Ultrafast Optical Physics II (SoSe 2020)
Lecture 2, May 8

Dielectric susceptibility and Helmholtz equation

$$\vec{P}(\vec{r},t) = \epsilon_0 \int dt' \, \chi(t-t') \, \vec{E}(\vec{r},t') \Longrightarrow \tilde{\vec{P}}(\vec{r},\omega) = \epsilon_0 \tilde{\chi}(\omega) \tilde{\vec{E}}(\vec{r},\omega)$$

$$\left(\Delta - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = \mu_0 \frac{\partial^2}{\partial t^2}\vec{P} \Longrightarrow \left(\Delta + \frac{\omega^2}{c_0^2}\right)\tilde{\vec{E}}(\omega) = -\omega^2 \mu_0 \epsilon_0 \tilde{\chi}(\omega) \tilde{\vec{E}}(\omega)$$

In a linear medium, dielectric susceptibility is independent of optical field

$$\left(\Delta + \frac{\omega^2}{c_0^2}(1+\tilde{\chi}(\omega))\right)\tilde{\vec{E}}(\omega) = 0$$

$$1 + \chi(\omega) = n^2(\omega)$$

Can be complex

Refractive Index

Medium speed of light (dependent on frequency):

$$c(\omega) = c_0/\tilde{n}(\omega)$$

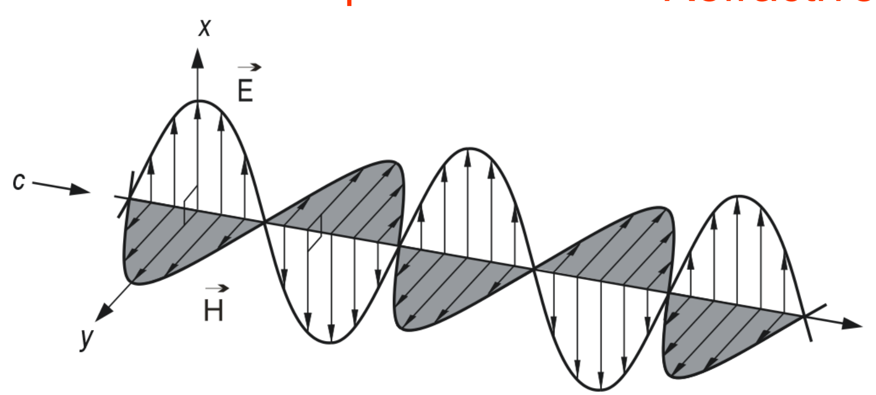


Figure 2.1: Transverse electromagnetic wave (TEM) [2]

Susceptibility calculated using Lorentz model

$$\underline{\tilde{\chi}}(\omega) = \tilde{\chi}_r(\omega) + \mathrm{j}\tilde{\chi}_i(\omega)$$

$$\omega_p = \left(\frac{Ne^2}{\varepsilon_0 m_0}\right)^{1/2}$$

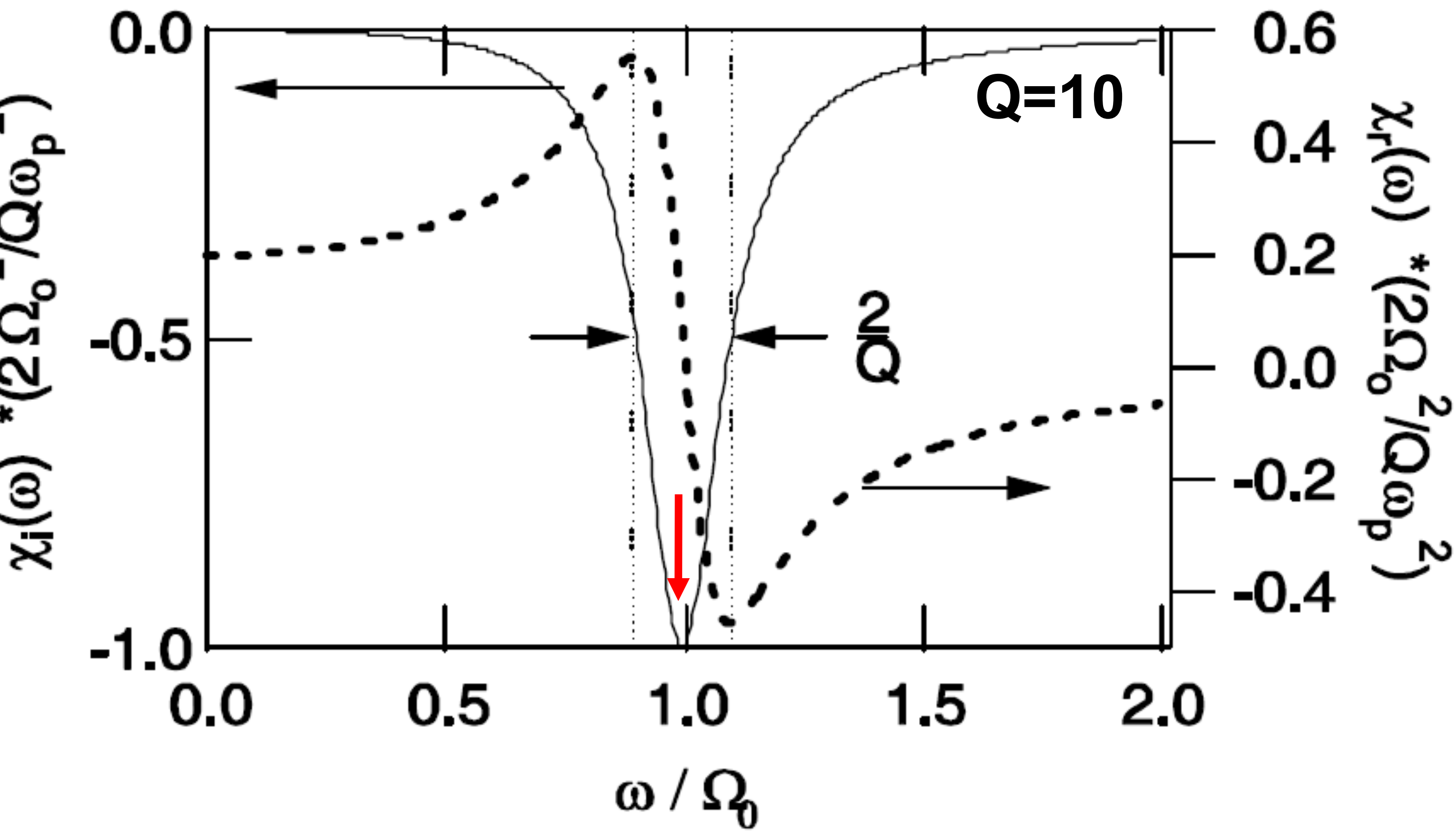
Plasma frequency

$$\tilde{\chi}_r(\omega) = \omega_p^2 \cdot \frac{\left(\Omega_0^2 - \omega^2\right)}{\left(\Omega_0^2 - \omega^2\right)^2 + \left(2\omega\frac{\Omega_0}{Q}\right)^2}$$

$$\tilde{\chi}_i(\omega) = -\omega_p^2 \cdot \frac{2\omega\frac{\Omega_0}{Q}}{\left(\Omega_0^2 - \omega^2\right)^2 + \left(2\omega\frac{\Omega_0}{Q}\right)^2}$$

Real and imaginary part of the susceptibility

$$\underline{\widetilde{\chi}}(\omega) = \widetilde{\chi}_r(\omega) + j\widetilde{\chi}_i(\omega)$$



Real part (dashed line) and imaginary part (solid line) of the susceptibility of the classical oscillator model for the dielectric polarizability

Real and Imaginary Part of the Susceptibility

$$\underline{\tilde{\chi}}(\omega) = \tilde{\chi}_r(\omega) + j\tilde{\chi}_i(\omega)$$

Example: EM-Wave polarized along x-axis and propagation along z-direction:

$$\underline{\vec{E}}(z,t) = \underline{E}_0 e^{j(\omega t - kz)}\ \vec{e}_x$$

$\sqrt{-1} = \mathrm{i}$	Physics notation
$\sqrt{-1} = \mathrm{j}$	Engineering notation

In general:

$$k(\omega) = \frac{\omega}{c_0}\tilde{n}(\omega) = \frac{\omega}{c_0}\sqrt{1+\tilde{\chi}(\omega)} = \frac{\omega}{c_0}\left(\tilde{n}_r(\omega) + j\tilde{n}_i(\omega)\right) = k_r(\omega) - j\alpha(\omega)$$

$$\boxed{\underline{\vec{E}}(z,t) = \underline{E}_0 e^{-\alpha \cdot z}\ e^{j(\omega t - k_r z)}\ \vec{e}_x}$$

damping

$$k(\omega) = \frac{\omega}{c_0}\sqrt{1+\tilde{\chi}(\omega)}$$

for: $\tilde{\chi}(\omega) \ll 1$

$$= \frac{\omega}{c_0}\left(1 + \frac{1}{2}\tilde{\chi}(\omega)\right) = \frac{\omega}{c_0}\left(1 + \frac{1}{2}\tilde{\chi}_r(\omega) + \frac{1}{2}j\tilde{\chi}_i(\omega)\right)$$

In a Metal

Free electrons between background ions

$$m\frac{d^2x}{dt^2} + \cancel{2\frac{\Omega_0}{Q}m\frac{dx}{dt}} + \cancel{m\Omega_0^2 x} = e_0 E(t), \tag{2.41}$$

In general:

$$\tilde{\chi}_r(\omega) = \omega_p^2 \cdot \frac{\left(\Omega_0^2 - \omega^2\right)}{\left(\Omega_0^2 - \omega^2\right)^2 + \left(2\omega\frac{\Omega_0}{Q}\right)^2} \rightarrow -\frac{\omega_p^2}{\omega^2}$$

$$\tilde{\chi}_i(\omega) = 0$$

$$k(\omega) = \frac{\omega}{c_0}\tilde{n}(\omega) = \frac{\omega}{c_0}\sqrt{1+\tilde{\chi}(\omega)} = \frac{\omega}{c_0}\sqrt{1-\frac{\omega_p^2}{\omega^2}}$$

$\omega < \omega_p$: Metal reflects and for $\omega > \omega_p$: "transparent"

2.5 Sellmeier Equations and Kramers-Kroenig Relations

Causality of medium impulse response: $\chi(t) = 0, \text{ for } t < 0$

Leads to relationship between real and imaginary part of susceptibility

$$\chi_r(\Omega) = \frac{2}{\pi}\int_0^\infty \frac{\omega\chi_i(\omega)}{\omega^2 - \Omega^2} d\omega = n_r^2(\Omega) - 1$$

$$\chi_i(\Omega) = -\frac{2}{\pi}\int_0^\infty \frac{\Omega\chi_r(\omega)}{\omega^2 - \Omega^2} d\omega.$$

Approximation for absorption spectrum in a medium:

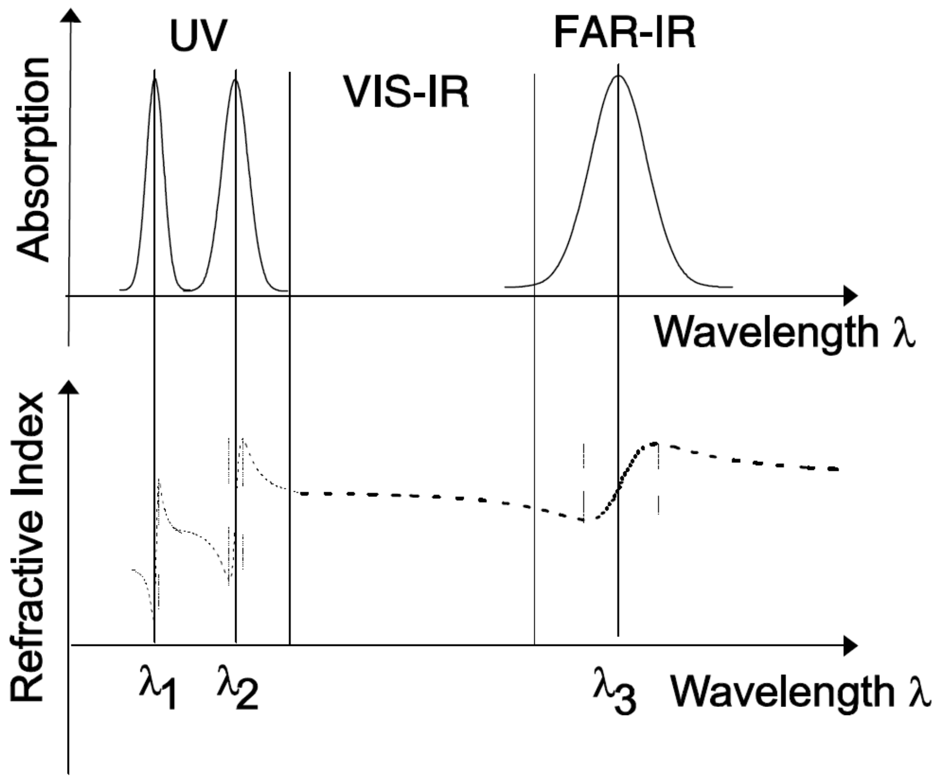
$$\chi_i(\Omega) = \sum A_i \delta\left(\omega - \omega_i\right)$$

$$n^2(\Omega) = 1 + \sum_i A_i \frac{\omega_i}{\omega_i^2 - \Omega^2} = 1 + \sum_i a_i \frac{\lambda}{\lambda^2 - \lambda_i^2}$$

$$\chi_r(\Omega)$$

In a Dielectric

Absorption and refractive index Vs. wavelength



$$\text{Classical Optics} \begin{cases} \frac{dn}{d\lambda} < 0 : \text{normal dispersion (blue refracts more than red)} \\ \frac{dn}{d\lambda} > 0 : \text{anomalous dispersion} \end{cases}$$

$$\text{Ultrafast Optics} \begin{cases} \frac{d^2n}{d\lambda^2} > 0 : \text{normal dispersion} \\ \text{short wavelengths slower than long wavelengths} \\ \frac{d^2n}{d\lambda^2} < 0 : \text{anomalous dispersion} \\ \text{short wavelengths faster than long wavelengths} \end{cases}$$

Example: Sellmeier Coefficients for Fused Quartz and Sapphire

	Fused Quartz	Sapphire
a_1	0.6961663	1.023798
a_2	0.4079426	1.058364
a_3	0.8974794	5.280792
λ_1^2	$4.679148 \cdot 10^{-3}$	$3.77588 \cdot 10^{-3}$
λ_2^2	$1.3512063 \cdot 10^{-2}$	$1.22544 \cdot 10^{-2}$
λ_3^2	$0.9793400 \cdot 10^{2}$	$3.213616 \cdot 10^{2}$

Table 2.3: Table with Sellmeier coefficients for fused quartz and sapphire.

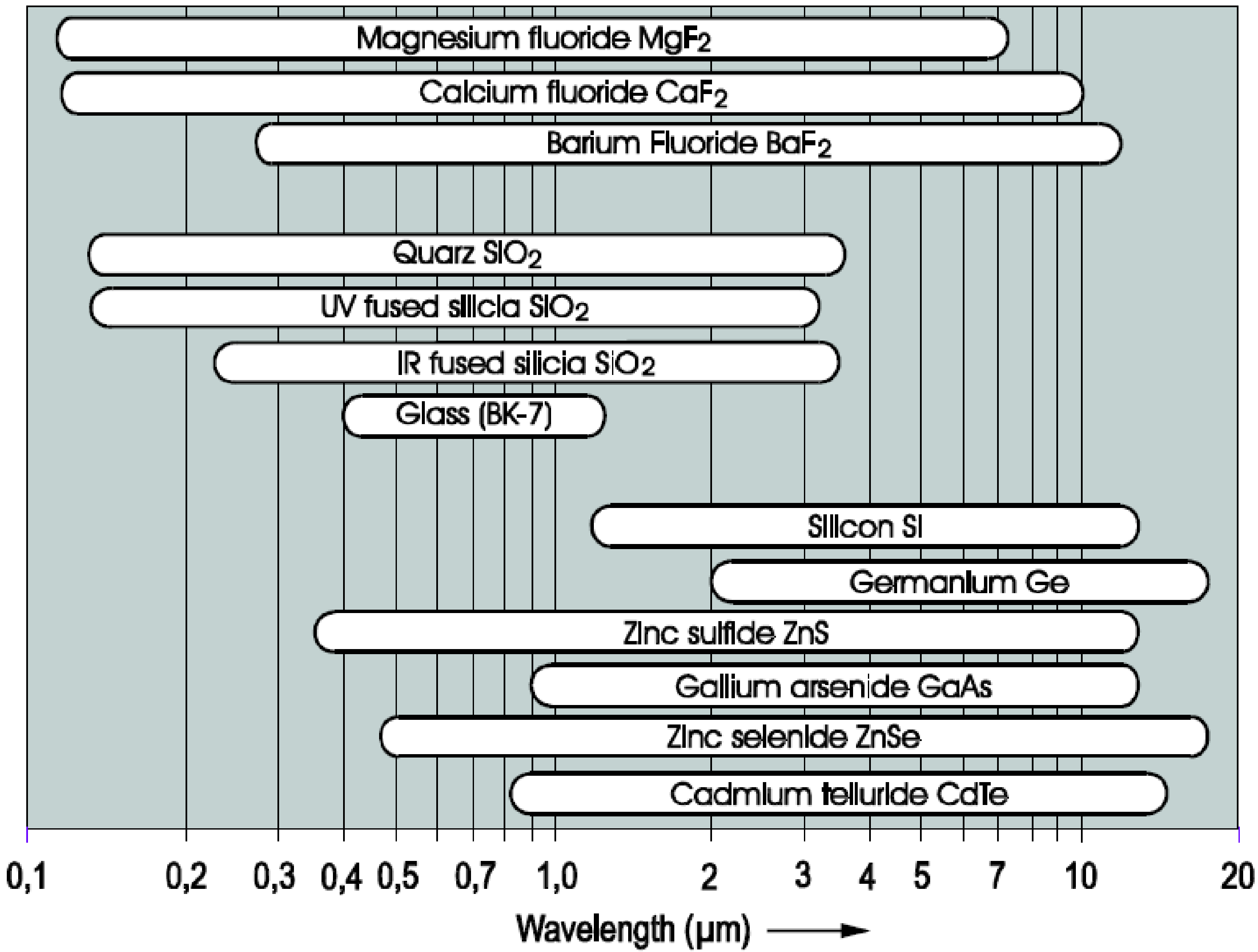


Figure 2.16: Transparency range of some materials according to Saleh and Teich, Photonics p. 175.

2.1.5 Optical Pulses (propagating along z-axis)

$$\underline{\vec{E}}(\vec{r},t) = \int_0^\infty \frac{d\Omega}{2\pi} \underline{\widetilde{E}}(\Omega) e^{\mathrm{j}(\Omega t - K(\Omega)z)} \ \vec{e}_x$$

$$\underline{\vec{H}}(\vec{r},t) = \int_0^\infty \frac{d\Omega}{2\pi Z_F(\Omega)} \underline{\widetilde{E}}(\Omega) e^{\mathrm{j}(\Omega t - K(\Omega)z)} \ \vec{e}_y$$

$$\vec{E}(\vec{r},t) = \frac{1}{2}\left(\underline{\vec{E}}(\vec{r},t) + \underline{\vec{E}}(\vec{r},t)^*\right)$$

$$\vec{H}(\vec{r},t) = \frac{1}{2}\left(\underline{\vec{H}}(\vec{r},t) + \underline{\vec{H}}(\vec{r},t)^*\right)$$

$|\underline{\tilde{E}}(\Omega)| e^{\mathrm{j}\varphi(\Omega)}$: Wave amplitude and phase

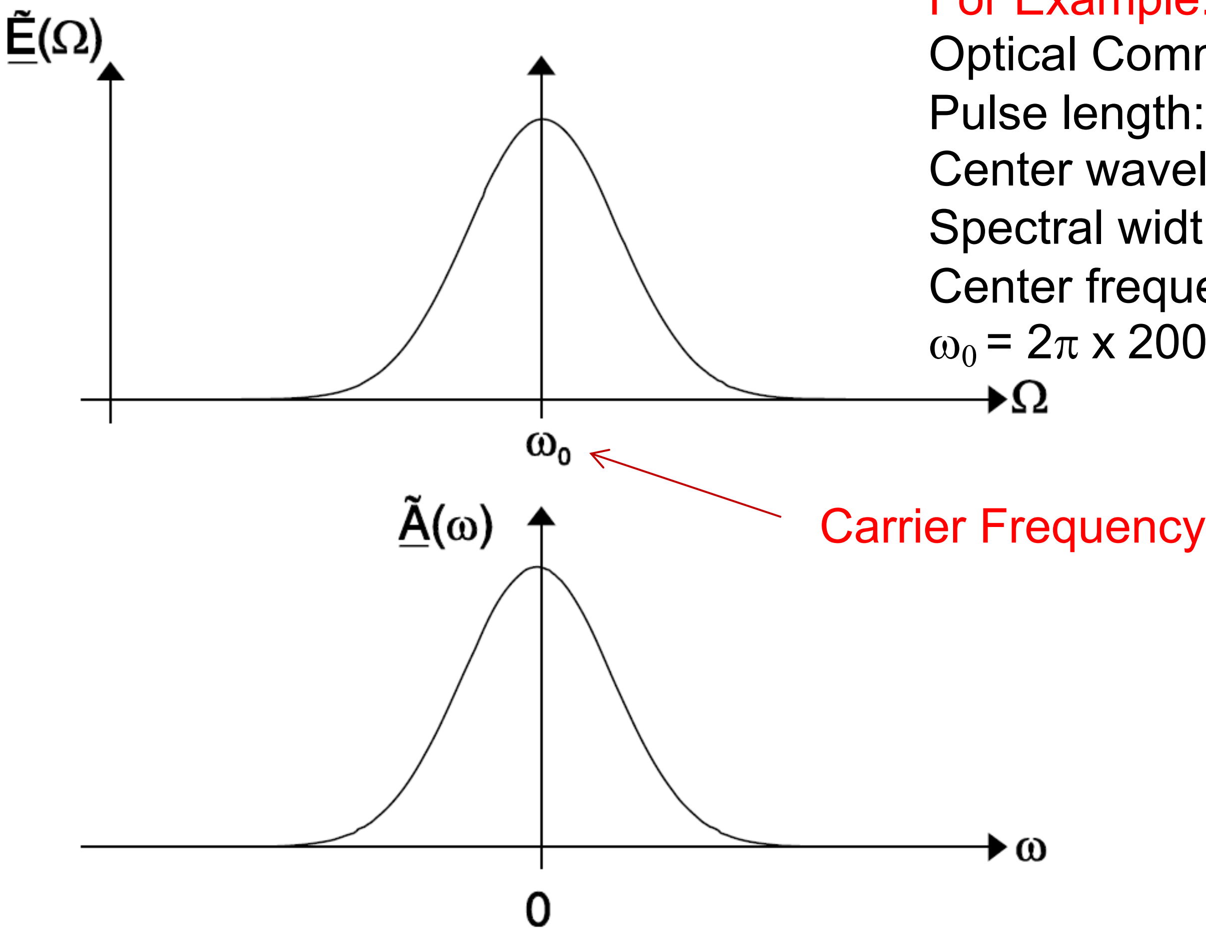
$K(\Omega) = \Omega / c(\Omega) = n(\Omega)\Omega / c_0$: Wave number

$c(\Omega) = \dfrac{c_0}{n(\Omega)}$: Phase velocity of wave

$$\tilde{n}^2(\Omega) = 1 + \tilde{\chi}(\Omega)$$

At z=0

$$\underline{E}(z=0,t)=\frac{1}{2\pi}\int_0^{\infty}\underline{\tilde{E}}(\Omega)e^{j\Omega t}d\Omega$$



For Example:

Optical Communication; 10Gb/s

Pulse length: 20 ps

Center wavelength : λ=1550 nm.

