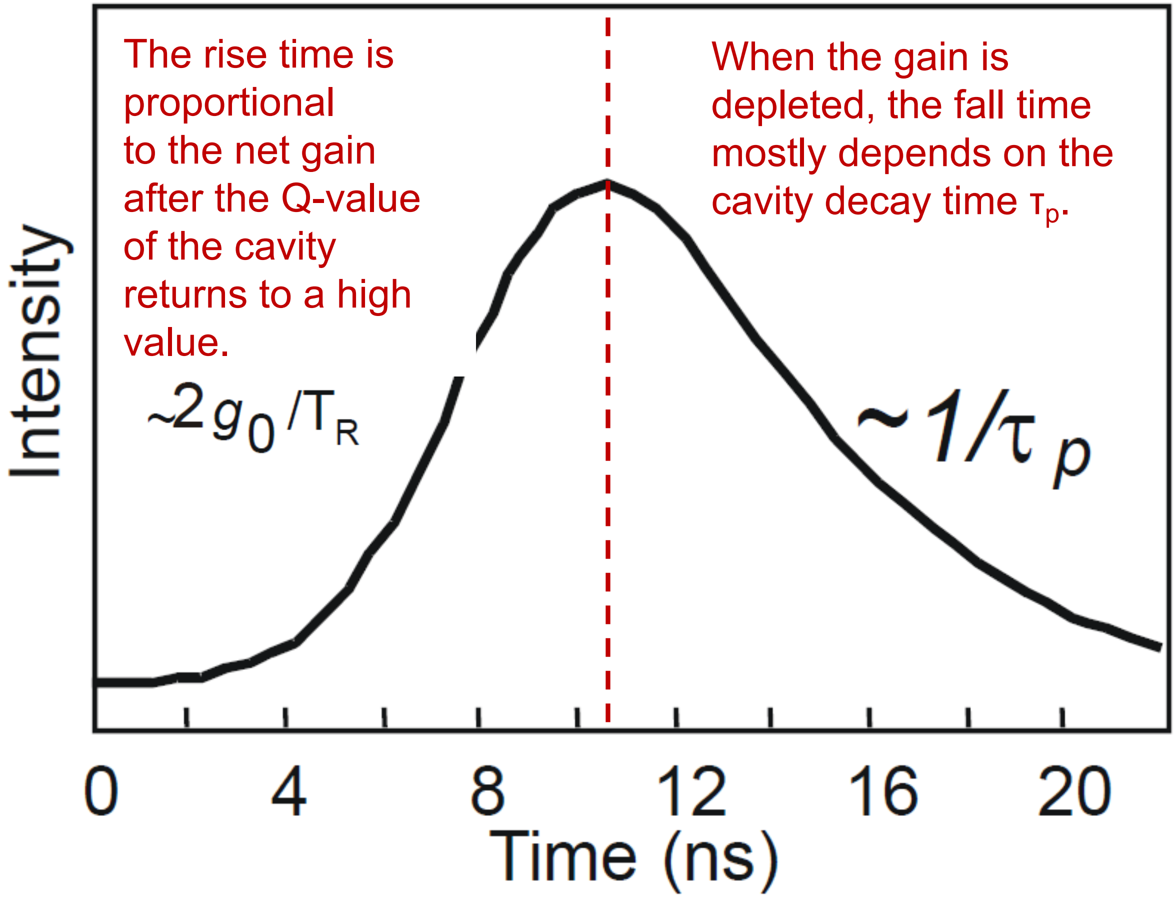

Gain and Loss Dynamics of Q-switched Laser

Output:

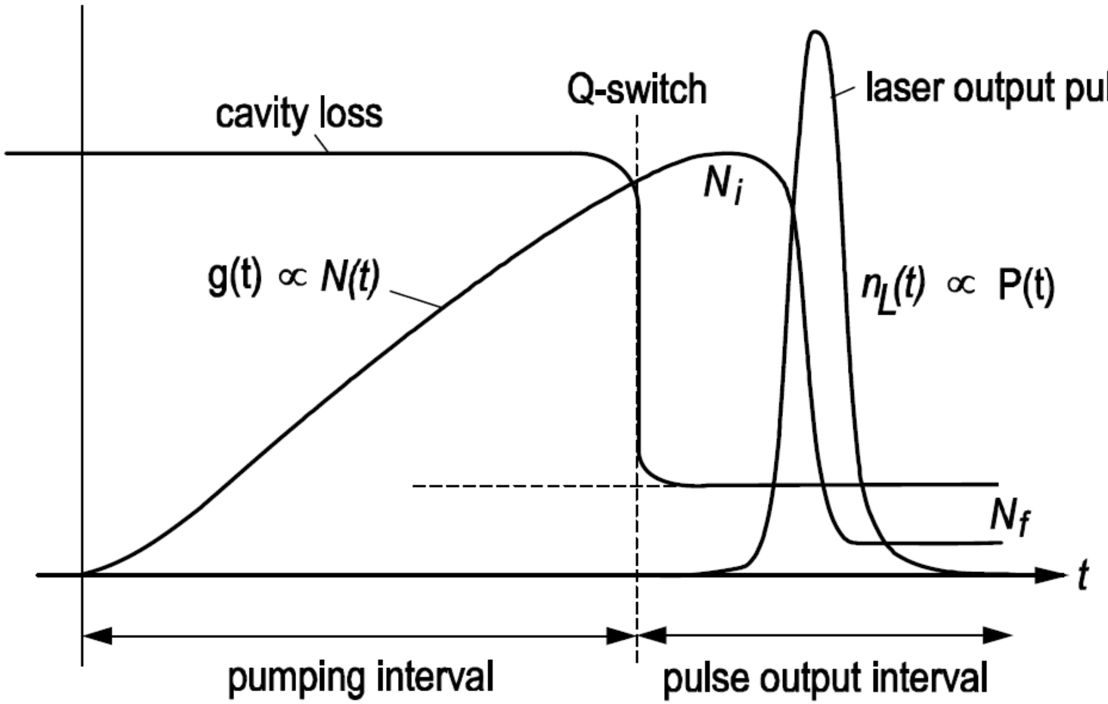



Only *single* Frequency gives a clean pulse!

## Asymmetric actively Q-switched pulse



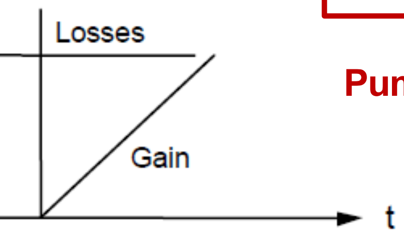
# Theory on active Q-switching




Active Q-switching dynamics assuming an instantaneous switching

# Theory on active Q-switching

**Rate equations:** $$\frac{d}{dt}g = -\frac{g-g_0}{\tau_L} - \frac{gP}{E_{sat}} \qquad \frac{d}{dt}P = -\frac{1}{\tau_p}P + \frac{2g}{T_R}(P+P_{vac})$$




**Pump interval with constant $R_p$:**

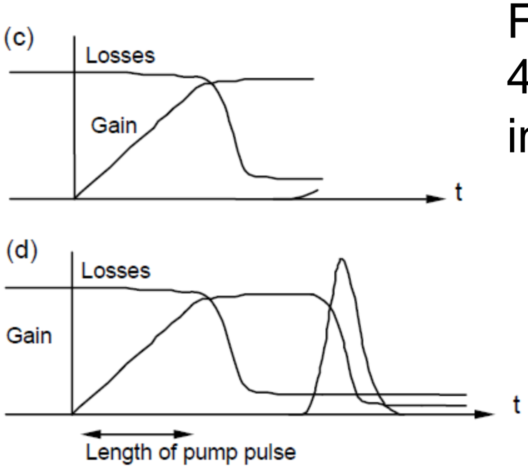
$$\frac{d}{dt}g = -\frac{g-g_0}{\tau_L} \;\rightarrow\; g(t) = g_0(1-e^{-t/\tau_L})$$

**Pulse built-up phase:**

Index is left away since there is only an upper state population.