Spectral width: ~ 50 GHz,

Center frequency: 200 THz,

$\omega_0 = 2\pi \times 200 \times 10^{-12}\ s^{-1}$

Figure 2.4: Spectrum of an optical wave packet described in absolute and relative frequencies

Carrier and Envelope

$$
\begin{aligned}
\underline{E}(z &= 0,t) = \frac{1}{2\pi}\int_{-\omega_0}^{\infty} \underline{\tilde{E}}(\omega_0+\omega)e^{j(\omega_0+\omega)t}d\omega \\
&= \frac{1}{2\pi}e^{j\omega_0 t}\int_{-\omega_0}^{\infty} \underline{\tilde{E}}(\omega_0+\omega)e^{j\omega t}d\omega \\
& A(t)e^{j\omega_0 t}.
\end{aligned}
$$

Carrier Frequency

Envelope:

$$
\begin{aligned}
\underline{A}(t) &= \frac{1}{2\pi}\int_{-\omega_0\to-\infty}^{\infty} \underline{\tilde{A}}(\omega)e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underline{\tilde{A}}(\omega)e^{j\omega t}d\omega,
\end{aligned}
$$

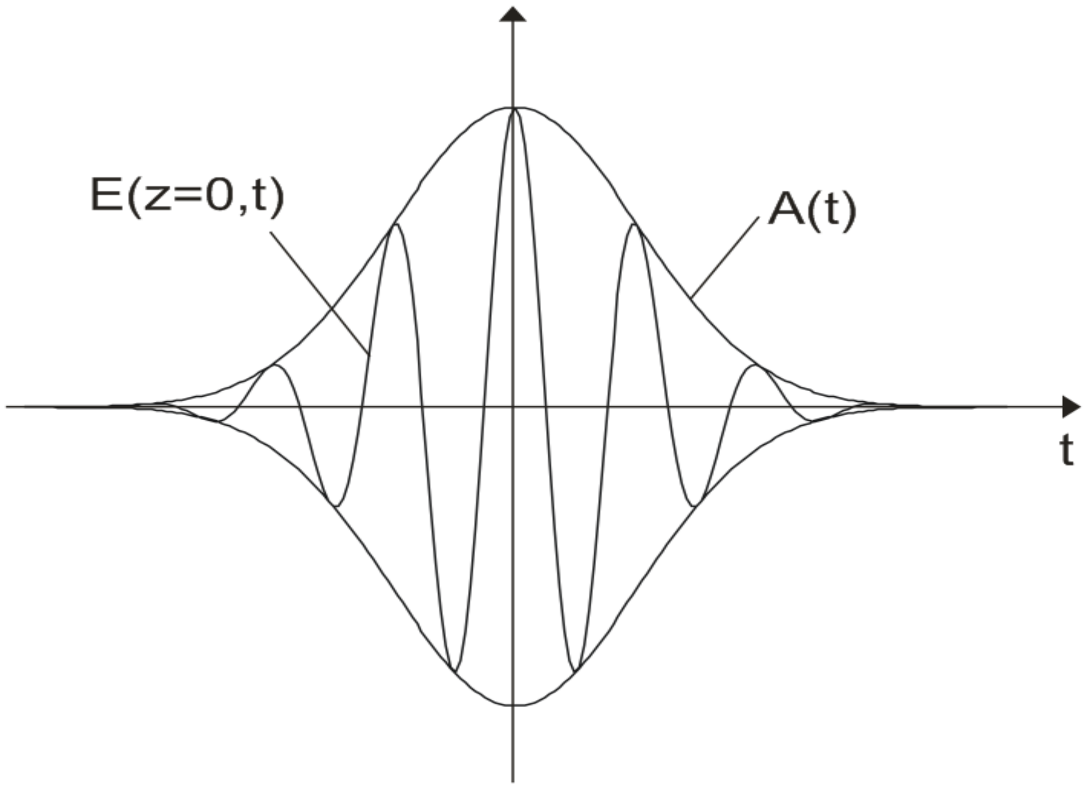


Figure 2.5: Electric field and envelope of an optical pulse

Pulse width: **Full Width at Half Maximum of $|A(t)|^2$**

Spectral width : **Full Width at Half Maximum of $|\underline{\tilde{A}}(\omega)|^2$**

2.4 Pulse Propagation

$$\underline{E}(z,t) = \frac{1}{2\pi}\int_0^\infty \underline{\tilde{E}}(\Omega)e^{\mathrm{j}(\Omega t - K(\Omega)z)}d\Omega.$$

$$\underline{E}(z,t) = \underline{A}(z,t)e^{\mathrm{j}(\omega_0 t - K(\omega_0)z)}$$

Envelope + Carrier Wave

$$\begin{aligned} \omega &= \Omega - \omega_0, \\ k(\omega) &= K(\omega_0+\omega) - K(\omega_0), \\ \underline{\tilde{A}}(\omega) &= \underline{\tilde{E}}(\Omega = \omega_0 + \omega). \end{aligned}$$

$$\underline{E}(z,t) = \frac{1}{2\pi}\int_{-\infty}^\infty \underline{\tilde{A}}(\omega)e^{\mathrm{j}(\omega t - k(\omega)z)}d\omega \; e^{\mathrm{j}(\omega_0 t - K(\omega_0)z)}$$

$$\underline{A}(z,t) = \frac{1}{2\pi}\int_{-\infty}^\infty \underline{\tilde{A}}(\omega)e^{\mathrm{j}(\omega t - k(\omega)z)}d\omega$$

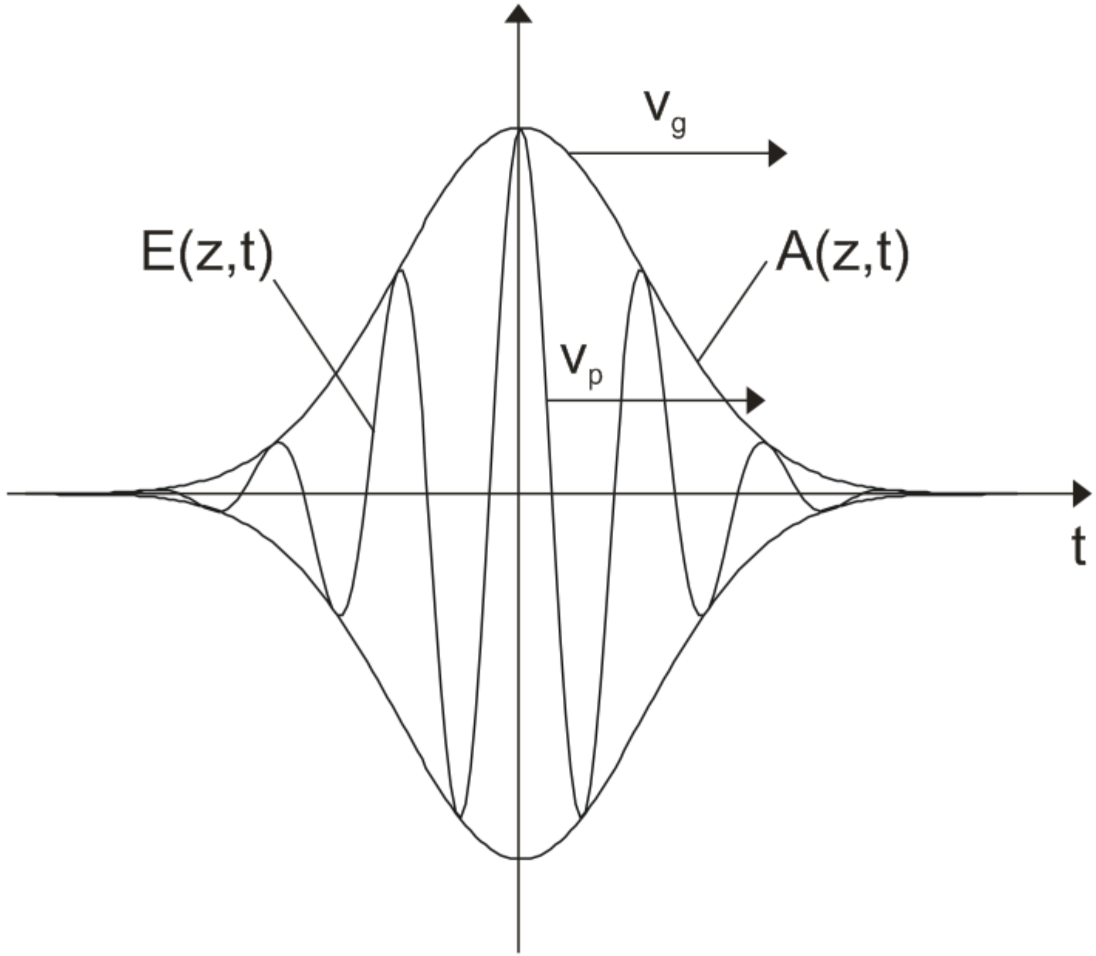


Figure 2.8: Electric field and pulse envelope in time domain

Linear pulse propagation

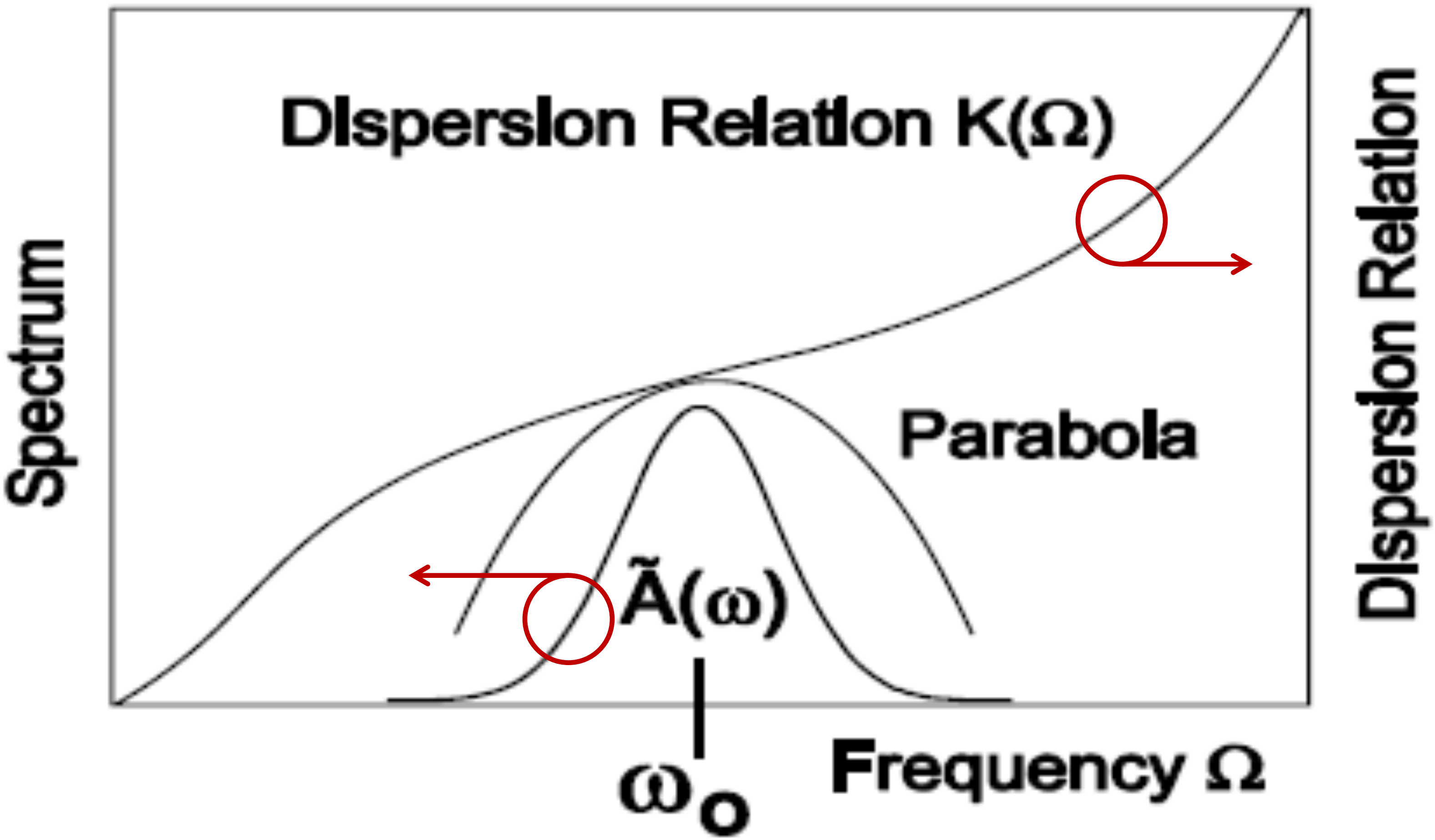


Figure 2.9: Taylor expansion of dispersion relation at the center frequency of the wave packet

2.4.1 Dispersion

In the frequency domain:

$$\underline{\tilde{A}}(z,\omega) = \underline{\tilde{A}}(z=0,\omega)e^{-\mathrm{j}k(\omega)z}$$

Taylor expansion of dispersion relation:

$$k(\omega) = k'\omega + \frac{k''}{2}\omega^2 + \frac{k^{(3)}}{6}\omega^3 + O(\omega^4)$$

Equation of motion in frequency domain:

$$\frac{\partial \underline{\tilde{A}}(z,\omega)}{\partial z} = -\mathrm{j}k(\omega)\underline{\tilde{A}}(z,\omega)$$

Equation of motion in time domain:

$$\frac{\partial \underline{A}(z,t)}{\partial z} = -\mathrm{j}\sum_{n=1}^{\infty}\frac{k^{(n)}}{n!}\left(-\mathrm{j}\frac{\partial}{\partial t}\right)^n \underline{A}(z,t)$$

i) Keep only linear term:

$$k(\omega) = k'\omega + \cancel{\frac{k''}{2}}\omega^2 + \cancel{\frac{k^{(3)}}{6}}\omega^3 + \cancel{O}(\omega^4)$$

$$\underline{\tilde{A}}(z,\omega) = \underline{\tilde{A}}(z=0,\omega)e^{-\mathrm{j}k'\omega z}$$

Time domain:

$$\underline{A}(z,t) = \underline{A}(0, t - z/\upsilon_{g0})$$

Group velocity:

$$\upsilon_{g0} = 1/k' = \left(\left.\frac{dk(\omega)}{d\omega}\right|_{\omega=0} \right)^{-1} = \left(\left.\frac{dK(\Omega)}{d\Omega}\right|_{\Omega=\omega_0} \right)^{-1}$$

Compare with phase velocity:

$$\upsilon_{p0} = \omega_0/K(\omega_0) = \left(\frac{K(\omega_0)}{\omega_0} \right)^{-1}$$

Retarded time: $t' = t - z/v_{g0}$

$$\underline{A}(z,t) = \underline{A}(0,t')$$

Or start from (2.63)

$$\frac{\partial \underline{A}(z,t)}{\partial z} + \frac{1}{v_{g0}} \frac{\partial \underline{A}(z,t)}{\partial t} = 0$$

Substitute:

$$\begin{aligned} z' &= z, \\ t' &= t - z/v_{g0}, \end{aligned} \qquad \begin{aligned} \frac{\partial}{\partial z} &= \frac{\partial}{\partial z'} - \frac{1}{v_{g0}} \frac{\partial}{\partial t'} \\ \frac{\partial}{\partial t} &= \frac{\partial}{\partial t'} \end{aligned}$$

$$\boxed{\frac{\partial \underline{A}(z',t')}{\partial z'} = 0}$$

ii) Keep up to second order term:

$$k(\omega) = k'\omega + \frac{k''}{2}\omega^2 + \cancel{\frac{k^{(3)}}{6}}\omega^3 + \cancel{O}(\omega^4)$$

$$\frac{\partial \underline{A}(z,t')}{\partial z} = \mathrm{j}\frac{k''}{2}\frac{\partial^2 \underline{A}(z,t')}{\partial t'^2}.$$

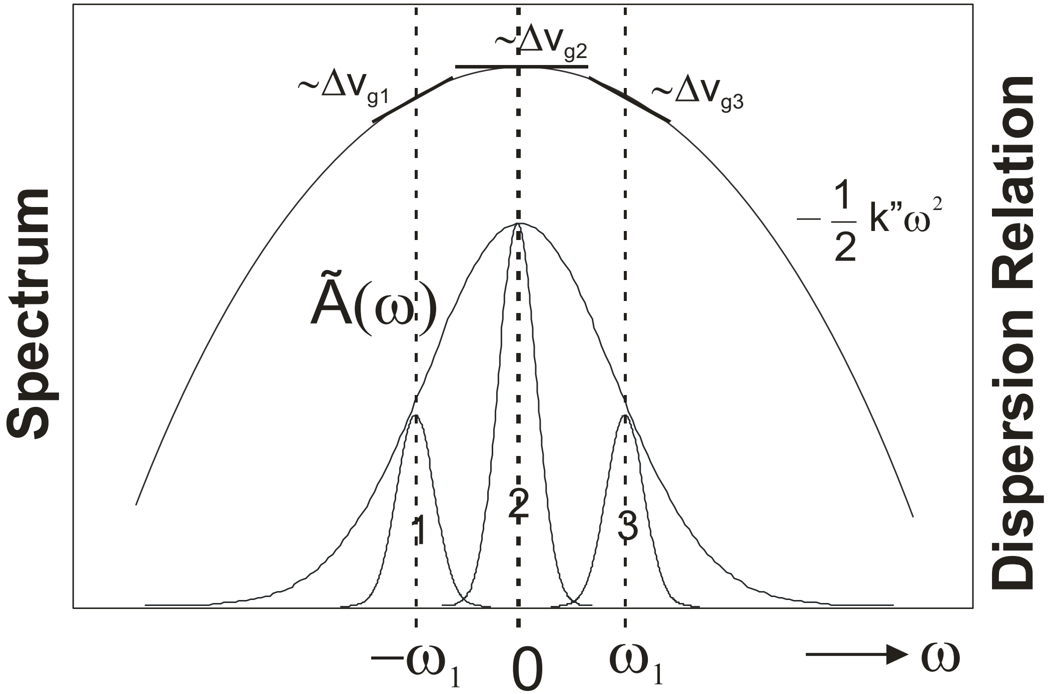


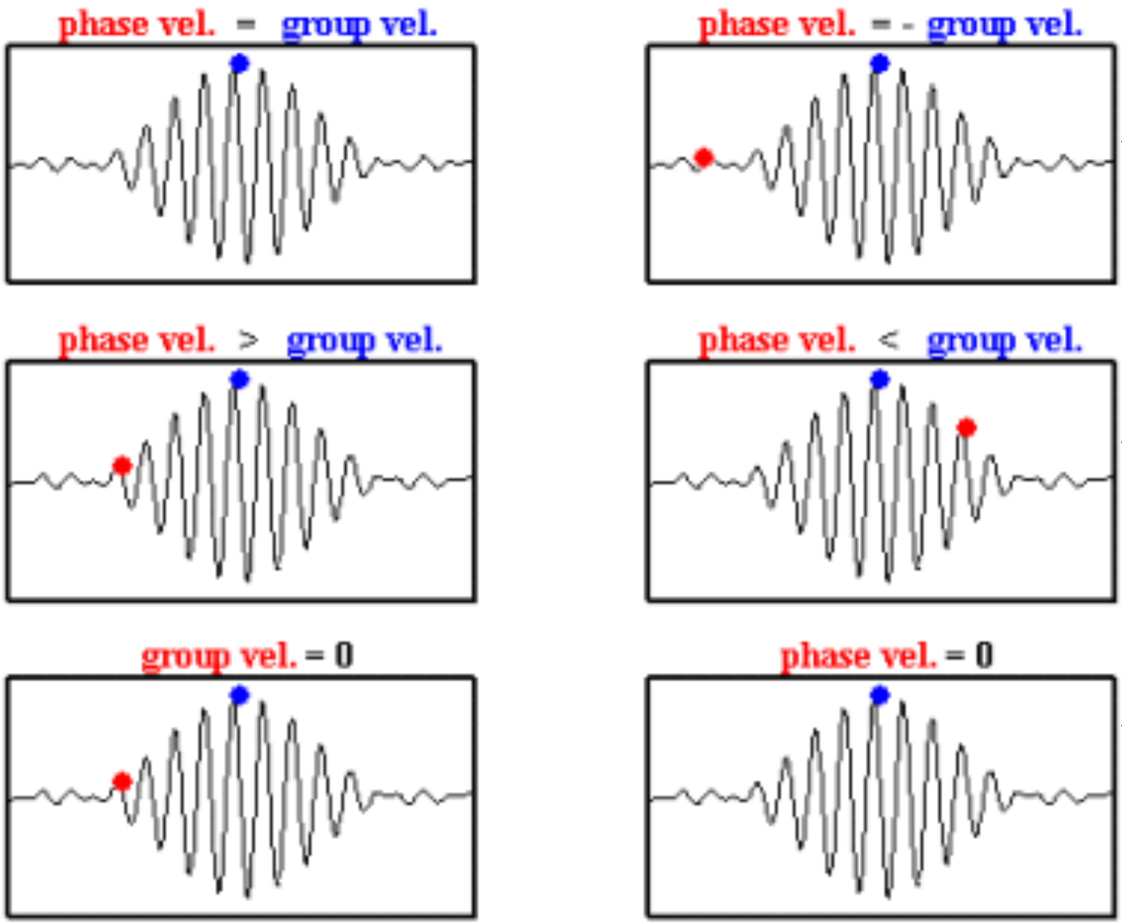
Figure 2.10: Decomposition of a pulse into wave packets with different center frequency. In a medium with dispersion the wave packets move at different relative group velocity

Group velocity Vs phase velocity

In vacuum

Most common case

Unrealistic



Unrealistic

Possible

Unrealistic

Adapted from Rick Trebino's course slides

Calculating group velocity vs. wavelength

We more often think of the refractive index in terms of wavelength, so let's write the group velocity in terms of the vacuum wavelength λ_0.

Use the chain rule: $\dfrac{dn}{d\omega} = \dfrac{dn}{d\lambda_0}\dfrac{d\lambda_0}{d\omega}$

Now, $\lambda_0 = 2\pi c_0 / \omega$, so: $\dfrac{d\lambda_0}{d\omega} = \dfrac{-2\pi c_0}{\omega^2} = \dfrac{-2\pi c_0}{(2\pi c_0/\lambda_0)^2} = \dfrac{-\lambda_0^2}{2\pi c_0}$

Recalling that: $\mathrm{v}_g = \left(\dfrac{c_0}{n}\right) \Big/ \left[1 + \dfrac{\omega}{n}\dfrac{dn}{d\omega}\right]$

we have: $\mathrm{v}_g = \left(\dfrac{c_0}{n}\right) \Big/ \left[1 + \dfrac{\cancel{2\pi c_0}}{n\lambda_0}\left\{\dfrac{dn}{d\lambda_0}\left(\dfrac{-\lambda_0^2}{\cancel{2\pi c_0}}\right)\right\}\right]$

or:

$$\boxed{\mathrm{v}_g = \left(\dfrac{c_0}{n}\right) \Big/ \left(1 - \dfrac{\lambda_0}{n}\dfrac{dn}{d\lambda_0}\right)}$$

<u>*Adapted from Rick Trebino's course slides*</u>

Group-velocity dispersion (GVD)

$$k^{(2)} = \frac{d}{d\omega}\left(\frac{1}{v_g}\right) = -\frac{1}{v_g{}^2(\omega_0)}\frac{dv_g}{d\omega} = \left(\frac{\lambda}{2\pi c}\right)\frac{\lambda^2}{c}\frac{d^2 n}{d\lambda^2}$$

Positive GVD or normal dispersion

$$k^{(2)} > 0$$

$$\frac{dv_g}{d\omega} < 0$$

Low frequency travels faster

Negative GVD or anomalous dispersion

$$k^{(2)} < 0$$

$$\frac{dv_g}{d\omega} > 0$$

High frequency travels faster

Effect of GVD on pulse propagation

Gaussian Pulse:

$$\underline{E}(z = 0, t) = \underline{A}(z = 0, t)e^{j\omega_0 t}$$

$$\underline{A}(z = 0, t = t') = \underline{A}_0 \exp\left[-\frac{1}{2}\frac{t'^2}{\tau^2}\right]$$

Pulse width

$$\frac{\partial\tilde{\underline{A}}(z,\omega)}{\partial z} = -j\frac{k''\omega^2}{2}\tilde{\underline{A}}(z,\omega)$$

Substitute:

$$\tilde{\underline{A}}(z,\omega) = \tilde{\underline{A}}(z = 0,\omega)\exp\left[-j\frac{k''\omega^2}{2}z\right]$$

Gaussian Integral:

$$\boxed{\frac{1}{\sqrt{2\pi\sigma}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma}}e^{-jx\varsigma}dx = e^{-\frac{\sigma}{2}\varsigma^2} \text{ for } \mathrm{Re}\{\sigma\} \geq 0}$$

$$\tilde{\underline{A}}(z = 0,\omega) = A_0\sqrt{2\pi}\tau\exp\left[-\frac{1}{2}\tau^2\omega^2\right]$$

Propagation of z distance:

$$\underline{\tilde{A}}(z,\omega) = A_0\sqrt{2\pi}\tau \exp\left[-\frac{1}{2}\left(\tau^2 + \mathrm{j}k''z\right)\omega^2\right]$$

$$\underline{A}(z,t') = A_0\left(\frac{\tau^2}{(\tau^2+\mathrm{j}k''z)}\right)^{1/2}\exp\left[-\frac{1}{2}\frac{t'^2}{(\tau^2+\mathrm{j}k''z)}\right]$$

Exponent Real and Imaginary Part:

$$\underline{A}(z,t') = A_0\underbrace{\left(\frac{\tau^2}{(\tau^2+\mathrm{j}k''z)}\right)^{1/2}}_{\text{z-dependent phase shift, independent on time}}\exp\left[\underbrace{-\frac{1}{2}\frac{\tau^2 t'^2}{\left(\tau^4+(k''z)^2\right)}}_{\text{determines pulse width}}+\underbrace{\mathrm{j}\frac{1}{2}k''z\frac{t'^2}{\left(\tau^4+(k''z)^2\right)}}_{\text{temporal quadratic phase}}\right]$$

FWHM Pulse width:

$$\exp\left[-\frac{\tau^2(\tau'_{FWHM}/2)^2}{\left(\tau^4+(k''z)^2\right)}\right] = 0.5$$

Initial pulse width:

$$\tau_{FWHM} = 2\sqrt{\ln 2}\;\tau$$

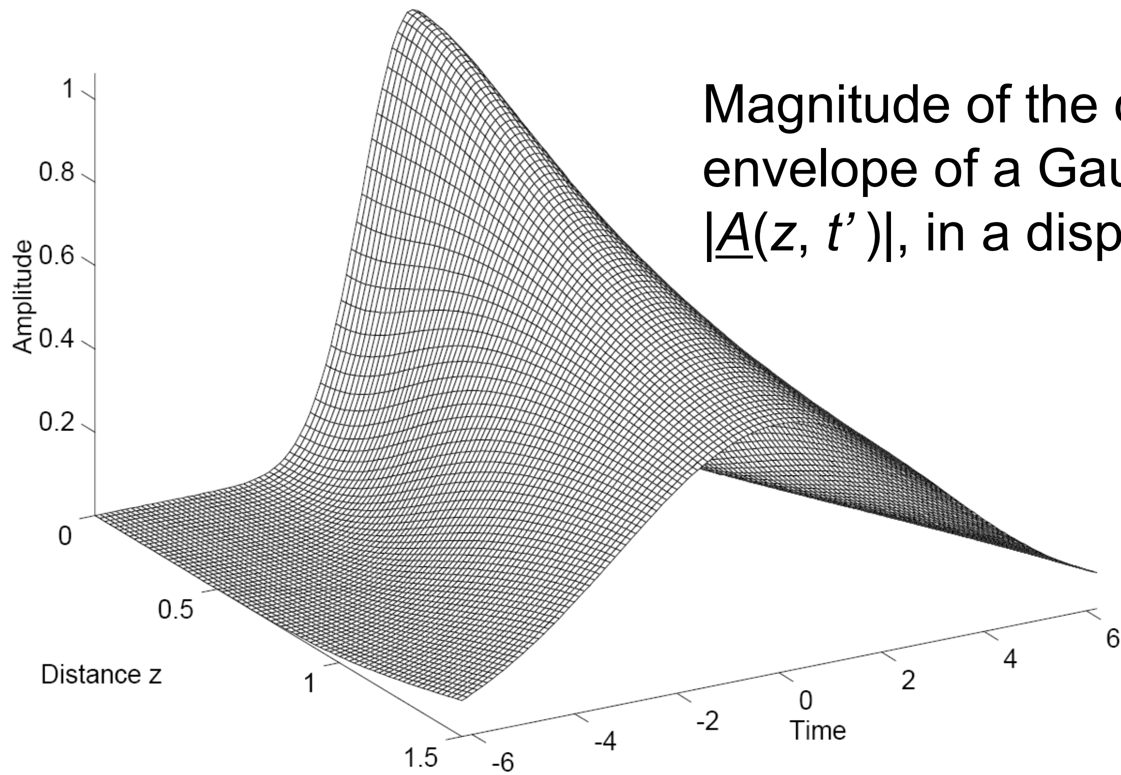
Initial pulse width:

$$\tau_{FWHM} = 2\sqrt{\ln 2}\,\tau$$

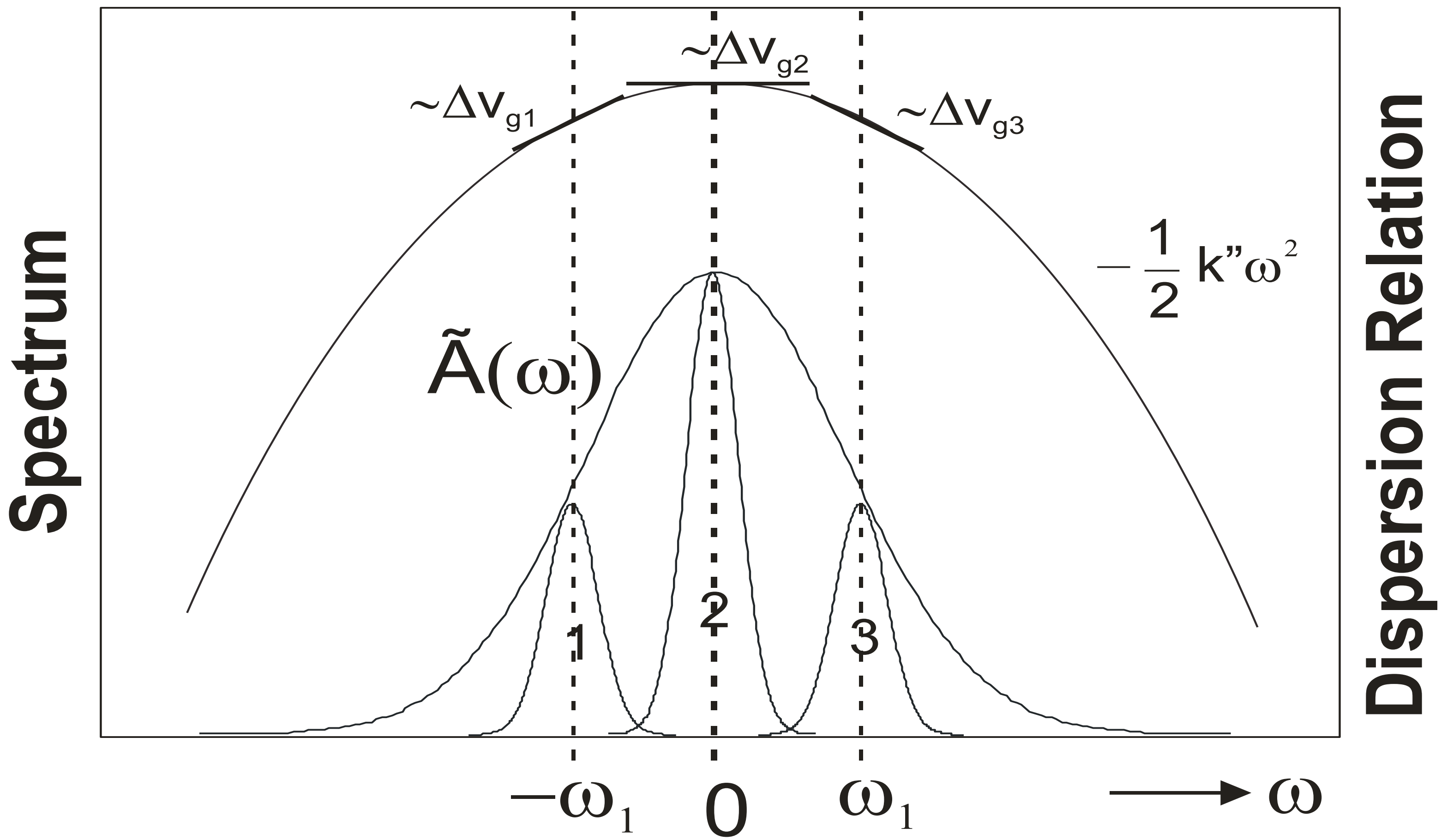
After propagation over a distance z=L:

$$\tau'_{FWHM} = 2\sqrt{\ln 2}\,\tau\sqrt{1+\left(\frac{k''L}{\tau^2}\right)^2} = \tau_{FWHM}\sqrt{1+\left(\frac{k''L}{\tau^2}\right)^2}$$

For large distances: $\tau'_{FWHM} = 2\sqrt{\ln 2}\left|\frac{k''L}{\tau}\right|$ for $\left|\frac{k''L}{\tau^2}\right| \gg 1$



Magnitude of the complex envelope of a Gaussian pulse, $|\underline{A}(z, t')|$, in a dispersive medium



Decomposition of a pulse into wave packets with different center frequency. In a medium with dispersion the wave packets move at different relative group velocity

Instantaneous frequency and chirp

$$\underline{A}(z,t') = A_0 \underbrace{\left(\frac{\tau^2}{(\tau^2 + \mathrm{j}k''z)}\right)^{1/2}}_{\text{z-dependent phase shift, independent on time}} \exp\left[-\underbrace{\frac{1}{2}\frac{\tau^2 t'^2}{\left(\tau^4 + (k''z)^2\right)}}_{\text{determines pulse width}} + \mathrm{j}\underbrace{\frac{1}{2}k''z\frac{t'^2}{\left(\tau^4 + (k''z)^2\right)}}_{\text{temporal quadratic phase}}\right]$$

After propagation of L distance: $\omega(z=L,t') = \dfrac{\partial}{\partial t'}\phi(L,t') = \dfrac{k''L}{\left(\tau^4 + (k''L)^2\right)}t'$

$$E(L,t') = \underline{A}(L,t')\exp(j\omega_0 t') \propto \exp[j\omega_0 t' + j\phi(L,t')]$$

$$\phi(z=L,t') = -\frac{1}{2}\arctan\left[\frac{k''L}{\tau^2}\right] + \frac{1}{2}k''L\frac{t'^2}{\left(\tau^4 + (k''L)^2\right)}$$

Instantaneous Frequency:

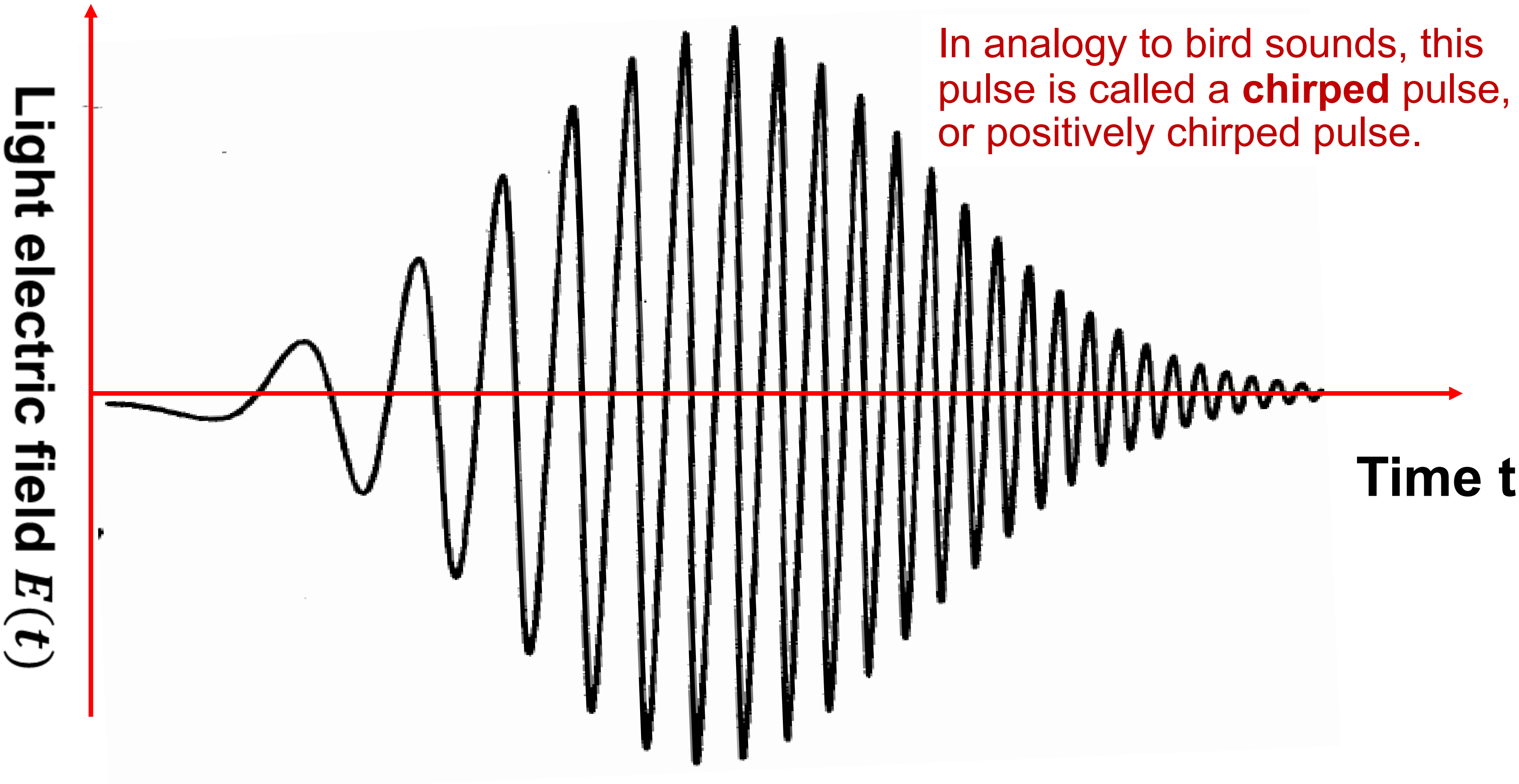
$$\omega_{inst}(t) \equiv \frac{\partial[\omega_0 t' + \phi(L,t')]}{\partial t'} = \omega_0 + \frac{\partial\phi(L,t')}{\partial t'}$$

$$= \omega_0 + \frac{k''L}{\left(\tau^4 + (k''L)^2\right)}t'$$

Linearly chirped Gaussian pulse: positive chirp

$$\omega_{ins}(t') = \omega_0 + \frac{k''L}{(\tau^4 + (k''L)^2)} t'$$

For positive GVD, i.e., k”>0, lower frequency travels faster, and the instantaneous frequency linearly **INCREASES** with time.

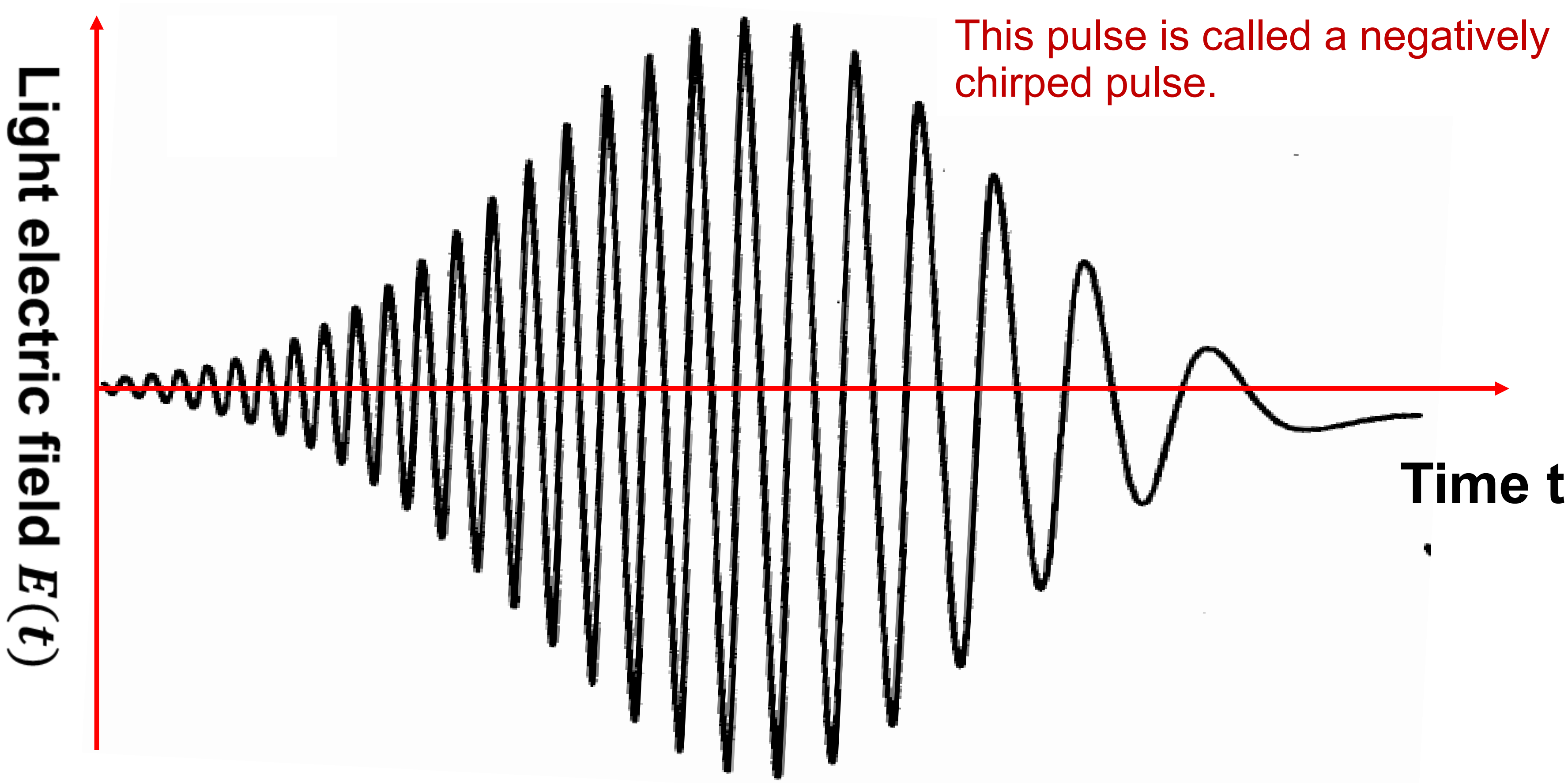


In analogy to bird sounds, this pulse is called a **chirped** pulse, or positively chirped pulse.

Linearly chirped Gaussian pulse: negative chirp

$$\omega_{ins}(t') = \omega_0 + \frac{k''L}{(\tau^4 + (k''L)^2)} t'$$

For negative GVD, i.e., k”<0, higher frequency travels faster.
The instantaneous frequency linearly **DECREASES** with time.



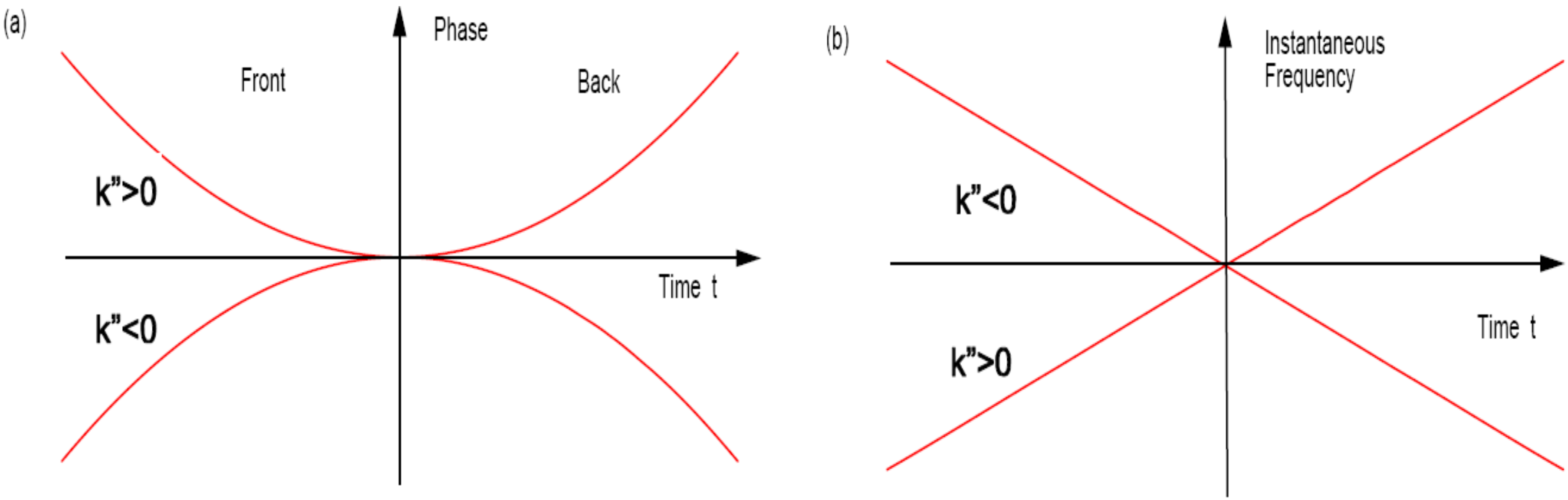
This pulse is called a negatively chirped pulse.