From Eqn 4.101 and 4.105, we know the initial gain:

$$g_i = hf_L N_{2i}/(2E_{sat}) = hf_L N_i/(2E_{sat})$$




Assume that during pulse buildup, stimulated emission rate is the dominant term changing the inversion:

$$\frac{d}{dt}g = -\frac{gP}{E_{sat}}$$
$$\frac{d}{dt}P = \frac{2(g-l)}{T_R}P$$

$$\longrightarrow \quad \frac{dP}{dg} = \frac{2E_{sat}}{T_R}\left(\frac{l}{g}-1\right)$$

# Theory on active Q-switching

**Initial conditions:** $g(t=0) = g_i = r \cdot l$ $\qquad P(t=0) = 0$

How many times the laser is pumped above the threshold after the Q-switch is operated.

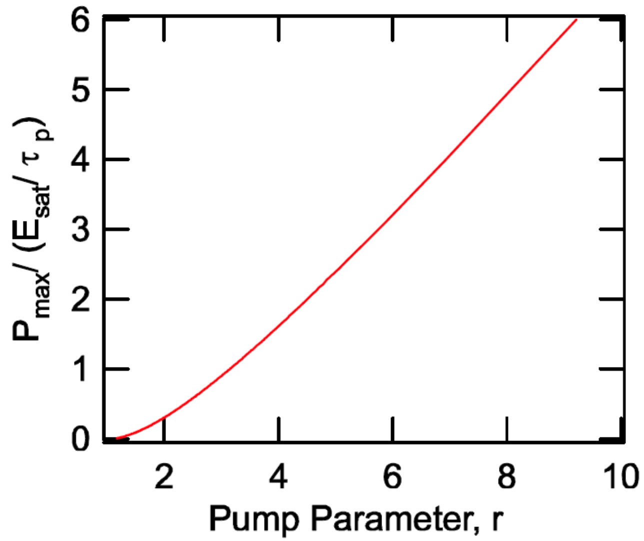
Intra-cavity loss after the Q-switch is operated.

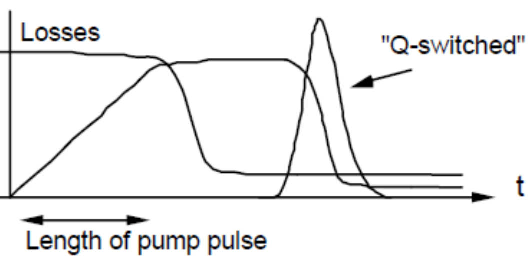
**Intra-cavity power evolution:**

$$P(t) = \frac{2E_{sat}}{T_R}\left(g_i - g(t) + l \ln \frac{g(t)}{g_i}\right)$$

**Maximum power as gain equals loss:**

$$P_{\max} = \frac{2lE_{sat}}{T_R}(r - 1 - \ln r) = \frac{E_{sat}}{\tau_p}(r - 1 - \ln r)$$



# Energy extraction

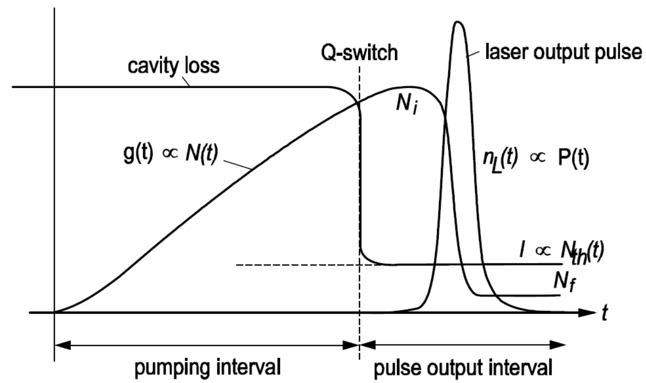
**Final gain when power vanishes $(P(t) = 0!)$:**