Chirp:

$$\phi(z=L,t') = -\frac{1}{2}\arctan\left[\frac{k''L}{\tau^2}\right] + \frac{1}{2}k''L\frac{t'^2}{(\tau^4 + (k''L)^2)}$$

Instantaneous Frequency:

$$\omega(z=L,t') = \frac{\partial}{\partial t'}\phi(L,t') = \frac{k''L}{(\tau^4 + (k''L)^2)}t'$$



k”>0: Postive Group Velocity Dispersion (GVD), low frequencies travel faster and are in front of the pulse

Figure 2.12: (a) Phase and (b) instantaneous frequency of a Gaussian pulse during propagation through a medium with positive or negative dispersion

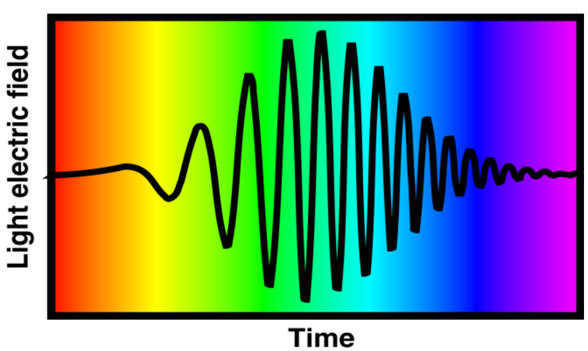
GVD changes the pulse duration and introduces chirp

$$k_2 = \frac{d}{d\omega}\left(\frac{1}{v_g}\right) = -\frac{1}{v_g^{\ 2}(\omega_0)}\frac{dv_g}{d\omega} = \left(\frac{\lambda}{2\pi c}\right)\frac{\lambda^2}{c}\frac{d^2 n}{d\lambda^2}$$

Positive GVD or normal dispersion

$$k_2 > 0 \qquad \frac{dv_g}{d\omega} < 0$$

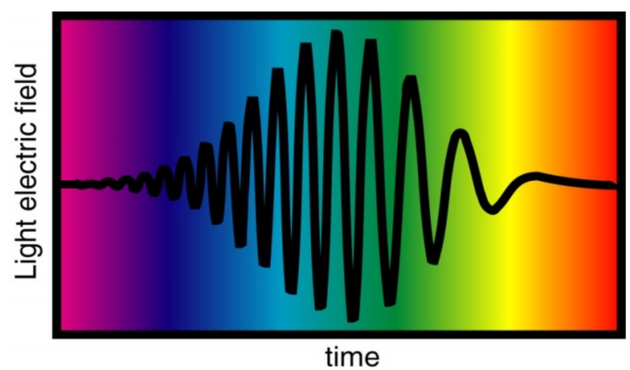
Red faster, positive chirp



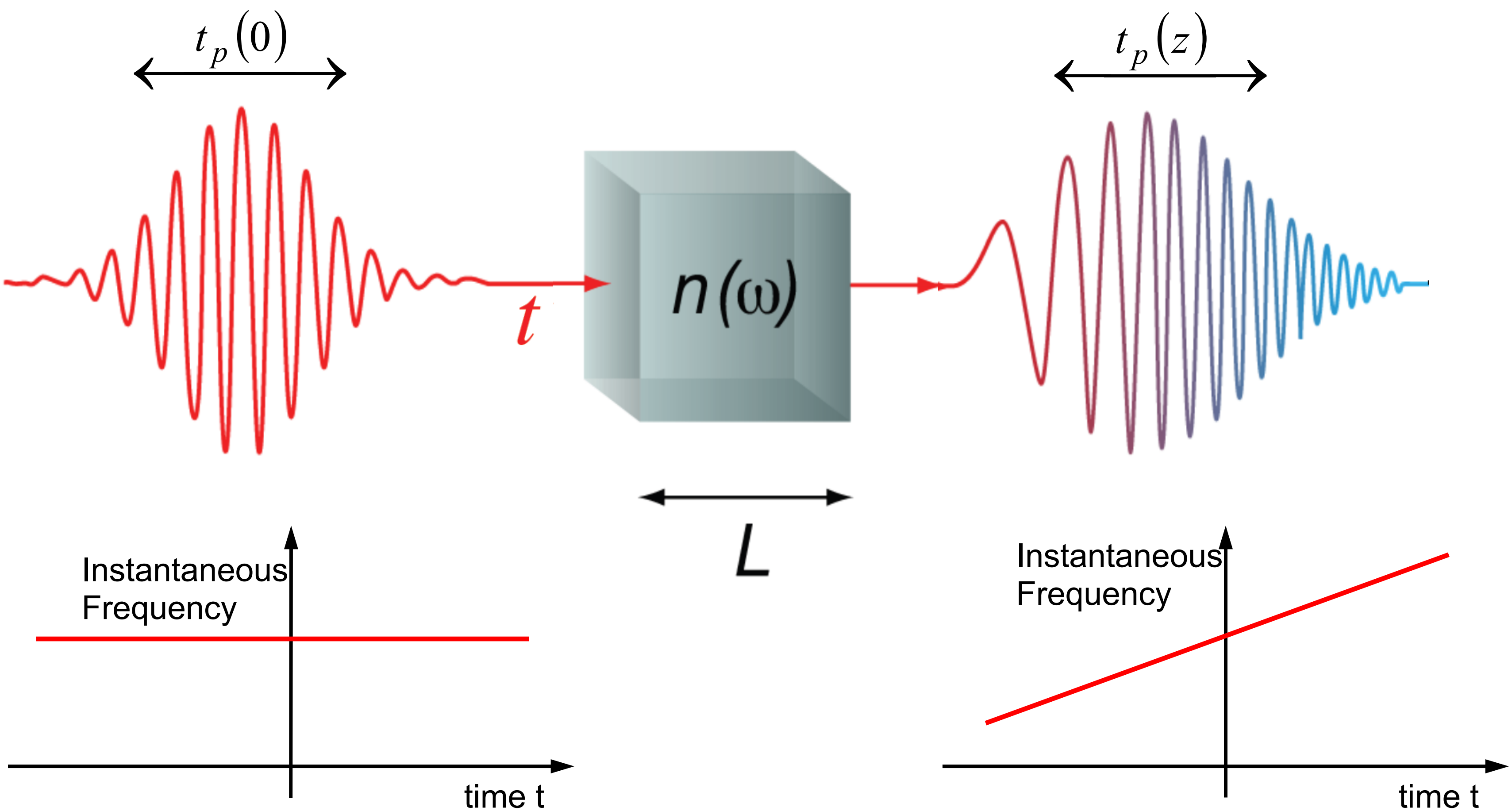
Negative GVD or anomalous dispersion

$$k_2 < 0 \qquad \frac{dv_g}{d\omega} > 0$$

Blue faster, negative chirp



Pulse travels through a dispersive bulk medium



Transform-limited pulse

Positive chirp

Group Delay & Group Delay Dispersion

$$\varphi(\omega) = k(\omega)z = \varphi_0 + \varphi_1(\omega-\omega_0) + \frac{1}{2}\varphi_2(\omega-\omega_0)^2 + \frac{1}{6}\varphi_3(\omega-\omega_0)^3 + \dots$$

$$\varphi_m = \left(\frac{d^m\varphi}{d\omega^m}\right)_{\omega=\omega_0}$$

$$\varphi_1 = \frac{z}{v_g} = \tau_g$$ Group delay, in fs

$$\varphi_2 = \frac{d\tau_g}{d\omega}$$ Group delay dispersion (GDD), in fs^2

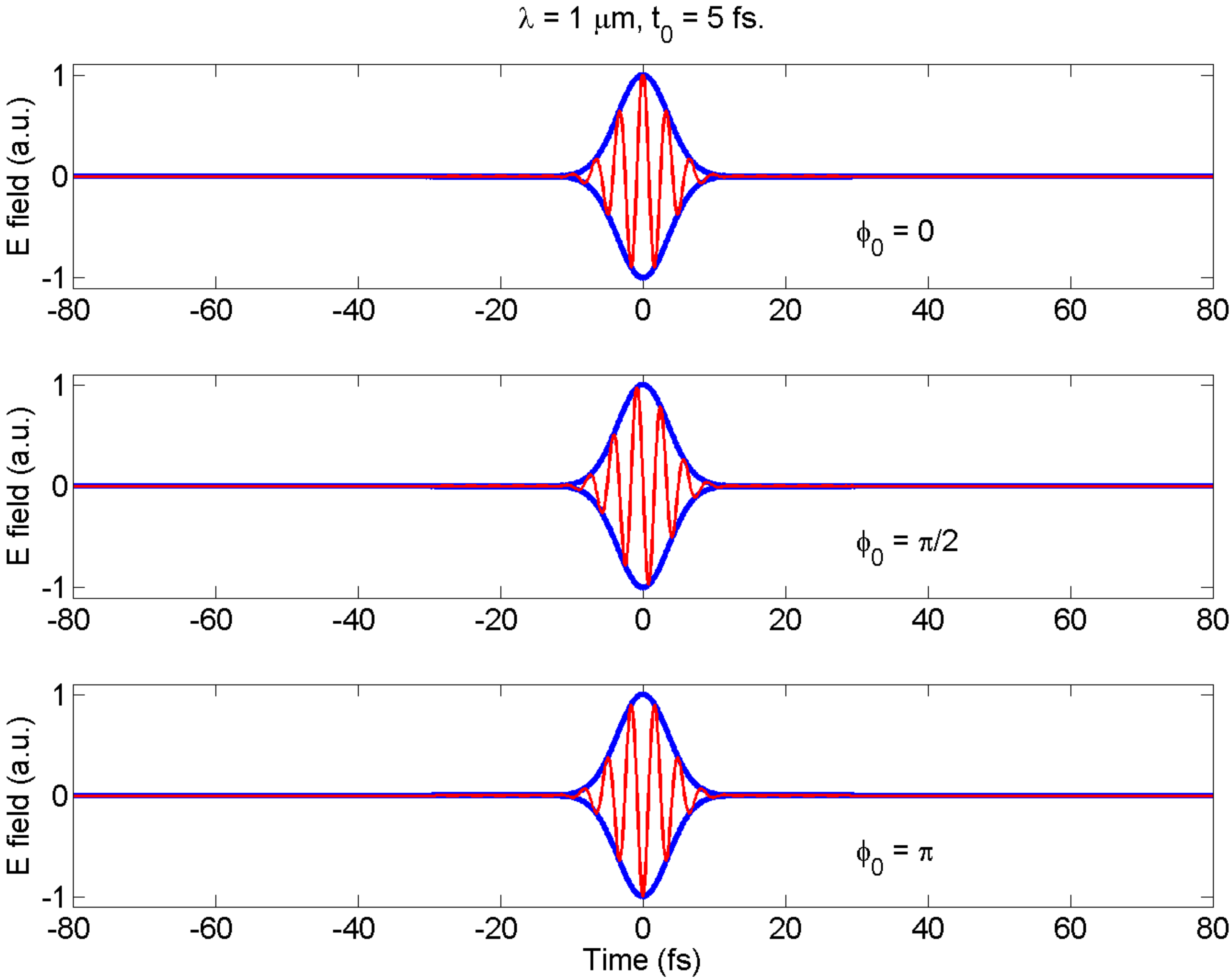
GDD > 0, positive dispersion
GDD < 0, negative dispersion

φ_3 Third order dispersion (TOD), in fs^3

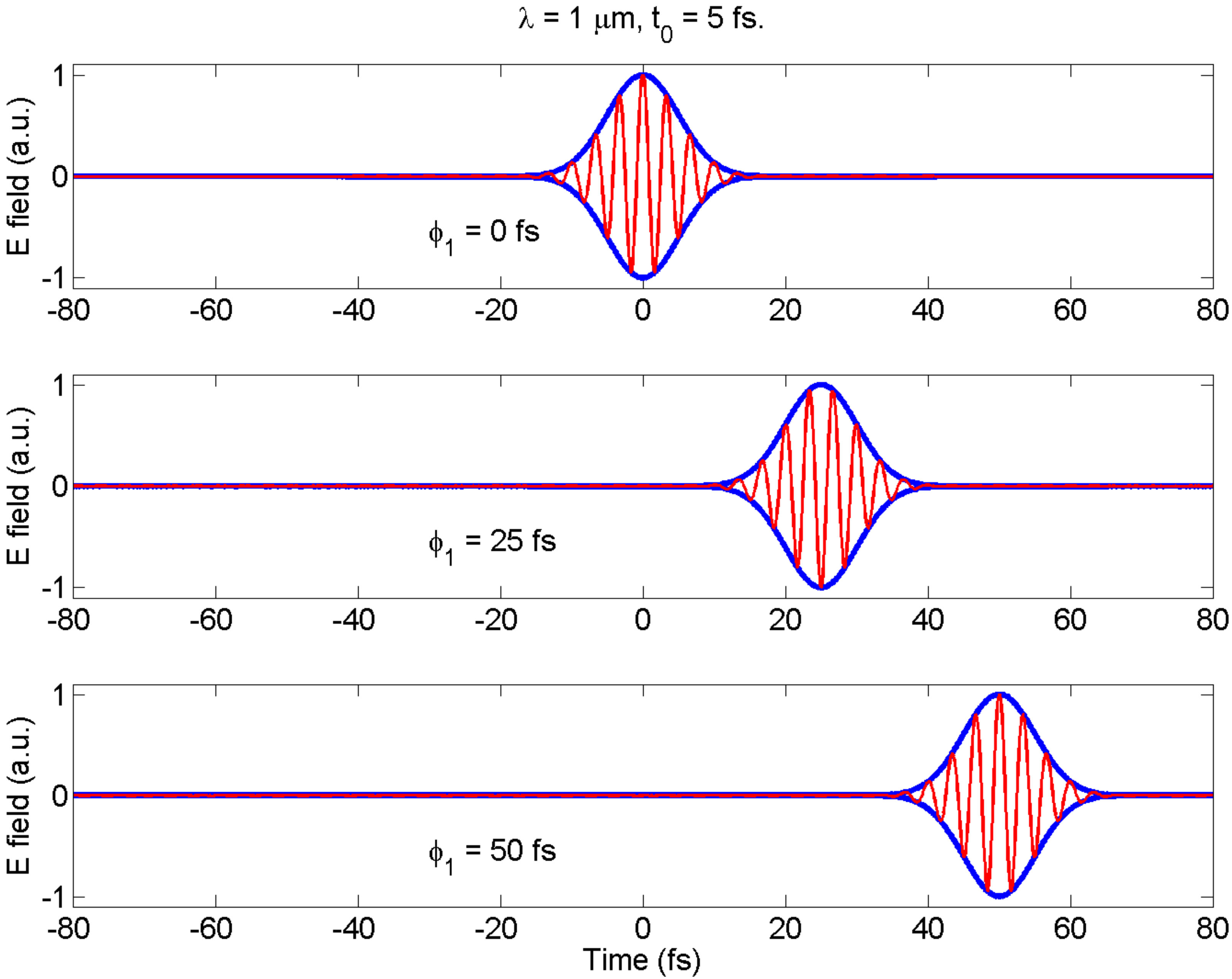
φ_4 Fourth order dispersion, in fs^4

<u>Group delay shift the time origin of the pulse envelope while GDD changes its shape.</u>