$$\left(g_i - g_f + l \ln\left(\frac{g_f}{g_i}\right)\right) = 0 \qquad r = g_i / l$$

$$\downarrow$$

$$1 - \frac{g_f}{g_i} + \frac{1}{r} \ln\left(\frac{g_f}{g_i}\right) = 0$$

$$1 - \frac{N_f}{N_i} + \frac{1}{r} \ln\left(\frac{N_f}{N_i}\right) = 0$$

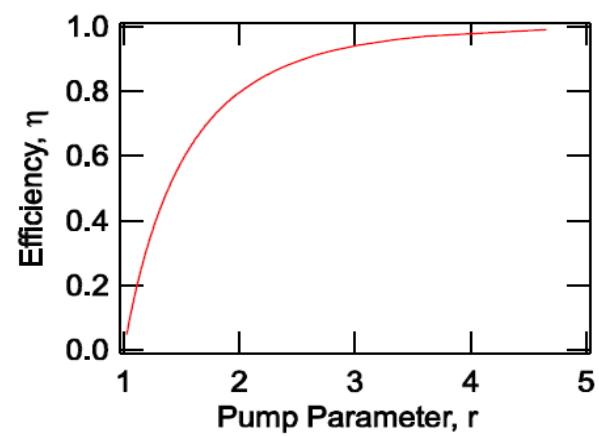



**Assuming no internal losses. the pulse energy is:** $E_P = (N_i - N_f)\, h f_L$

**Energy extraction efficiency:** $\eta = \dfrac{N_i - N_f}{N_i}$

$$\eta + \frac{1}{r} \ln(1 - \eta) = 0$$

Energy extraction efficiency only depends on the pump parameter r.



## Emitted pulse peak power:

### Estimate of pulse width

We can estimate the pulse width of the emitted pulse by the ratio between pulse energy and peak power.

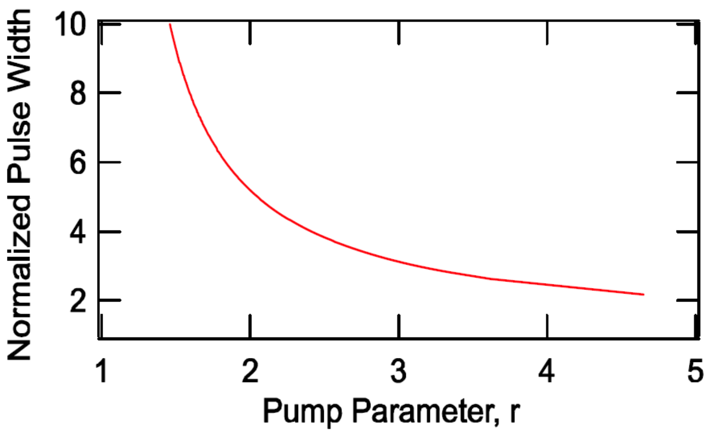
**Emitted pulse energy can be written as** $E_P = \eta(r) N_i h f_L$

Rechteck Puls

$$P_{\max} = \frac{E_{sat}}{\tau_p}(r - 1 - \ln r)$$

$$\begin{aligned}
\tau_{Pulse} &= \frac{E_P}{2l P_{peak}} \\
&= \tau_p \frac{\eta(r)}{(r-1-\ln r)} \frac{N_i h f_L}{2l E_{sat}} \\
&= \tau_p \frac{\eta(r)}{(r-1-\ln r)} \frac{g_i}{l} \\
&= \tau_p \frac{\eta(r) \cdot r}{(r-1-\ln r)}
\end{aligned}$$

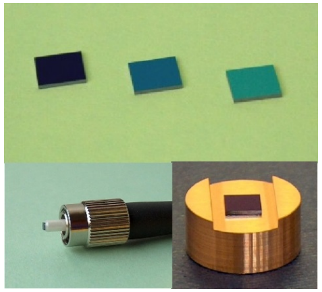
$$\boxed{g_i = h f_L N_i / (2E_{sat})}$$

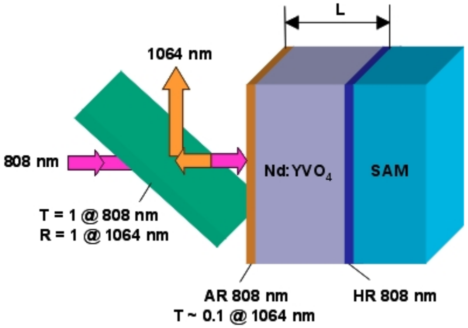


# Passive Q-Switching: manage cavity loss using saturable absorber

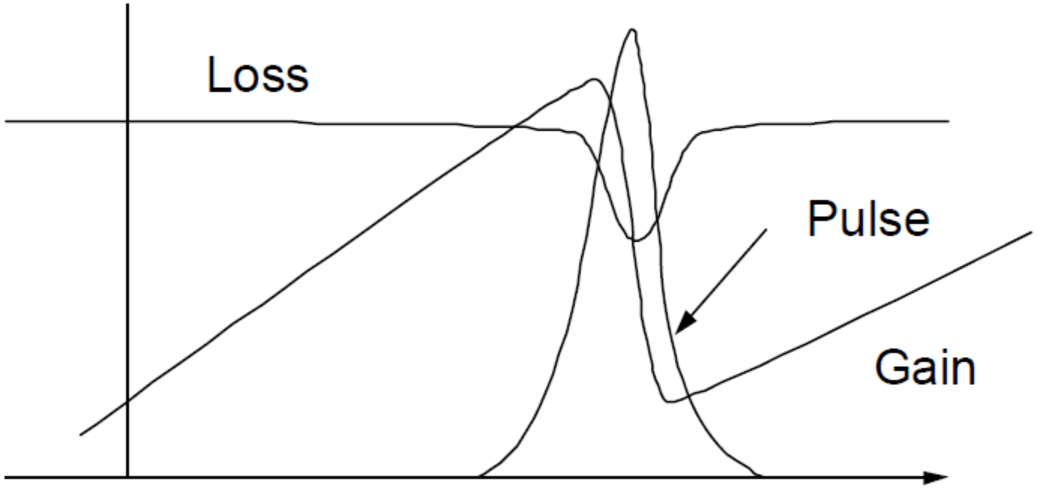
**Saturable absorber**: an optical passive device, which introduces large loss for low optical intensities and small loss at high optical intensities.

$$T_R \frac{dq}{dt} = -\frac{q - q_0}{T_A} - \frac{q T_R P}{E_A}$$

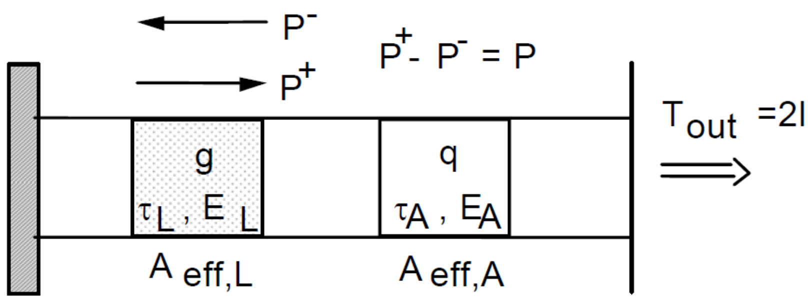





http://www.batop.de/products/products.html



# Modeling of passively Q-switched laser




We assume small output coupling so that the laser power within one roundtrip can be considered position inde