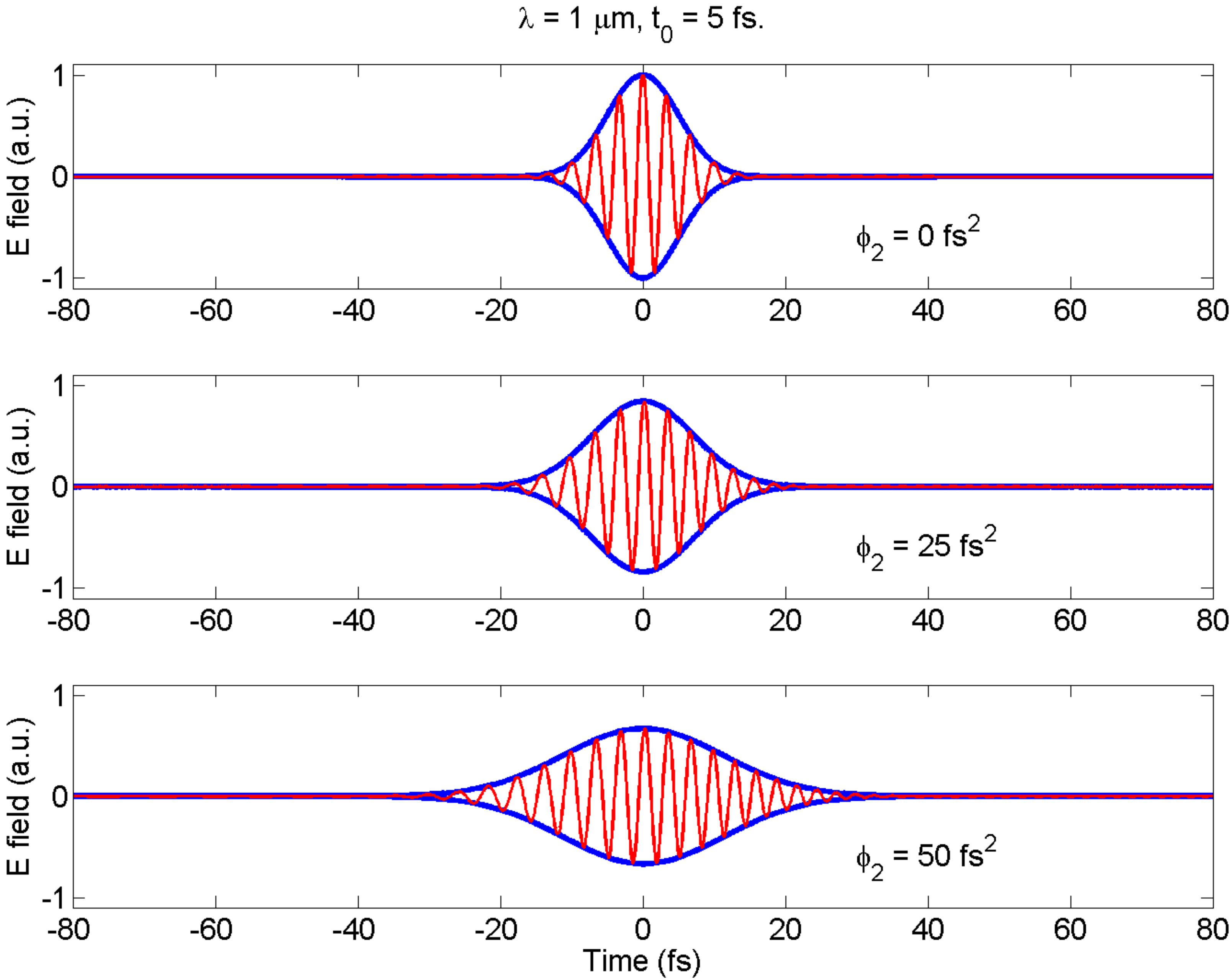
Effect of absolute phase



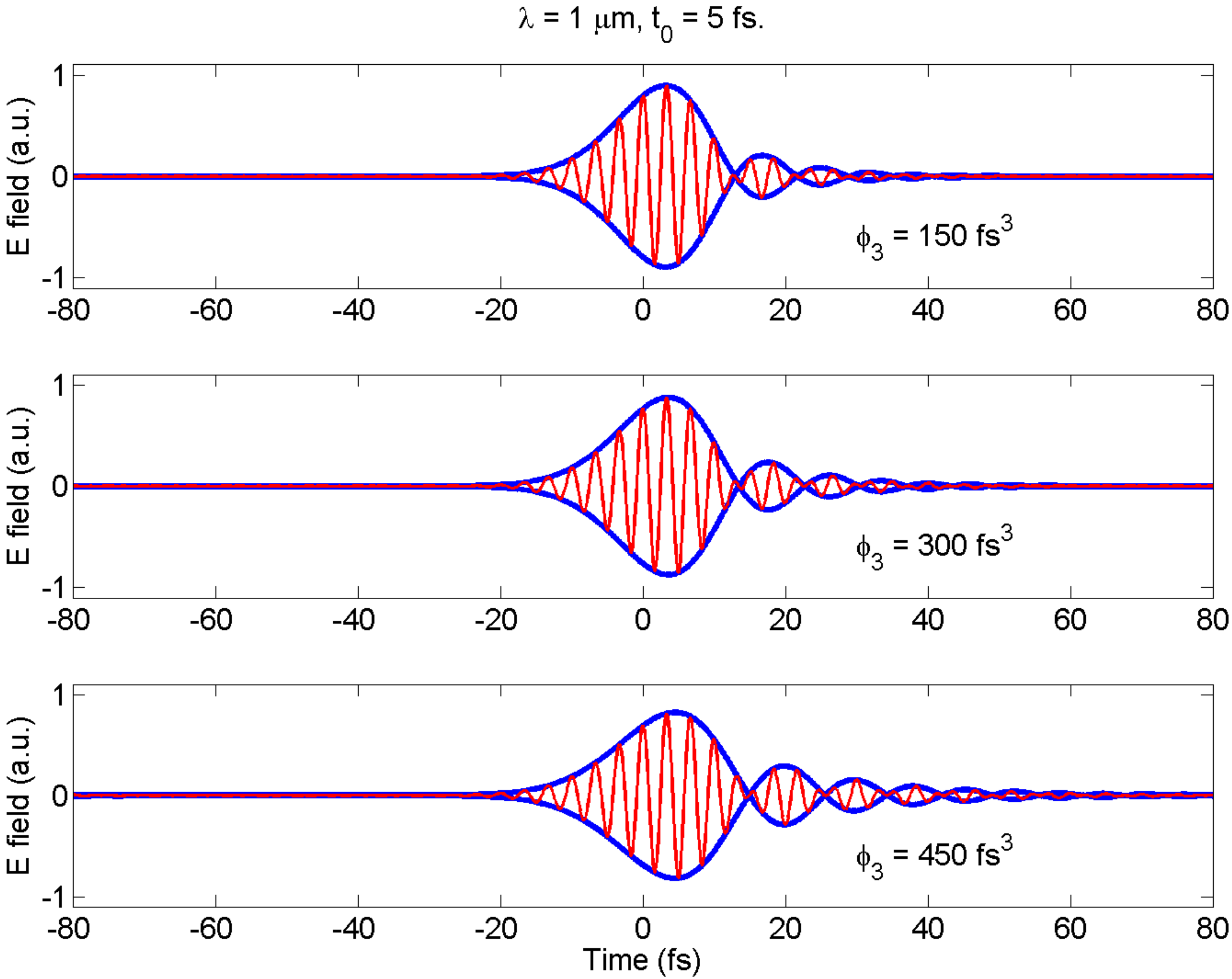
Effect of group delay



Effect of positive 2nd order dispersion

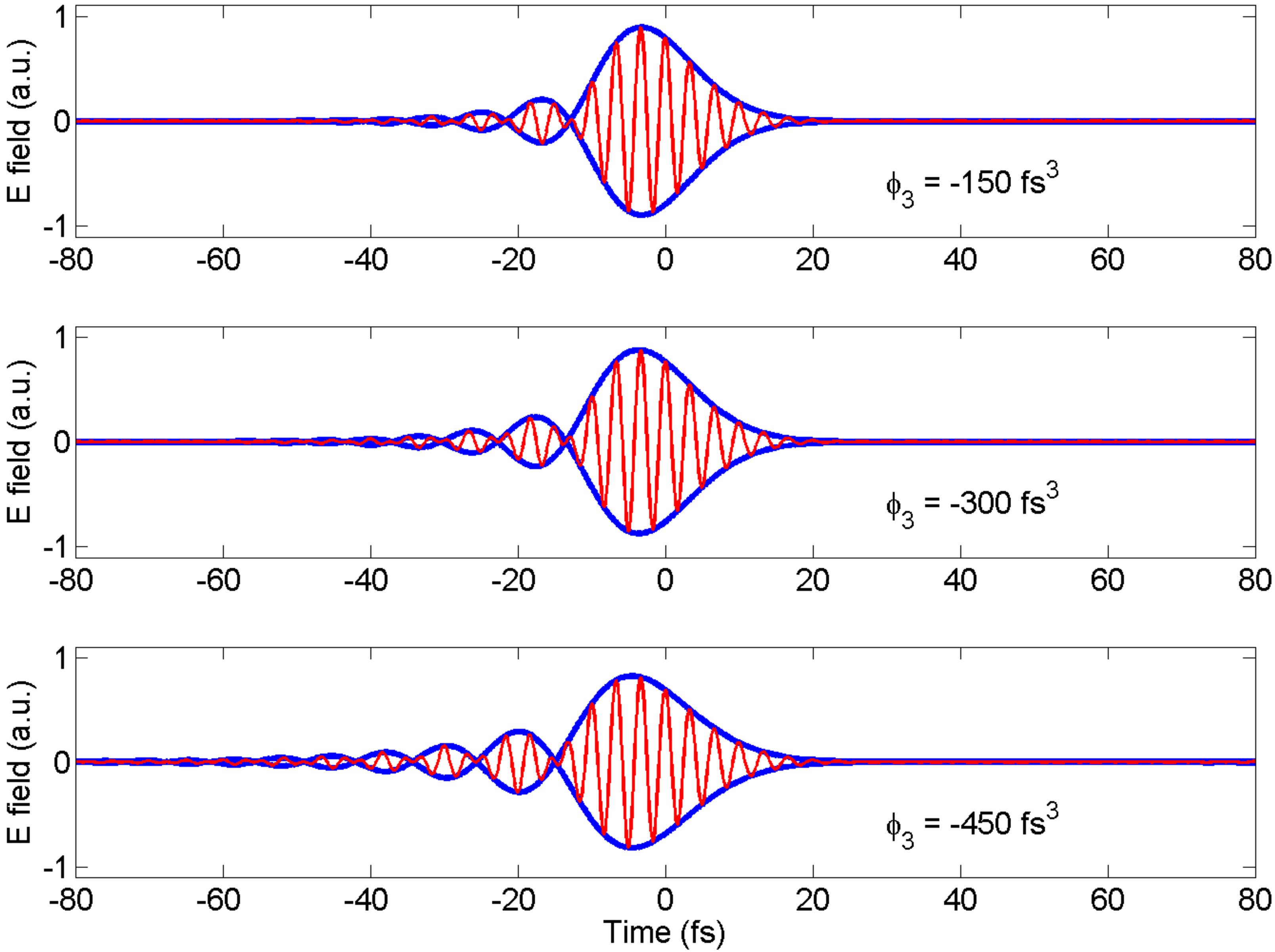


Effect of positive 3rd order dispersion

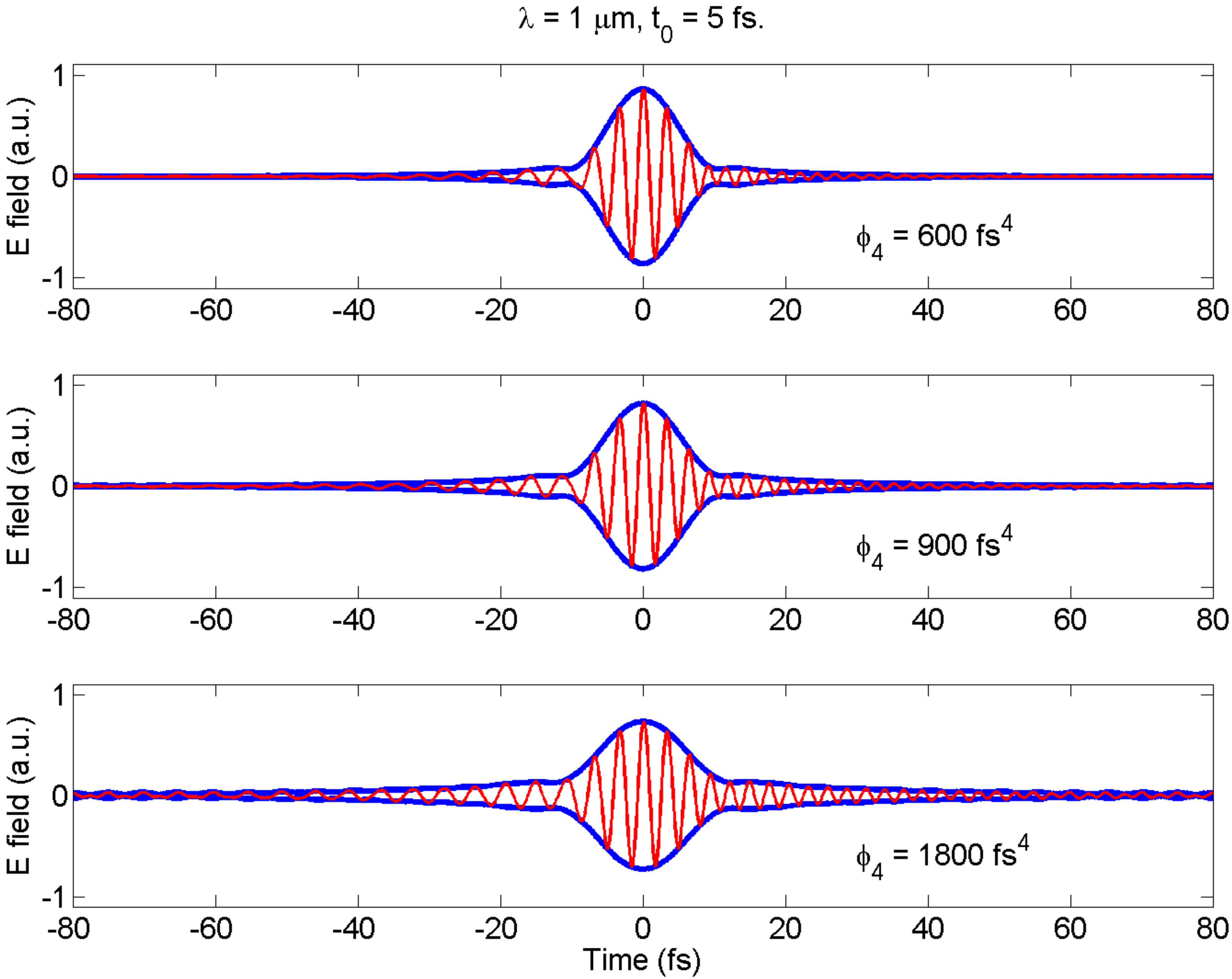


Effect of negative 3rd order dispersion

λ = 1 μm, t_0 = 5 fs.



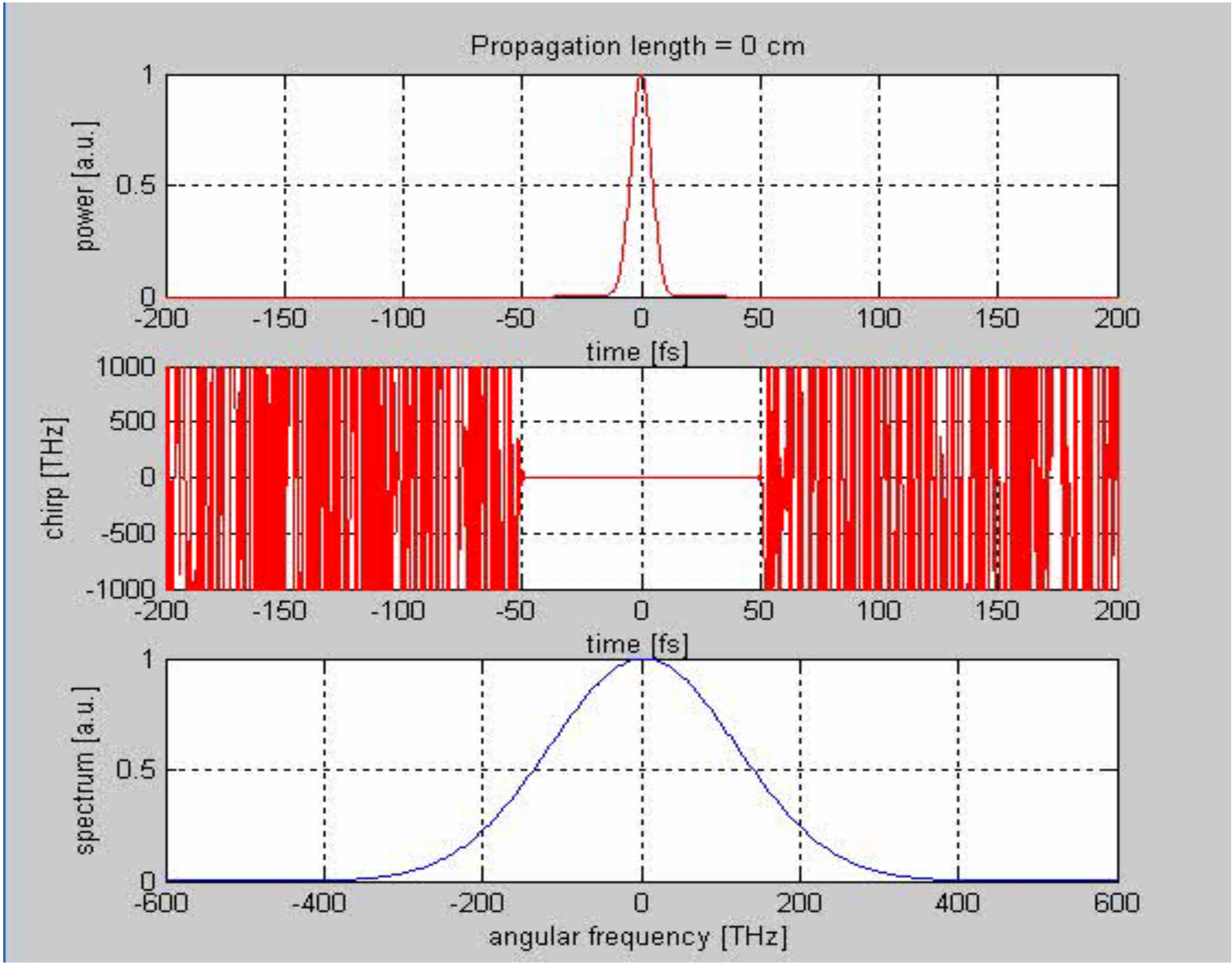
Effect of positive 4th order dispersion



Dispersion parameters for various materials

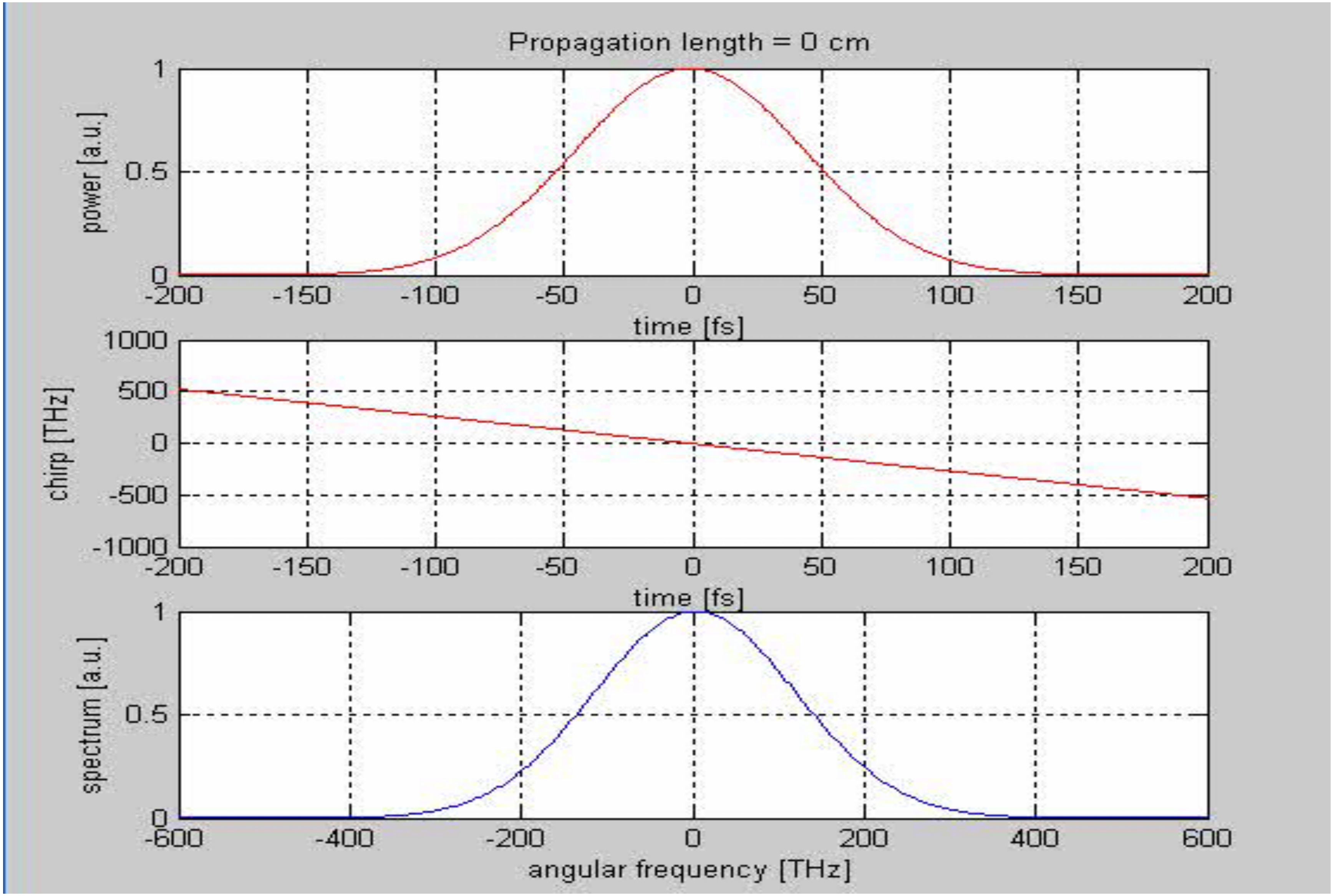
material	λ [nm]	$n(\lambda)$	$\frac{dn}{d\lambda} \cdot 10^{-2} \left[\frac{1}{\mu m}\right]$	$\frac{d^2n}{d\lambda^2} \cdot 10^{-1} \left[\frac{1}{\mu m^2}\right]$	$\frac{dn^3}{d\lambda^3} \left[\frac{1}{\mu m^3}\right]$	$T_g \left[\frac{fs}{mm}\right]$	$GDD \left[\frac{fs^2}{mm}\right]$	$TOD \left[\frac{fs^3}{mm}\right]$
BK7	400	1,5308	-13,17	10,66	-12,21	5282	120,79	40,57
	500	1,5214	-6,58	3,92	-3,46	5185	86,87	32,34
	600	1,5163	-3,91	1,77	-1,29	5136	67,52	29,70
	800	1,5108	-1,97	0,48	-0,29	5092	43,96	31,90
	1000	1,5075	-1,40	0,15	-0,09	5075	26,93	42,88
	1200	1,5049	-1,23	0,03	-0,04	5069	10,43	66,12
SF10	400	1,7783	-52,02	59,44	-101,56	6626	673,68	548,50
	500	1,7432	-20,89	15,55	-16,81	6163	344,19	219,81
	600	1,7267	-11,00	6,12	-4,98	5980	233,91	140,82
	800	1,7112	-4,55	1,58	-0,91	5830	143,38	97,26
	1000	1,7038	-2,62	0,56	-0,27	5771	99,42	92,79
	1200	1,6992	-1,88	0,22	-0,10	5743	68,59	107,51
Sapphire	400	1,7866	-17,20	13,55	-15,05	6189	153,62	47,03
	500	1,7743	-8,72	5,10	-4,42	6064	112,98	39,98
	600	1,7676	-5,23	2,32	-1,68	6001	88,65	37,97
	800	1,7602	-2,68	0,64	-0,38	5943	58,00	42,19
	1000	1,7557	-1,92	0,20	-0,12	5921	35,33	57,22
	1200	1,7522	-1,70	0,04	-0,05	5913	13,40	87,30
Quartz	300	1,4878	-30,04	34,31	-54,66	5263	164,06	46,49
	400	1,4701	-11,70	9,20	-10,17	5060	104,31	31,49
	500	1,4623	-5,93	3,48	-3,00	4977	77,04	26,88
	600	1,4580	-3,55	1,59	-1,14	4934	60,66	25,59
	800	1,4533	-1,80	0,44	-0,26	4896	40,00	28,43
	1000	1,4504	-1,27	0,14	-0,08	4880	24,71	38,73
	1200	1,4481	-1,12	0,03	-0,03	4875	9,76	60,05

Effect of negative GVD



GVD $\beta_2 = -25\, ps^2 / km$ **Input pulse duration:10fs**

Effect of positive GVD



GVD $\beta_2 = 25\, ps^2 / km$ The output of last slide is taken as the input here.

Real and imaginary part of the susceptibility

$$\underline{\tilde{\chi}}(\omega) = \tilde{\chi}_r(\omega) + j\tilde{\chi}_i(\omega)$$

Example: EM-Wave polarized along x-axis and propagation along z-direction:

$$\underline{\vec{E}}(z,t) = \underline{E}_0 e^{j(\omega t - kz)}\ \vec{e}_x$$

In general:

$$k(\omega) = \frac{\omega}{c_0}\tilde{n}(\omega) = \frac{\omega}{c_0}\sqrt{1+\tilde{\chi}(\omega)} = \frac{\omega}{c_0}\left(\tilde{n}_r(\omega) + j\tilde{n}_i(\omega)\right) = k_r(\omega) - j\alpha(\omega)$$

$$\boxed{\underline{\vec{E}}(z,t) = \underline{E}_0 e^{-\alpha\cdot z}\ e^{j(\omega t - k_r z)}\ \vec{e}_x}$$

Dispersion relation:

$$k_r(\omega) = \frac{\omega}{c_0} n_r(\omega)$$

2.4.2 Loss and Gain

$$\underline{\tilde{n}}(\Omega) = n_r(\Omega) + jn_i(\Omega)$$

Refractive index + gain and/or loss

$$\underline{\tilde{n}}(\Omega) = \sqrt{1 + \underline{\tilde{\chi}}(\Omega)}$$

for: $\left|\underline{\tilde{\chi}}(\Omega)\right| << 1$

$$\underline{\tilde{n}}(\Omega) \approx 1 + \frac{\underline{\tilde{\chi}}(\Omega)}{2}$$

Complex Lorentzian close to resonance : $\Omega \approx \Omega_0$

$$\underline{\chi}(\omega) = \frac{\omega_p^2}{(\Omega_0^2 - \omega^2) + 2j\omega\frac{\Omega_0}{Q}} \longrightarrow \underline{\tilde{\chi}}(\Omega) = \frac{-j\chi_0}{1 + jQ\frac{\Omega-\Omega_0}{\Omega_0}}$$

Maximum absorption: $\chi_0 = Q\frac{\omega_p^2}{2\Omega_0^2}$

Half Width Half Maximum linewidth (HWHM): $\Delta\Omega = \frac{\Omega_0}{Q}$

Real and imaginary parts:

$$\tilde{\chi}_r(\Omega) = \frac{-\chi_0 \frac{(\Omega-\Omega_0)}{\Delta\Omega}}{1+\left(\frac{\Omega-\Omega_0}{\Delta\Omega}\right)^2},$$

$$\tilde{\chi}_i(\Omega) = \frac{-\chi_0}{1+\left(\frac{\Omega-\Omega_0}{\Delta\Omega}\right)^2},$$

Complex wave number in lossy medium:

$$\underline{\tilde{K}}(\Omega) = \frac{\Omega}{c_0}\left(1+\frac{1}{2}\left(\tilde{\chi}_r(\Omega)+\mathrm{j}\tilde{\chi}_i(\Omega)\right)\right)$$

Redefine group velocity: e.g. at line center:

$$v_g^{-1} = \left.\frac{\partial K_r(\Omega)}{\partial\Omega}\right|_{\Omega_0} = \frac{1}{c_0}\left(1-\frac{\chi_0}{2}\frac{\Omega_0}{\Delta\Omega}\right)$$

Change in group velocity can be positive or negative

Absorption:

$$K = \frac{\Omega}{c_0} \qquad \alpha(\Omega) = -\frac{K}{2}\tilde{\chi}_i(\Omega)$$

For a wavepacket (optical pulse) with carrier frequency $\omega_0 = \Omega_0 \quad K_0 = \frac{\Omega_0}{c_0}$

$$\left.\frac{\partial \underline{\tilde{A}}(z,\omega)}{\partial z}\right|_{(loss)} = -\alpha(\Omega_0+\omega)\tilde{A}(z,\omega) = \frac{-\chi_0 K_0/2}{1+\left(\frac{\omega}{\Delta\Omega}\right)^2}\underline{\tilde{A}}(z,\omega)$$

Parabolic loss or gain approximation:

$$\left.\frac{\partial \underline{A}(z,t')}{\partial z}\right|_{(loss)} = -\frac{\chi_0 K_0}{2}\left(1+\frac{1}{\Delta\Omega^2}\frac{\partial^2}{\partial t^2}\right)\underline{A}(z,t')$$

Gain: $g = -\frac{\chi_0 K_0}{2}$

$$\left.\frac{\partial \underline{A}(z,t')}{\partial z}\right|_{(gain)} = g\left(1+\frac{1}{\Omega_g^2}\frac{\partial^2}{\partial t^2}\right)\underline{A}(z,t')$$

HWHM – gain bandwidth

Group Velocity and Group Delay Dispersion

$$GVD = \left.\frac{d^2k(\omega)}{d\omega^2}\right|_{\omega=0} = \left.\frac{d}{d\omega}\frac{1}{\upsilon_g(\omega)}\right|_{\omega=0}$$

$$GDD = \left.\frac{d^2k(\omega)}{d\omega^2}\right|_{\omega=0} L = \left.\frac{d}{d\omega}\frac{L}{\upsilon_g(\omega)}\right|_{\omega=0} = \frac{d}{d\omega}T_g(\omega)|_{\omega=0}$$

Group Delay: $T_g(\omega) = L/\upsilon_g(\omega)$

Dispersion Characteristic	Definition	Comp. from $n(\lambda)$
medium wavelength: λ_n	$\frac{\lambda}{n}$	$\frac{\lambda}{n(\lambda)}$
wavenumber: k	$\frac{2\pi}{\lambda_n}$	$\frac{2\pi}{\lambda}n(\lambda)$
phase velocity: υ_p	$\frac{\omega}{k}$	$\frac{c_0}{n(\lambda)}$
group velocity: υ_g	$\frac{d\omega}{dk}$; $d\lambda = \frac{-\lambda^2}{2\pi c_0}d\omega$	$\frac{c_0}{n}\left(1 - \frac{\lambda}{n}\frac{dn}{d\lambda}\right)^{-1}$
group velocity dispersion: GVD	$\frac{d^2k}{d\omega^2}$	$\frac{\lambda^3}{2\pi c_0^2}\frac{d^2n}{d\lambda^2}$
group delay: $T_g = \frac{L}{\upsilon_g} = \frac{d\phi}{d\omega}$	$\frac{d\phi}{d\omega} = \frac{d(kL)}{d\omega}$	$\frac{n}{c_0}\left(1 - \frac{\lambda}{n}\frac{dn}{d\lambda}\right)L$
group delay dispersion: GDD	$\frac{dT_g}{d\omega} = \frac{d^2(kL)}{d\omega^2}$	$\frac{\lambda^3}{2\pi c_0^2}\frac{d^2n}{d\lambda^2}L$

Table 2.4: Table with important dispersion characteristics and how to compute them from the wavelength dependent refractive index $n(\lambda)$.

3. Nonlinear Pulse Propagation

3.1 The Optical Kerr Effect

Without derivation, there is a nonlinear contribution to the refractive index:

$$n = n(\omega, |A|^2) \approx n_0(\omega) + n_{2,L}|A|^2 \longleftarrow$$ **Polarization dependent**

Material	Refractive index n	$n_{2,L}[cm^2/W]$
Sapphire (Al_2O_3)	1.76 @ 850 nm	$3 \cdot 10^{-16}$
Fused Quarz	1.45 @ 1064 nm	$2.46 \cdot 10^{-16}$
Glass (LG-760)	1.5 @ 1064 nm	$2.9 \cdot 10^{-16}$
YAG ($Y_3Al_5O_{12}$)	1.82 @ 1064 nm	$6.2 \cdot 10^{-16}$
YLF ($LiYF_4$), n_e	1.47 @ 1047 nm	$1.72 \cdot 10^{-16}$
Si	3.3 @ 1550 nm	$4 \cdot 10^{-14}$

Table 3.1: Nonlinear refractive index of some materials

1) A variety of effects give rise to a nonlinear refractive index.
2) Those that yield a large n_2 typically have a slow response.
3) Nonlinear coefficient can be negative.

Intensity dependent nonlinear refractive index

The refractive index in the presence of linear and nonlinear polarizations:

$$n = \sqrt{1+\chi^{(1)} + \chi^{(3)} |E|^2}$$

Now, the usual refractive index (which we'll call n_0) is: $n_0 = \sqrt{1+\chi^{(1)}}$

So:

$$n = \sqrt{n_0^2 + \chi^{(3)} |E|^2} = n_0 \sqrt{1 + \chi^{(3)} |E|^2 / n_0^2}$$

Assume that the nonlinear term << n_0:

$$n \approx n_0 \left[1 + \chi^{(3)} |E|^2 / 2n_0^2\right]$$

So:

$$n \approx n_0 + \chi^{(3)} |E|^2 / 2n_0$$

since: $I \propto |E|^2$

Usually we define a "nonlinear refractive index", $n_{2,L}$:

$$\boxed{n = n_0 + n_{2,L} I}$$

<u>Kerr effect: refractive index linearly dependent on light intensity.</u>

Who is Kerr?

John Kerr (1824-1907) was a Scottish physicist. He was a student in Glasgow from 1841 to 1846, and at the Theological College of the Free Church of Scotland, in Edinburgh, in 1849. Starting in 1857 he was mathematical lecturer at the Free Church Training College in Glasgow.

He is best known for the discovery in 1875 of what is now called Kerr effect—the first nonlinear optical effect to be observed. In the Kerr effect, a change in refractive index is proportional to the square of the electric field. The Kerr effect is exploited in the *Kerr cell*, which is used in applications such as shutters in high-speed photography, with shutter-speeds as fast as 100 ns.

John Kerr, c. 1860, photograph by Thomas Annan

3.2 Self-Phase Modulation (SPM)

$$\frac{\partial A(z,t)}{\partial z} = -jk_0 n_{2,L}|A(z,t)|^2 A(z,t) = -j\delta|A(z,t)|^2 A(z,t).$$

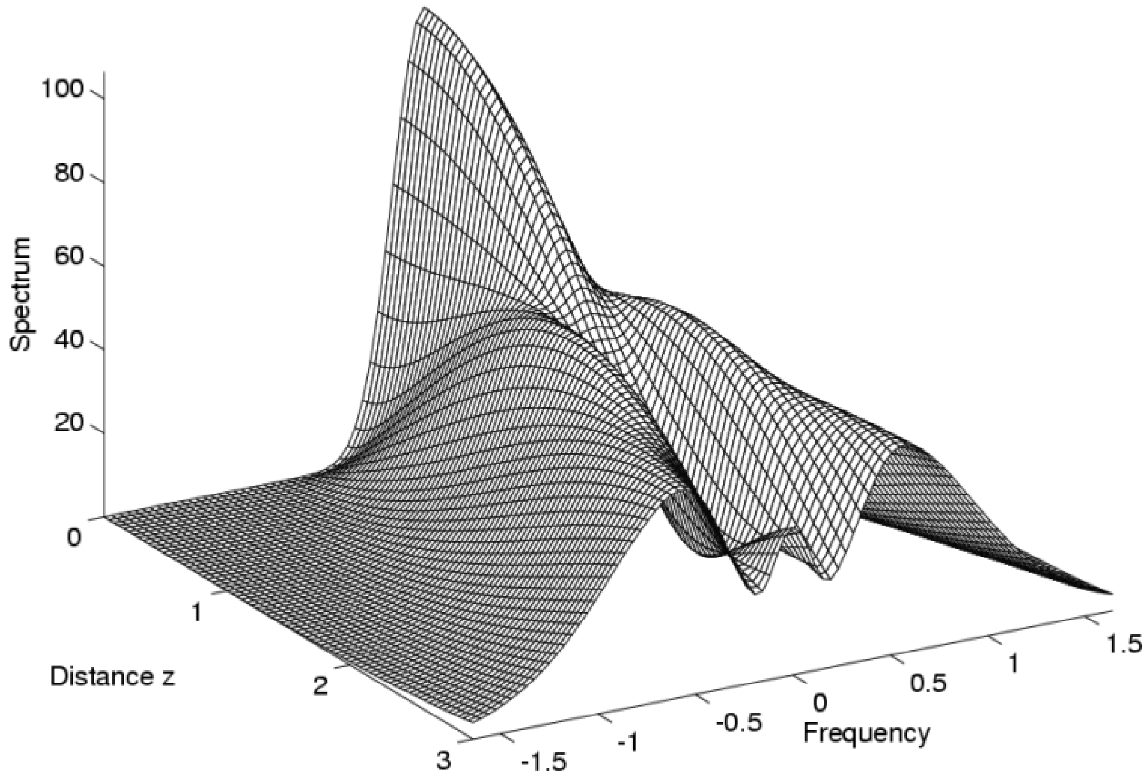


Figure 3.1: Intensity spectrum of a Gaussian pulse subject to self-phase modulation

Kerr effect for an optical pulse: self-phase modulation

In a purely one dimensional propagation problem, the intensity dependent refractive index imposes an additional self-phase shift on the pulse envelope during propagation, which is proportional to the instantaneous intensity of the pulse:

$$\frac{\partial A(z,t)}{\partial z} = -jk_0 n_{2,L}|A(z,t)|^2 A(z,t)$$
$$= -j\delta|A(z,t)|^2 A(z,t)$$

$$\delta = k_0 n_{2,L}$$

Note, here the pulse profile has been re-normalized so that its square gives intensity:

$$\boxed{\begin{aligned} A(z,t) &= A(0,t)e^{j\varphi_{NL}} \\ &= A(0,t)e^{-j\delta|A(z,t)|^2 z} \end{aligned}}$$

Pulse shape does not change, but the pulse acquires nonlinear phase:

$$|A(z,t)| = |A(0,t)|$$

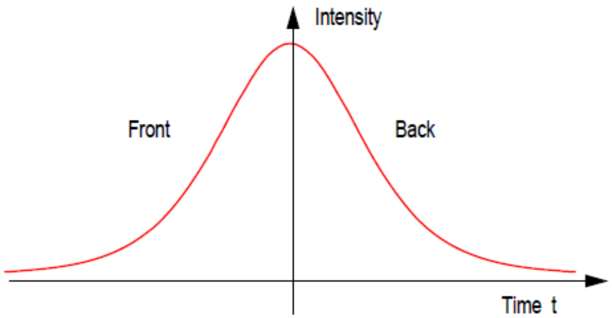
$$\varphi_{NL} = -\delta|A(z,t)|^2 z$$

Self-phase modulation (SPM):

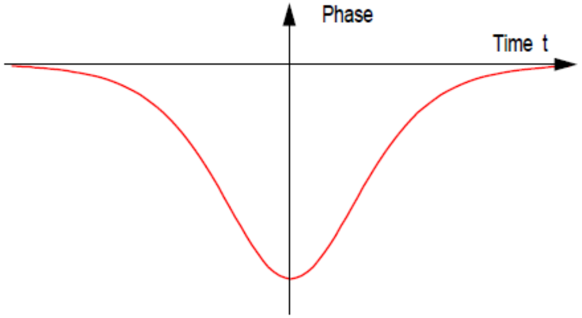
Nonlinear phase modulation of a pulse, caused by its own intensity via the Kerr effect.

SPM induces positive chirp

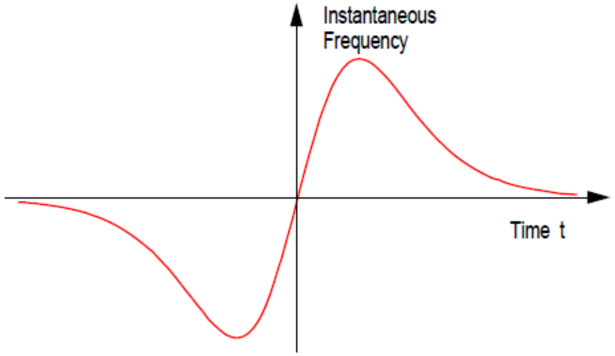
$$|A(z,t)|^2$$



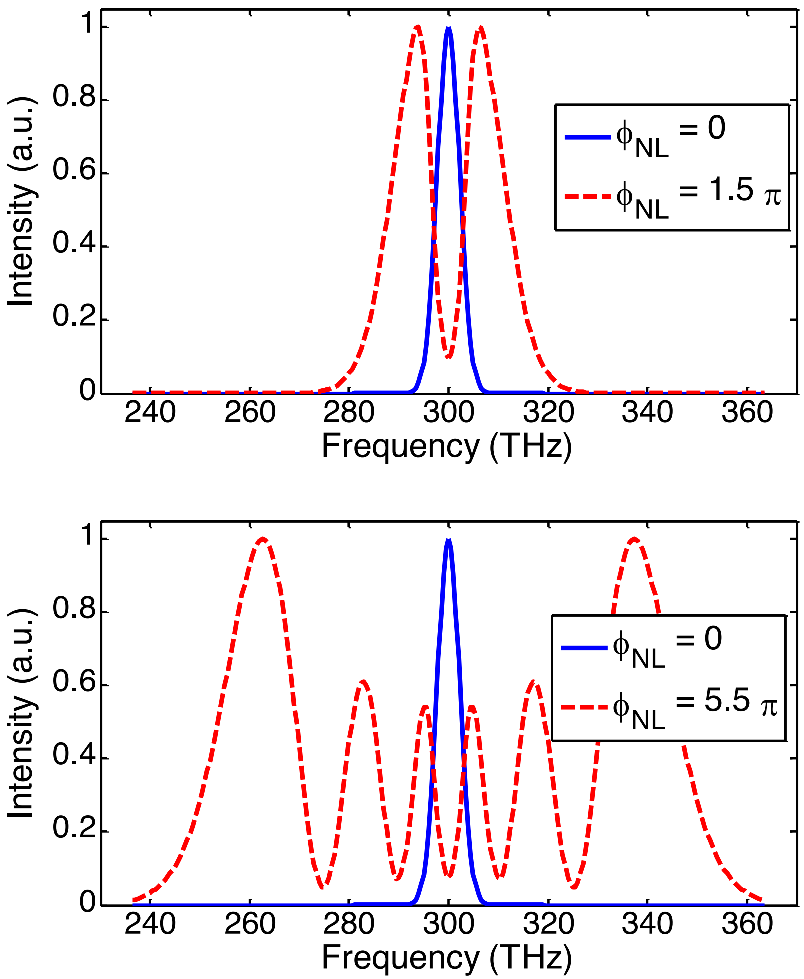
$$\varphi_{NL} = -\delta|A(z,t)|^2 z$$

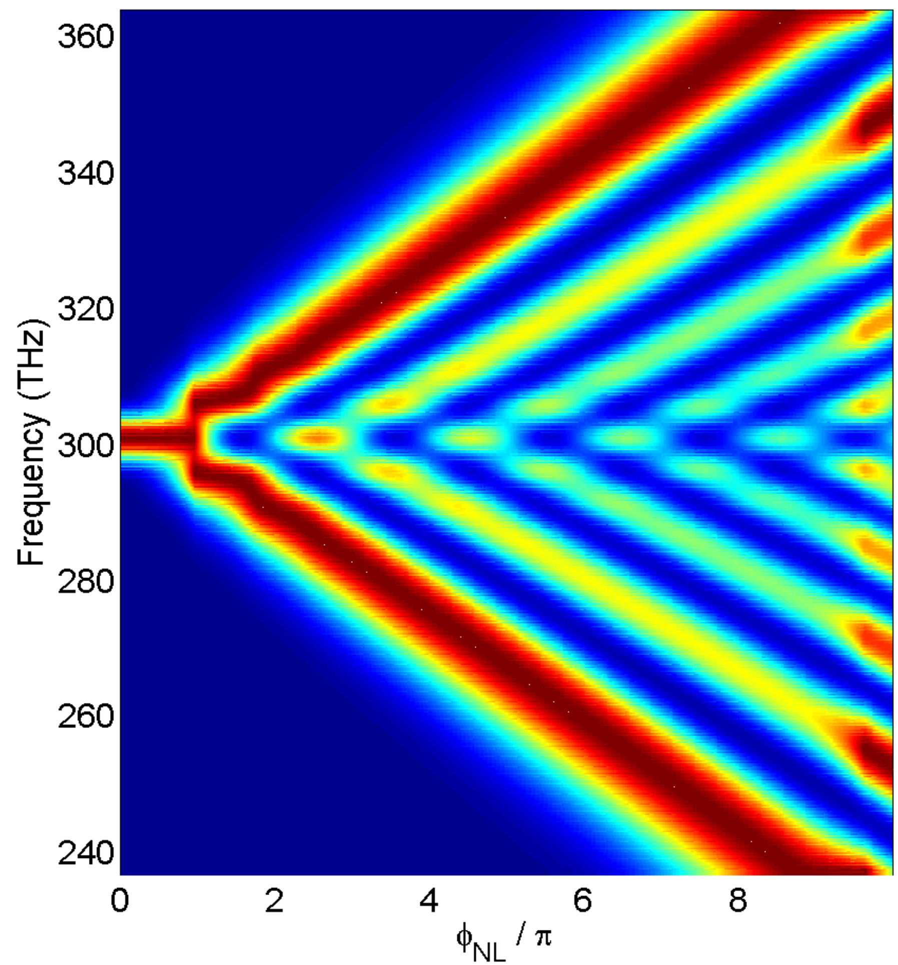


$$\Delta\omega = \frac{d\varphi_{NL}}{dt} = -\delta z \frac{d|A(z,t)|^2}{dt}$$



SPM modifies spectrum

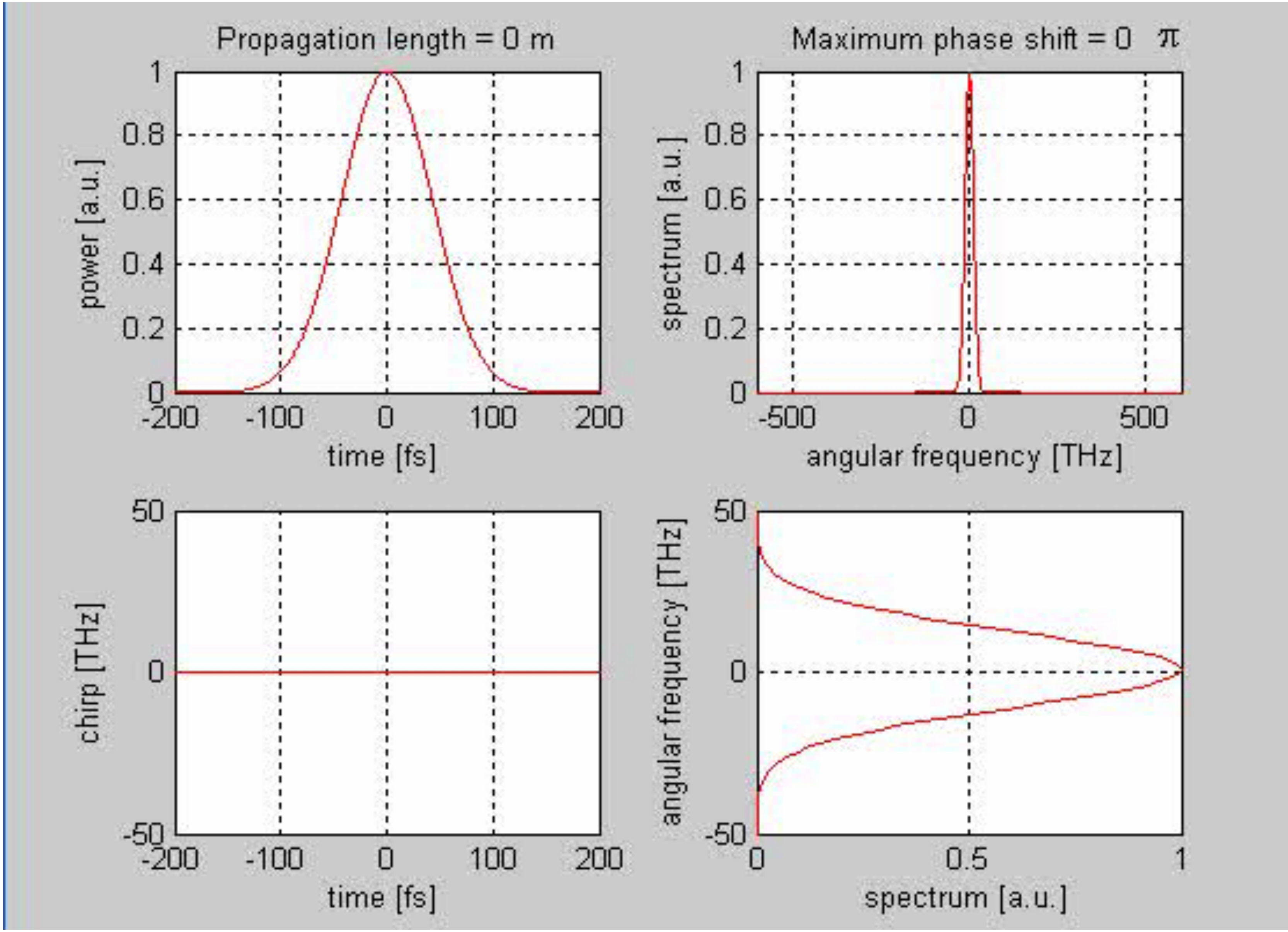




Spectral bandwidth is proportional to the amount of nonlinear phase accumulated inside the fiber.

$$\phi_{NL} \approx (M - \frac{1}{2}) \times \pi$$

M is the number of spectral peaks.



Input: Gaussian pulse, Pulse duration = 100 fs, Peak power = 1 kW

Pulse propagation: pure dispersion Vs pure SPM

- Pure dispersion $$j\frac{\partial A(z,t)}{\partial z} = -D_2 \frac{\partial^2 A}{\partial t^2} \qquad D_2 = \frac{\beta_2}{2}$$

$=k^{(2)}$

$\beta_2 \rightarrow$ GVD

(1) Pulse's spectrum acquires phase.

(2) Spectrum profile does not change.

(3) In the time domain, pulse may be stretched or compressed depending on its initial chirp .

- Pure SPM $$j\frac{\partial A(z,t)}{\partial z} = \delta |A|^2 A$$

(1) Pulse acquires phase in the time domain.

(2) Pulse profile does not change.

(3) In the frequency domain, pulse's spectrum may be broadened or narrowed depending on its initial chirp.

Nonlinear Schrödinger Equation (NLSE)

$$j\frac{\partial A(z,t)}{\partial z} = -D_2\frac{\partial^2 A}{\partial t^2} + \delta|A|^2 A \qquad D_2 = \frac{\beta_2}{2}$$

Positive GVD (normal dispersion) + SPM:

GVD and SPM both act to shift the red frequency to the front of the pulse. Therefore the pulse will spread faster than it would in the purely linear case.

Negative GVD (anomalous dispersion) + SPM:

GVD and SPM shift frequency in the opposite direction. At a certain condition, the dispersive spreading of the pulse is exactly balanced by the compression due to the opposite chirp induced by SPM. A steady-state pulse can propagate without changing its shape. (i.e. soliton regime)

NLSE has soliton solution.

$$j\frac{\partial A'(z',t)}{\partial z'} = \frac{\partial^2 A'}{\partial t^2} + 2|A'|^2 A'$$

3.3.2 The Fundamental Soliton

$$A_s(z,t) = A_0 \text{sech}\left(\frac{t}{\tau}\right) e^{-j\theta}$$

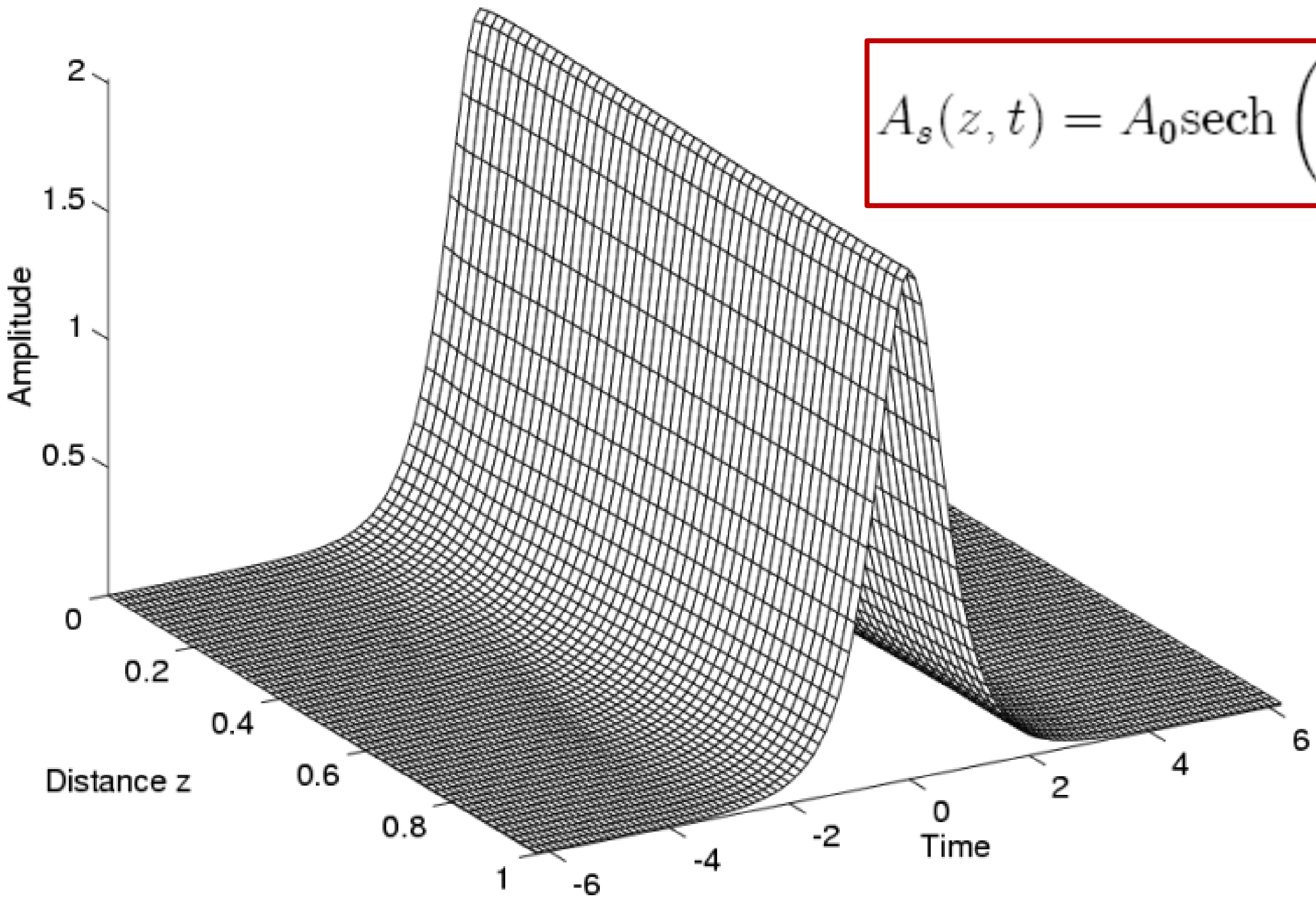


Figure 3.3: Propagation of a fundamental soliton

Important Relations

$$\boxed{\delta A_0^2 = \frac{2|D_2|}{\tau^2} \left(= \frac{|\beta_2|}{\tau^2}\right)} \Longrightarrow \boxed{A_s(z,t) = A_0 \mathrm{sech}\left(\frac{t}{\tau}\right) e^{-j\theta}}$$

(Balance between dispersion and nonlinearity)

Nonlinear phase shift soliton acquires during propagation:

$$\theta = \frac{1}{2}\delta A_0^2 z = \frac{|D_2|}{\tau^2} z$$

Area Theorem

$$\text{Pulse Area} = \int_{-\infty}^{\infty} |A_s(z,t)| dt = \pi A_0 \tau = \pi \sqrt{\frac{2|D_2|}{\delta}}.$$

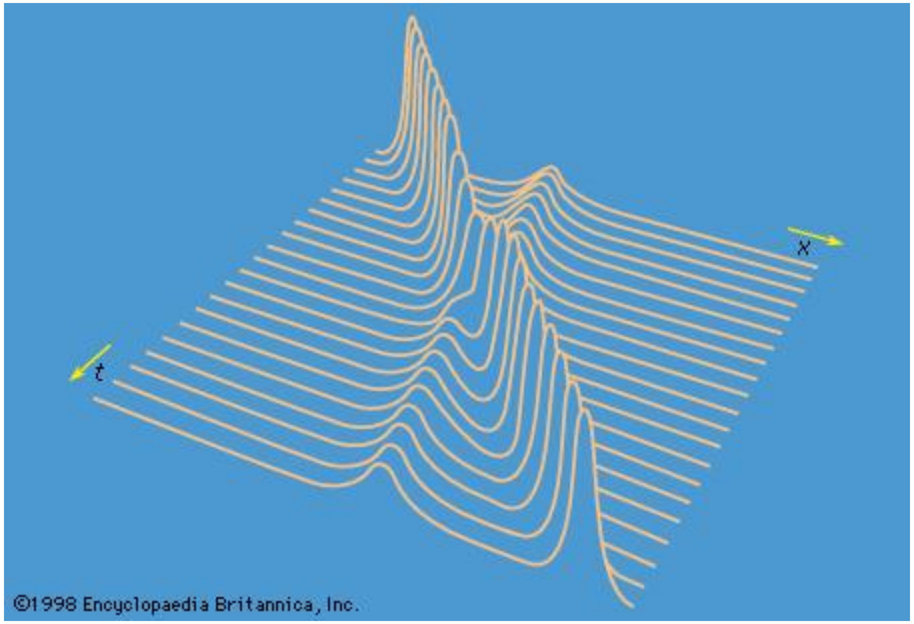
Soliton Energy: $w = \int_{-\infty}^{\infty} |A_s(z,t)|^2 dt = 2 A_0^2 \tau$ **Pulse width:** $\tau = \dfrac{4|D_2|}{\delta w}$

General properties of soliton

In mathematics and physics, a **soliton** is a self-reinforcing solitary wave (a wave packet or pulse) that maintains its shape while it travels at constant speed. Solitons are caused by a cancellation of nonlinear and dispersive effects in the medium. ---Wiki

- When two solitons get closer, they gradually collide and merge into a single wave packet.