## Rate equations for a passively Q-switched laser

Assume that the changes in the laser intensity, gain and saturable absorption are small on a time scale on the order of the round-trip time $T_R$ in the cavity, (i.e. less than

$$T_R \frac{dP}{dt} = 2(g - l - q)P$$

$$T_R \frac{dg}{dt} = -\frac{g - g_0}{T_L} - \frac{g T_R P}{E_L}$$

$$T_R \frac{dq}{dt} = -\frac{q - q_0}{T_A} - \frac{q T_R P}{E_A}$$

Normalized upper-state lifetime of the gain medium and the absorber recovery time

$$T_L = \tau_L / T_R$$

$$T_A = \tau_A / T_R$$

Saturation energies of the gain and the absorber

$$E_L = h\nu A_{eff,L} / 2^* \sigma_L$$

$$E_A = h\nu A_{eff,A} / 2^* \sigma_A$$

# Passively Q-switched laser: fast saturable absorber

**Typical solid-state lasers:** $\tau_L$=100 μs  $T_R$=10 ns  $\tau_A$=1-100 ps

$T_L \approx 10^4 \quad T_A \approx 10^{-4} \text{ to } 10^{-2}$

**Fast Saturable Absorber**: $T_A << T_L$, the absorber will follow the instantaneous laser power:

$$T_R \frac{dq}{dt} = -\frac{q - q_0}{T_A} - \frac{q T_R P}{E_A} \xrightarrow{\text{Adiabatic solution}} q = \frac{q_0}{1 + P/P_A} \quad \text{with} \quad P_A = \frac{E_A}{\tau_A}$$

saturation power

**New equations of motion:**

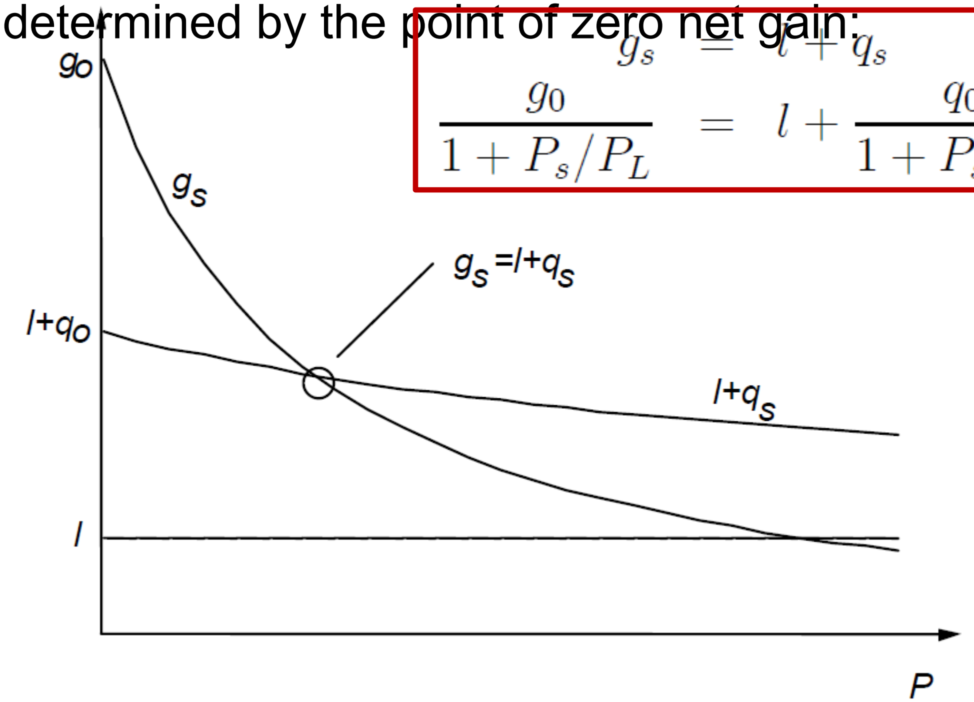
$$T_R \frac{dP}{dt} = 2(g - l - q(P))P$$

$$T_R \frac{dg}{dt} = -\frac{g - g_0}{T_L} - \frac{g T_R P}{E_L}$$

## Passively Q-switched laser: stationary solution

As in the case for the cw-running laser the stationary operation point of the laser is determined by the point of zero net gain:

$$\boxed{\begin{aligned} g_s &= l + q_s \\ \frac{g_0}{1+P_s/P_L} &= l + \frac{q_0}{1+P_s/P_A} \end{aligned}}$$