- This packet soon splits into two solitons with the same shape and velocity before "collision".

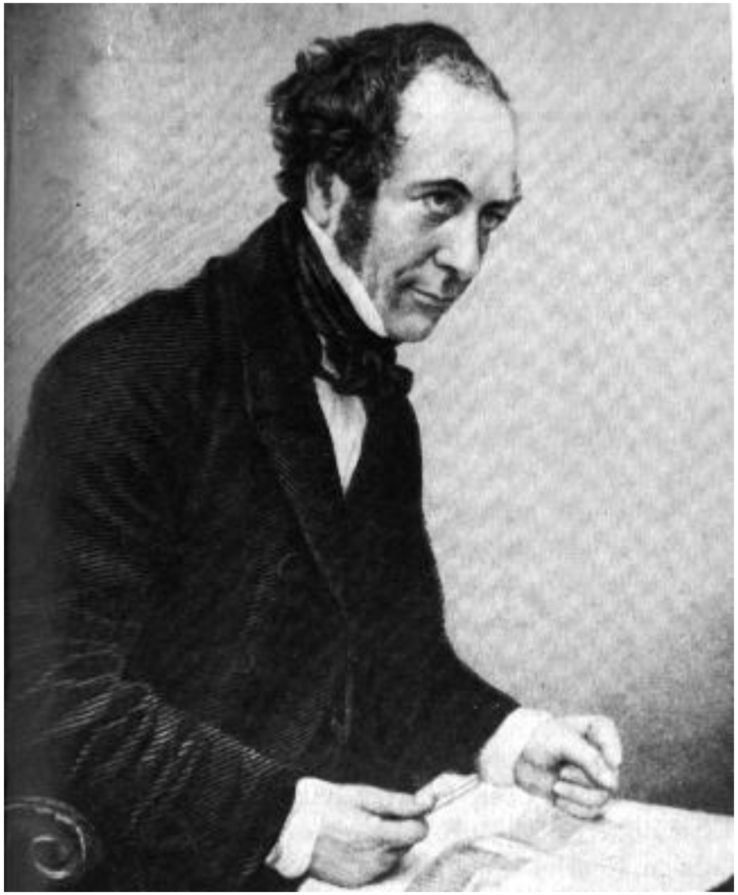


Who discovered solitons?

John Scott Russell (1808 – 1882) was a Scottish civil engineer, naval architect and shipbuilder.

In 1834, while conducting experiments to determine the most efficient design for canal boats, John Scott Russell made a remarkable scientific discovery, leading to a conference paper—Report on Waves.

Report of the fourteenth meeting of the British Association for the Advancement of Science, York, September 1844 (London 1845), pp 311-390, Plates XLVII-LVII).



John Scott Russell (1808-1882)

Russell's report

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed."

"I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation."

Report of the fourteenth meeting of the British Association for the Advancement of Science, York, September 1844 (London 1845), pp 311-390, Plates XLVII-LVII).

Water wave soliton in Scott Russell Aqueduct

89.3m long, 4.13m wide,1.52m deep, On the union Canal, Near Heroit-Watt Univ.

*www.spsu.edu/math/txu/research/presentations/**soliton/talk**.**ppt***

Water wave soliton in Scott Russell Aqueduct

www.spsu.edu/math/txu/research/presentations/**soliton/talk**.**ppt**

A brief history (mainly for optical solitons)

- 1838 – soliton observed in water
- 1895 – KdV equation: mathematical description of waves on shallow water surfaces.
- 1972 – optical solitons arising from NLSE and Inverse Scattering Theory
- 1980 – experimental demonstration in optical fibers
- 1990's – development of techniques to control solitons
- 2000's – understanding solitons in the context of supercontinuum generation

Soliton solution of NLSE：fundamental soliton

$$j\frac{\partial A(z,t)}{\partial z} = -D_2\frac{\partial^2 A(z,t)}{\partial t^2} + \delta\left|A(z,t)\right|^2 A(z,t)$$

The NLSE possesses the following genereral fundamental soliton solution:

$$A_s(z,t) = A_0\text{sech}(x(z,t))e^{-j\theta(z,t)} \qquad \text{sech}(x) = \frac{2}{e^x + e^{-x}}$$

$$\left.\begin{aligned} x &= \frac{1}{\tau}(t - 2|D_2|p_0 z - t_0) \\ \theta &= p_0(t-t_0) + |D_2|\left(\frac{1}{\tau^2} - p_0^2\right)z + \theta_0 \\ \frac{\delta A_0^2}{2} &= \frac{|D_2|}{\tau^2} \end{aligned}\right\}$$

Four degrees of freedom:

energy fluence w or amplitude A_0

carrier frequency p_0

phase θ_0

origin t_0

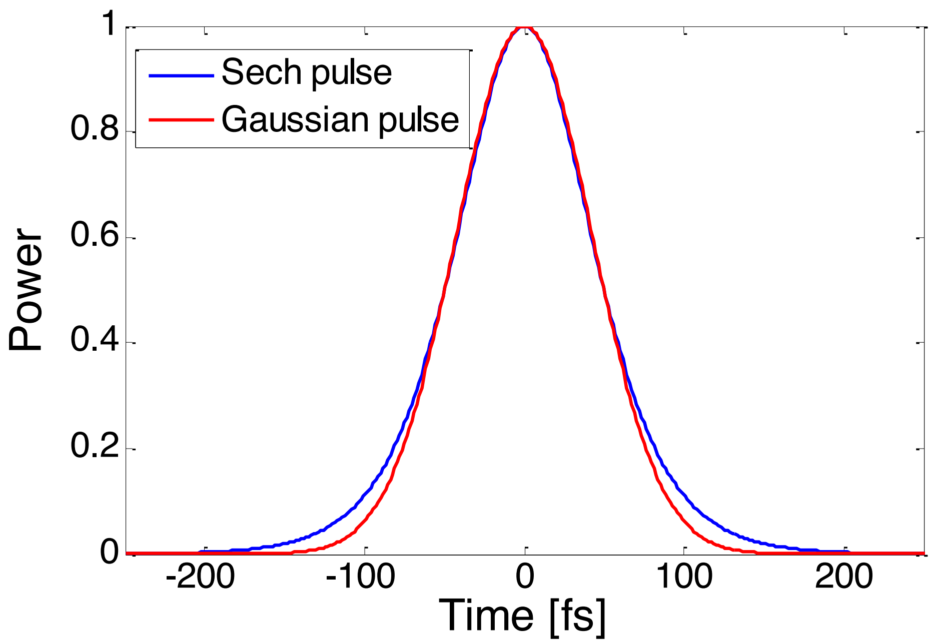
These 4 parameters can be arbitrarily chosen, e.g.,

arbitrary τ $\quad p_0 = 0 \quad \theta_0 = 0 \quad t_0 = 0$

Soliton solution of NLSE：fundamental soliton

$$j\frac{\partial A(z,t)}{\partial z} = -D_2 \frac{\partial^2 A(z,t)}{\partial t^2} + \delta \left| A(z,t) \right|^2 A(z,t)$$

$$A_s(z,t) = A_0 \mathrm{sech}\left(\frac{t}{\tau}\right) e^{-j\theta} \qquad \mathrm{sech}(x) = \frac{2}{e^x + e^{-x}}$$

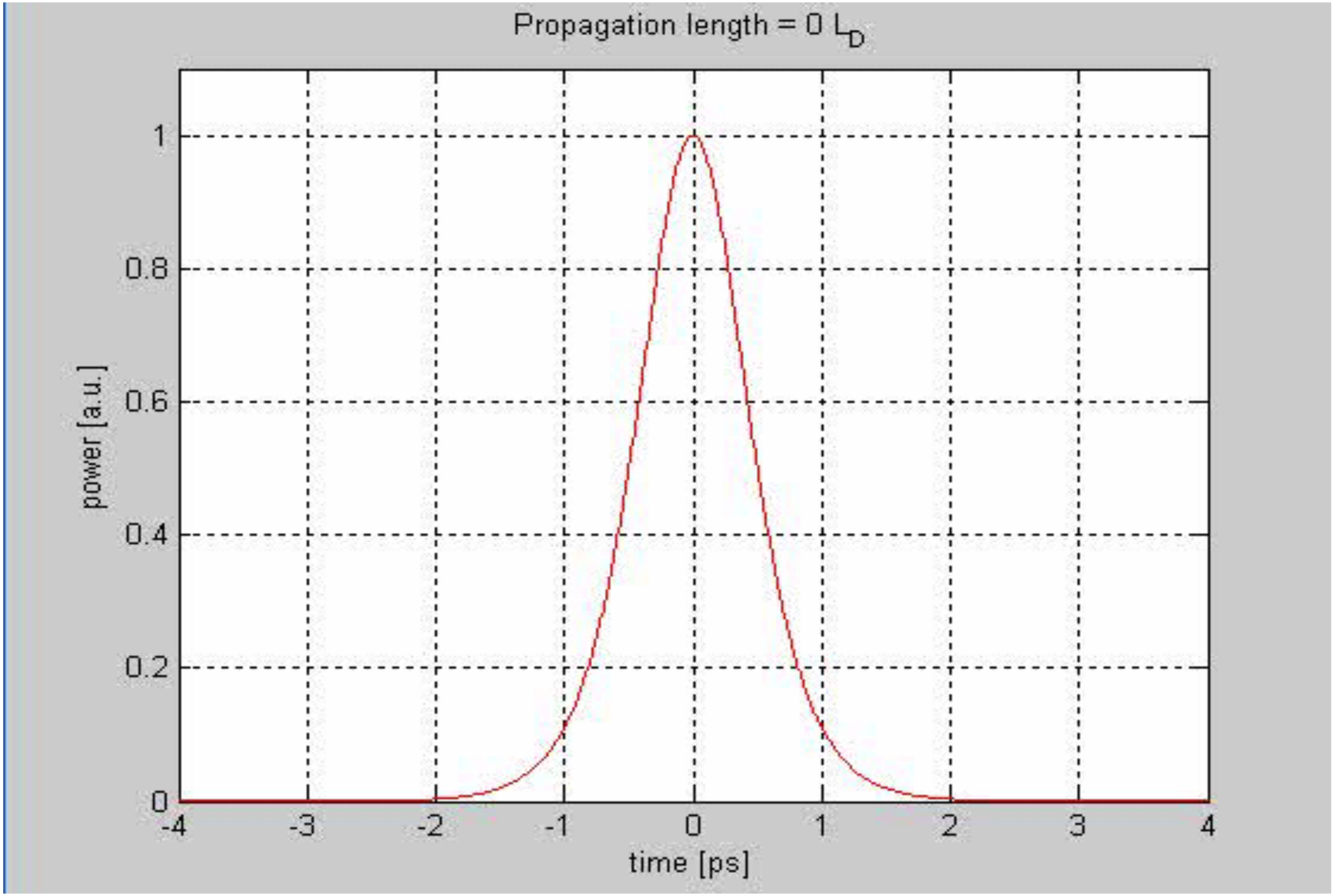


$$\underbrace{\frac{\delta A_0^2}{2}}_{\text{nonlinearity}} = \underbrace{\frac{|D_2|}{\tau^2}}_{\text{dispersion}}$$

Soliton is the result of balance between nonlinearity and dispersion.

A phase linearly proportional to propagation distance: $\theta = \frac{1}{2}\delta A_0^2 z$

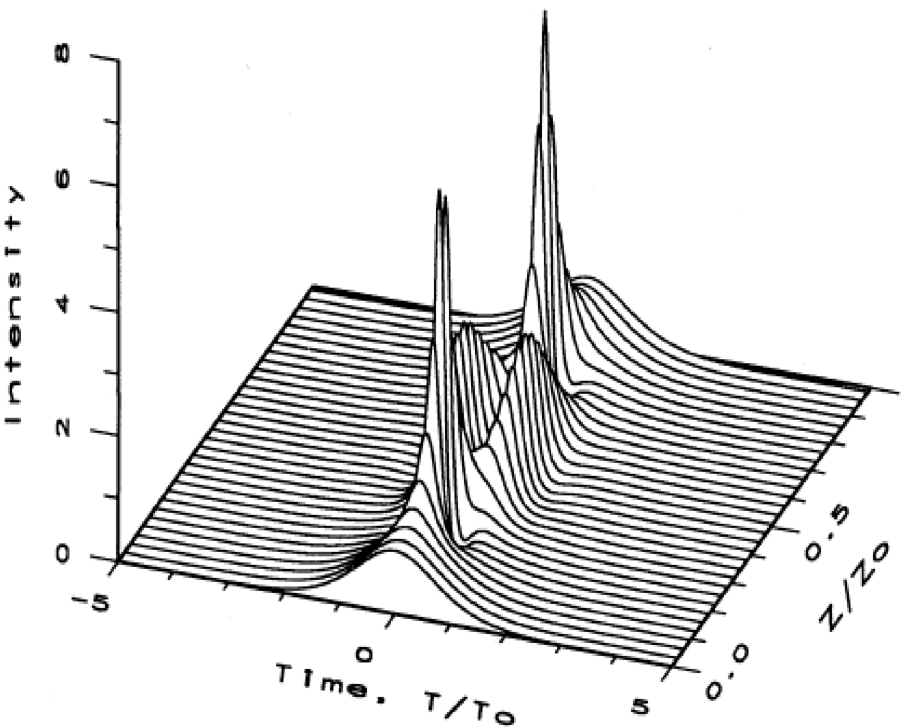
Propagation of fundamental soliton

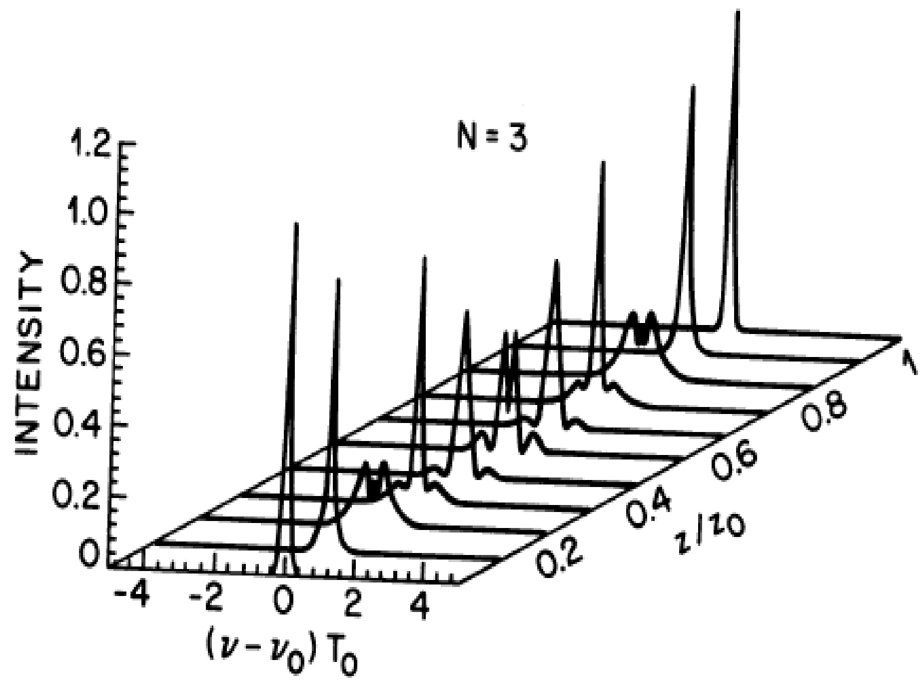


Input: 1ps soliton centered at 1.55 um; medium: single-mode fiber

Higher-order Solitons: periodical evolution in both the time and the frequency domain

$$A_0\tau = N\sqrt{\frac{2|D_2|}{\delta}}, N = 1,2,3\ldots \qquad A_s(z,t) = NA_0\mathrm{sech}(\frac{t}{\tau})e^{-j\theta}$$





G. P. Agrawal, *Nonlinear fiber optics* (2001)

3.3.3 Higher Order Soliton (Breather Soliton)

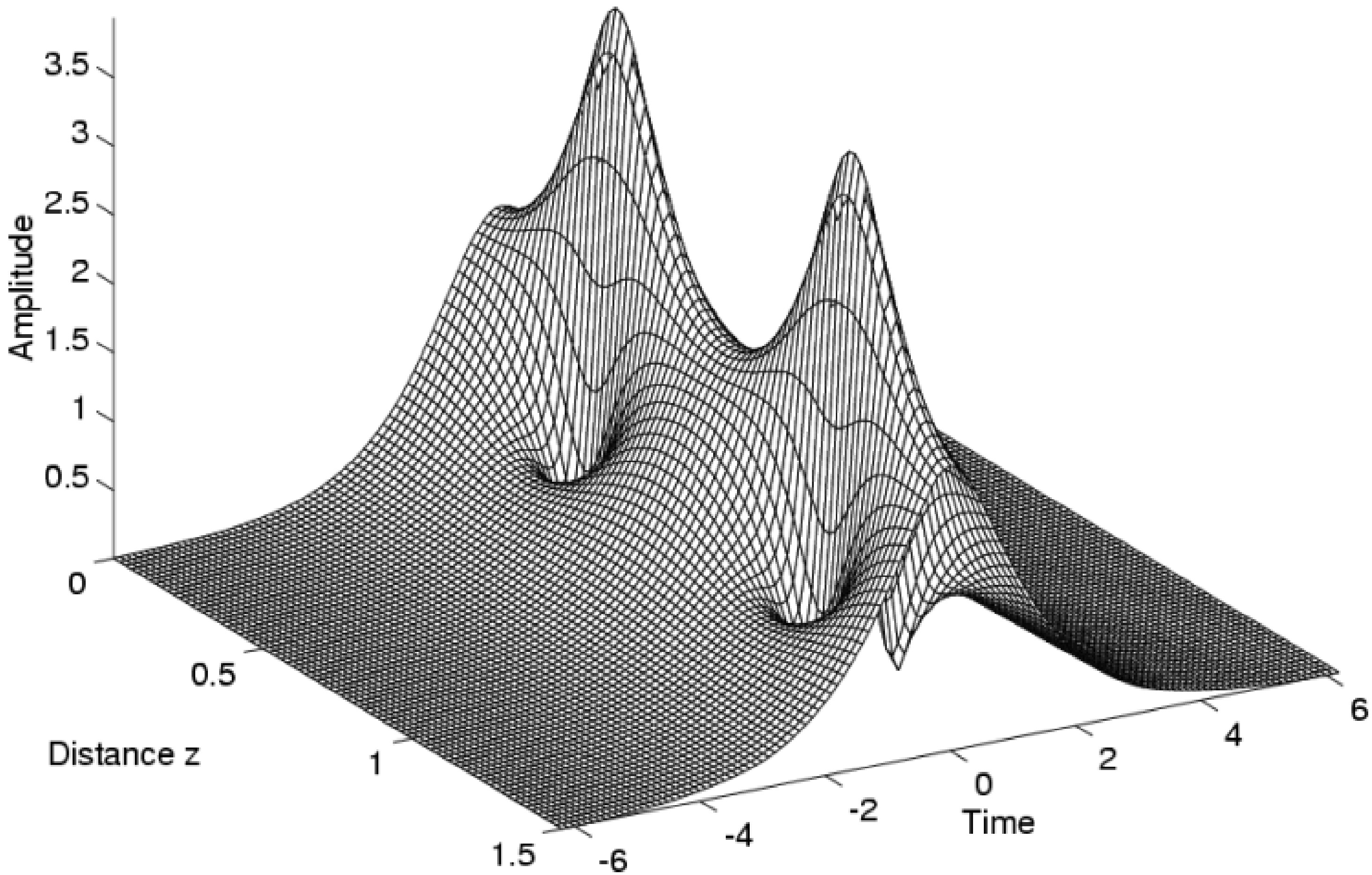


Figure 3.5a: Amplitude of higher order soliton composed of two fundamental solitons with the same carrier freuqency

3.3.3 Higher Order Soliton (Breather Soliton)

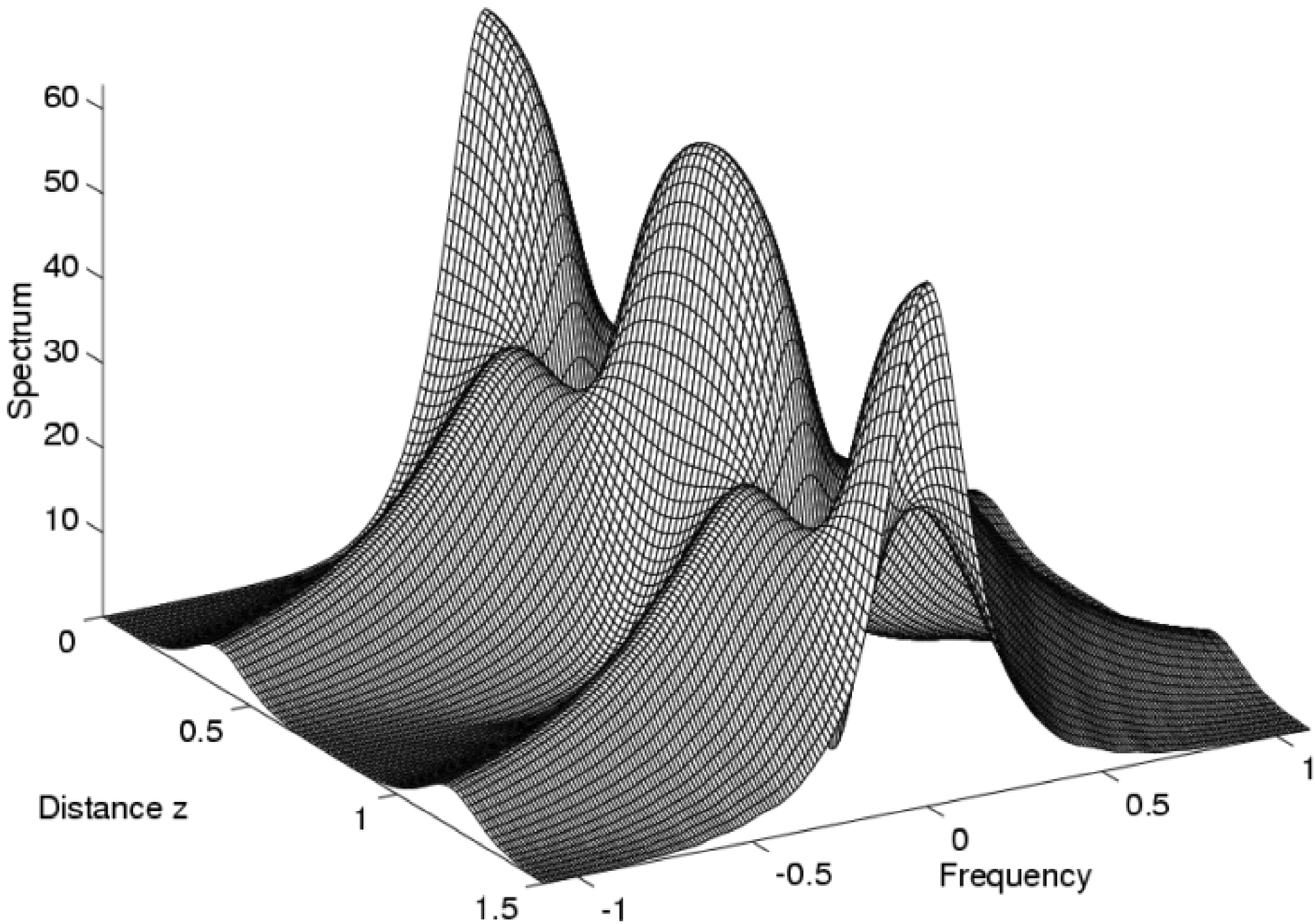


Figure 3.5b: Spectrum of higher order soliton composed of two fundamental solitons with the same carrier freuqency