Graphical solution of the stationary operating point

## Stability of stationary operating point: Passive Q-switching

To find the stability criterion, we linearize the system just as we have done for laser CW operation:

$$T_R \frac{d}{dt}\begin{pmatrix}\Delta P_0 \\ \Delta g_0\end{pmatrix} = A \begin{pmatrix}\Delta P_0 \\ \Delta g_0\end{pmatrix}, \text{ with } A = \begin{pmatrix} -2\left.\frac{dq}{dP}\right|_{cw} P_s & 2P_s \\ -\frac{g_s T_R}{E_L} & -\frac{T_R}{\tau_{stim}} \end{pmatrix}$$

We look for the eigen solution:

$$\frac{d}{dt}\begin{pmatrix}\Delta P_0(t) \\ \Delta g_0(t)\end{pmatrix} = s \begin{pmatrix}\Delta P_0(t) \\ \Delta g_0(t)\end{pmatrix}$$

$$\boxed{s = \frac{1}{2}\left(\gamma_Q - \frac{1}{\tau_{stim}}\right) \pm j\sqrt{\gamma_Q \frac{r}{\tau_L} + \frac{r-1}{\tau_p \tau_L} - \left(\frac{\gamma_Q - \frac{1}{\tau_{stim}}}{2}\right)^2}}$$

Growth rate introduced by the saturable absorber that destabilizes the laser relaxation oscillation:

$$\gamma_Q = -\frac{2}{T_R}\left.\frac{dq}{dP}\right|_{cw} P_s$$

Q-switching happens w $\gamma_Q > \dfrac{1}{\tau_{stim}}$

## Passive Q-switching: a numerical example

$\tau_L$=250µs, $T_R$=4ns, 2l=0.1, $2q_o$=0.005, $2g_o$=2, $P_L/P_A$=100.

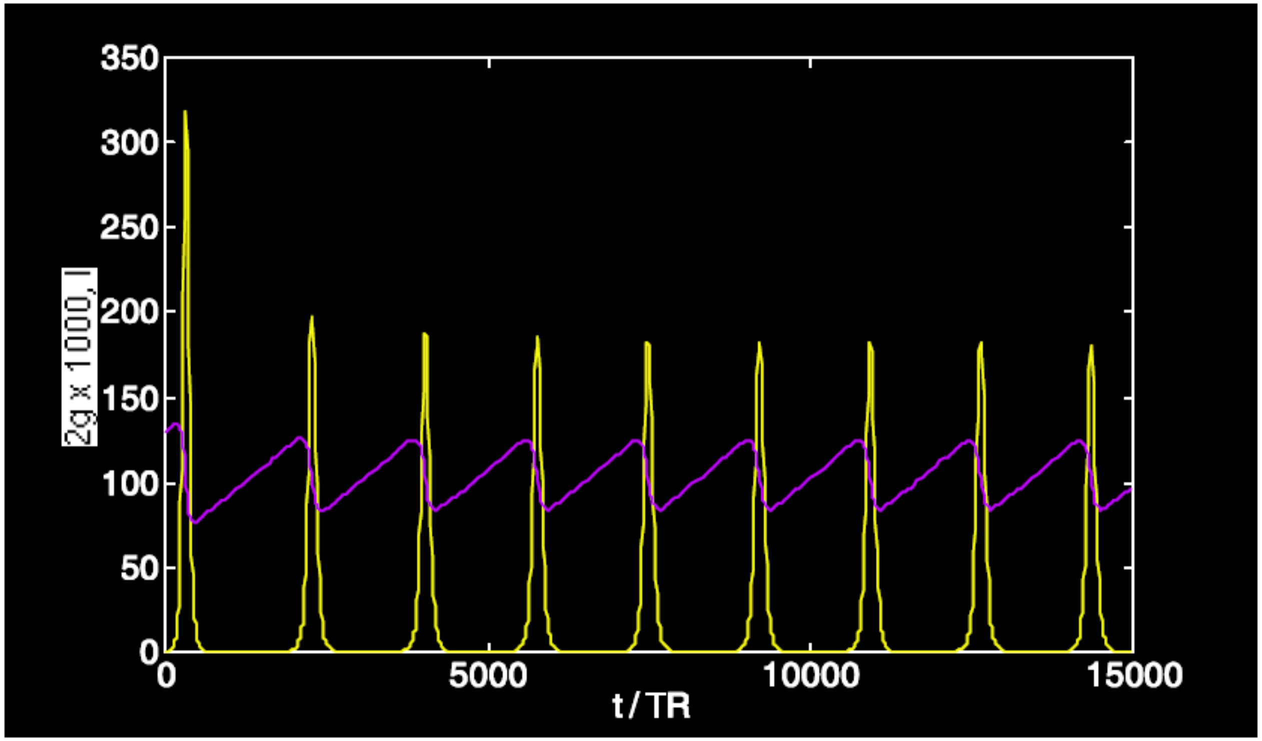
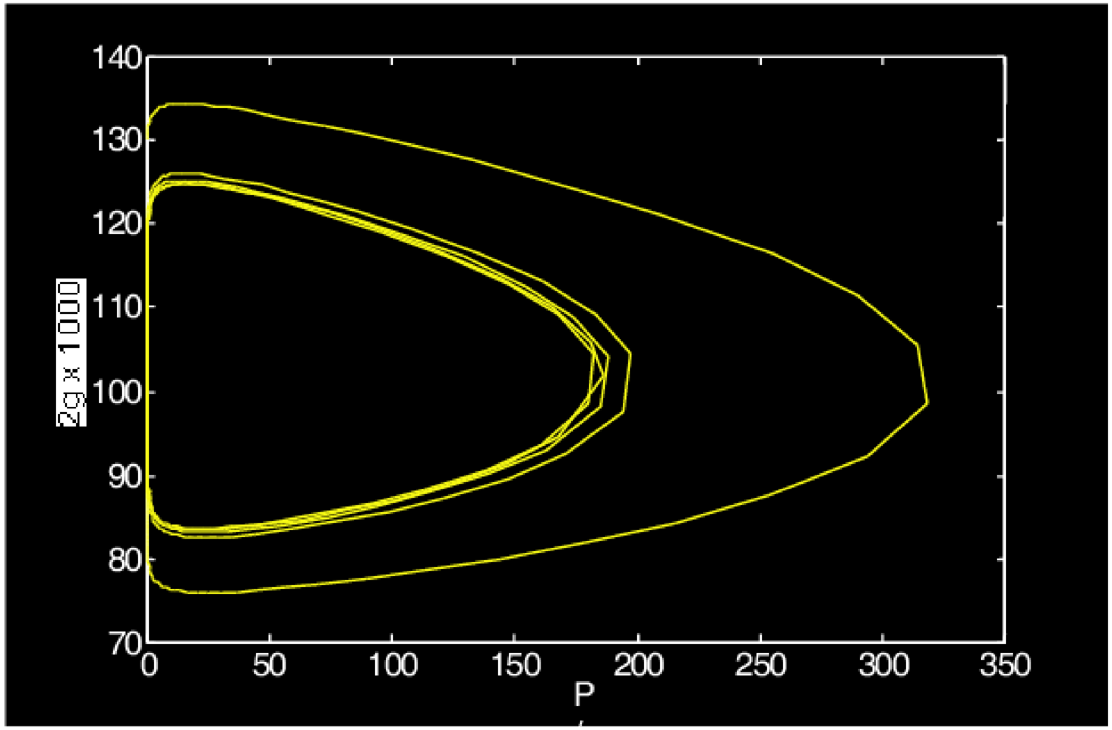



Fig. 4.28: Gain and output power as a function of time.




$\tau_L$=250μs, $T_R$=4ns, 2l=0.1, $2q_o$=0.005, $2g_o$=2, $P_L/P_A$=100.

Fig. 4.27: Phase space solution for rate equations.