Interaction between solitons (soliton collision)

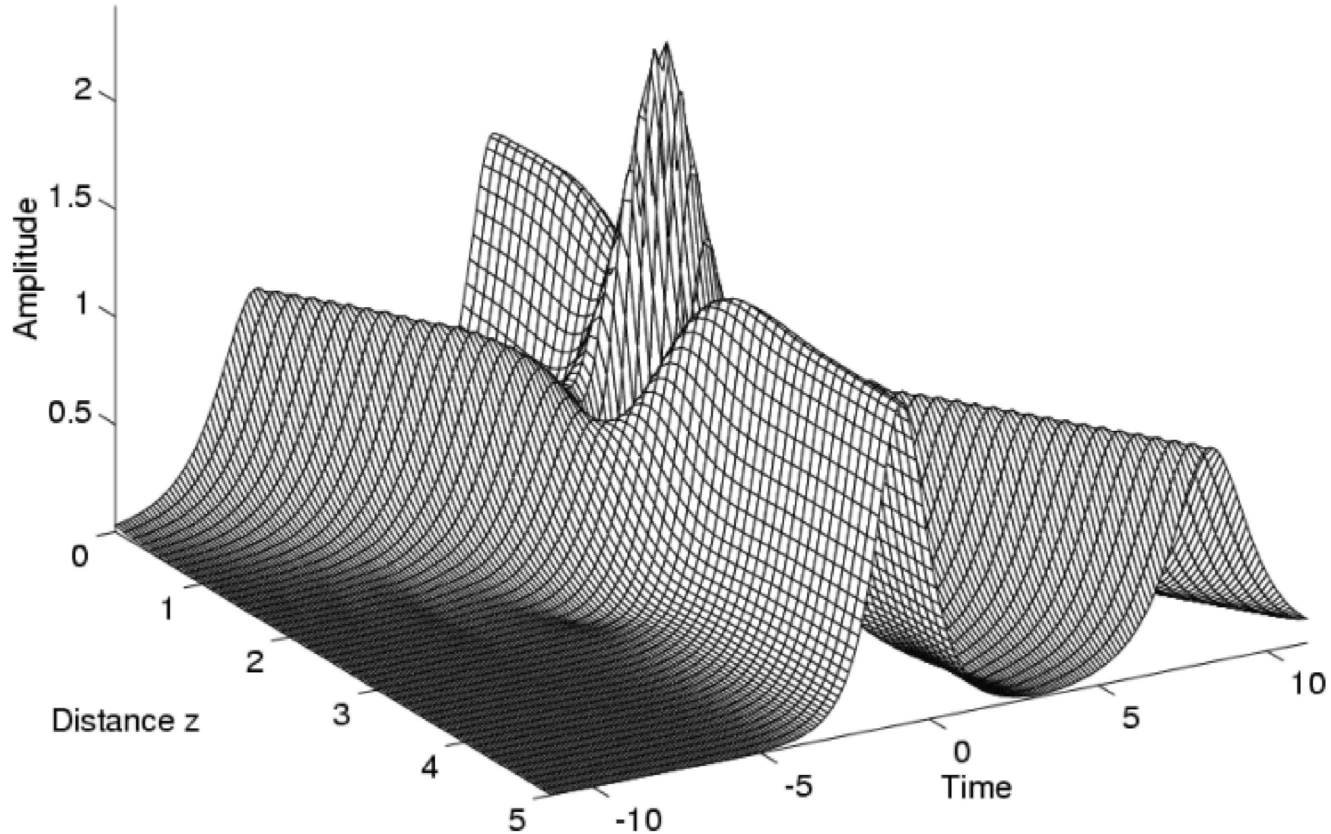
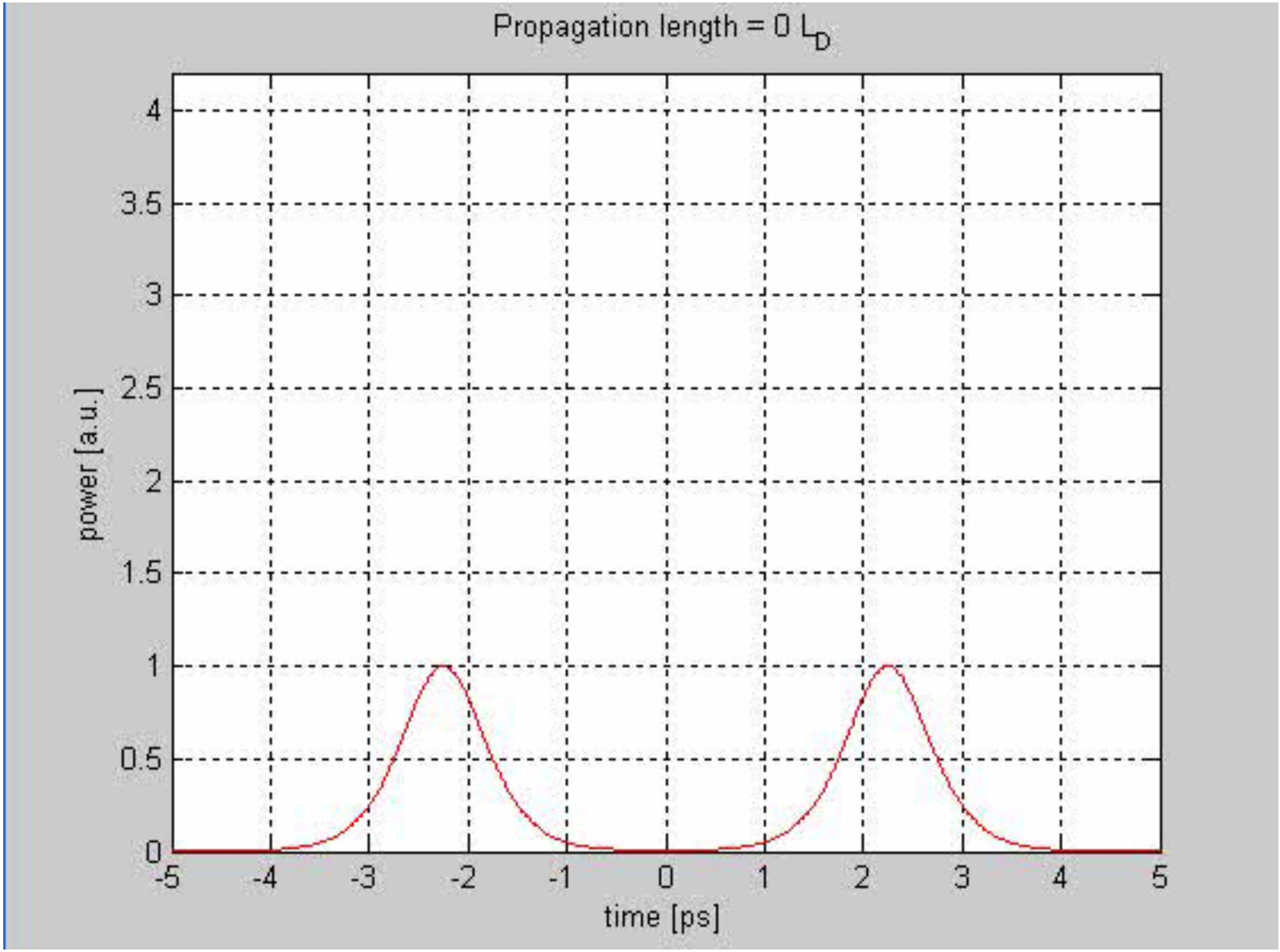
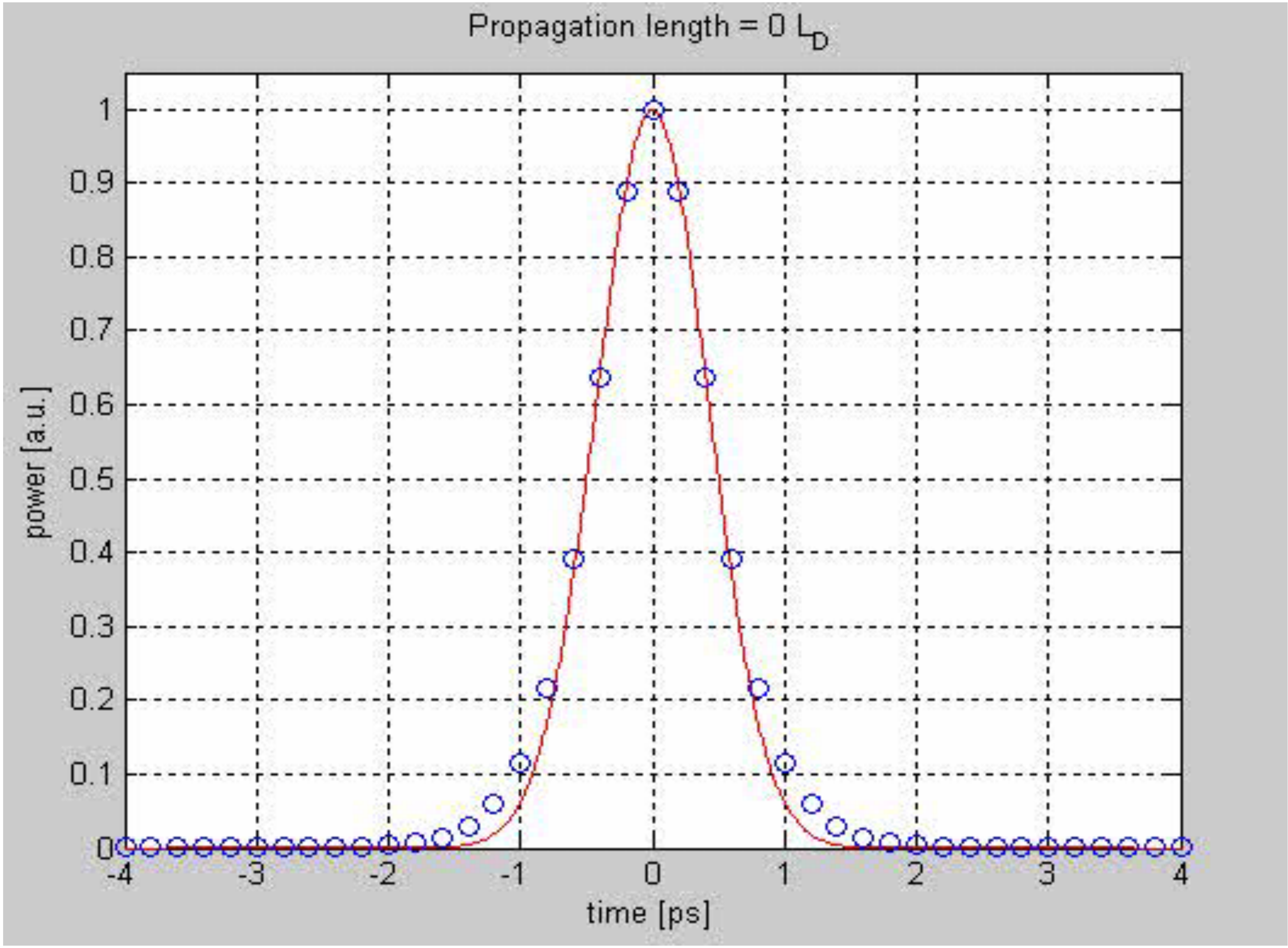


Figure 3.4: A soliton with high carrier frequency collides with a soliton of lower carrier frequency.

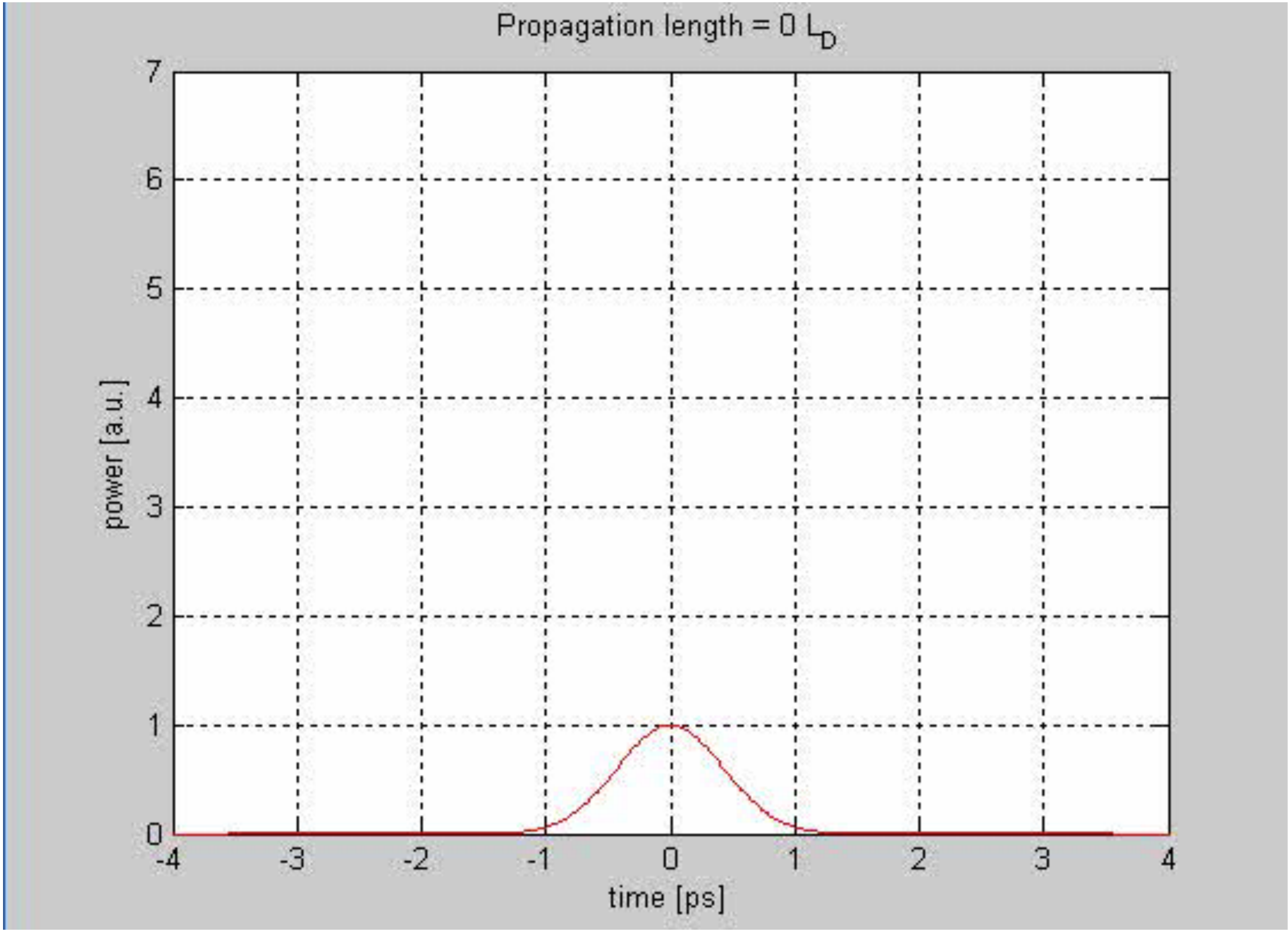
Interactions of two fundamental solitons



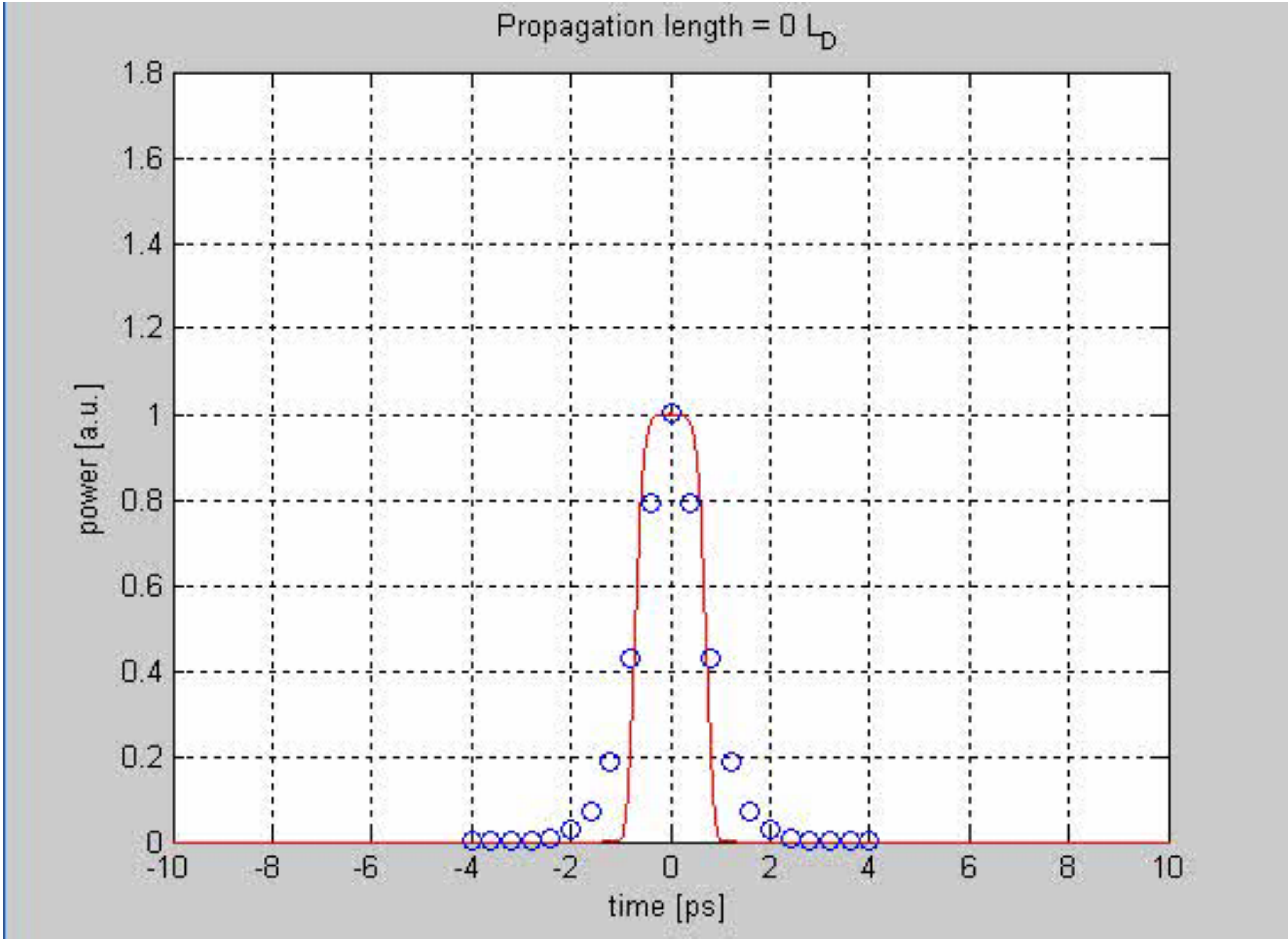
From Gaussian pulse to fundamental soliton



Gaussian pulse to 3-order soliton



Evolution of a super-Gaussian pulse to soliton



Soliton perturbation theory: a very brief introduction

$$\frac{\partial A(z,t)}{\partial z} = \underbrace{-j\left[|D_2|\frac{\partial^2 A}{\partial t^2} + \delta|A|^2 A\right]}_{\text{Perfect World}} + \underbrace{F(A, A^*, z)}_{\text{Reality: Perturbations}}$$

Without perturbations

$$\left.\begin{aligned}
&A_s(z,t) = A_0 \mathrm{sech}(x(z,t)) e^{-j\theta(z,t)} \\
&x = \frac{1}{\tau}(t - 2|D_2|p_0 z - t_0) \\
&\theta = p_0(t - t_0) + |D_2|\left(\frac{1}{\tau^2} - p_0^2\right) z + \theta_0 \\
&\frac{\delta A_0^2}{2} = \frac{|D_2|}{\tau^2}
\end{aligned}\right\}$$

Four degrees of freedom:

energy fluence w or amplitude A_0

carrier frequency p_0

phase θ_0

origin t_0

What happens to the soliton in the presence of perturbations? Will it fall apart?

Is it just kicked around? If yes, can we understand how it is kicked around?

Soliton perturbation theory: a very brief introduction

$$\frac{\partial A(z,t)}{\partial z} = -j\left[|D_2|\frac{\partial^2 A}{\partial t^2} + \delta |A|^2 A\right] + F(A, A^*, z)$$

Ansatz: Solution of perturbed equation is a soliton + a small component:

$$A(z,t) = \left[a(\frac{t}{\tau}) + \Delta A(z,t)\right] e^{-jk_s z}$$ with: $$a(\frac{t}{\tau}) = A_0 \operatorname{sech}(\frac{t}{\tau}) \quad k_s = \frac{1}{2}\delta A_0^2$$

Any deviation ΔA can be decomposed into a contribution that leads to a soliton with a shift in the four soliton parameters and a continuum contribution:

$$\Delta A(z) = \underbrace{\Delta w(z) f_w}_{\text{Energy fluctuation}} + \underbrace{\Delta \theta(z) f_\theta}_{\text{Optical phase fluctuation}} + \underbrace{\Delta p(z) f_p}_{\text{Center frequency fluctuation}} + \underbrace{\Delta t(z) f_t}_{\text{Timing fluctuation}} + \underbrace{a_c(z)}_{\text{Continuum background}}$$

$$f_w = \frac{\partial A}{\partial w}$$

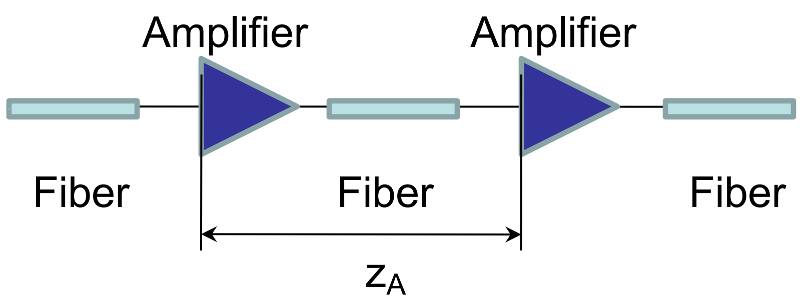
$$f_\theta = \frac{\partial A}{\partial \theta}$$

$$f_p = \frac{\partial A}{\partial p}$$

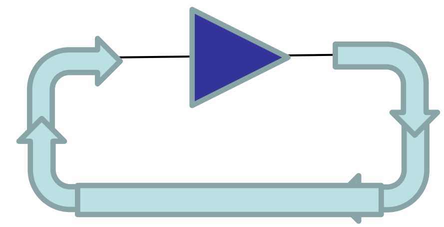
$$f_t = \frac{\partial A}{\partial t}$$

Soliton instabilities by periodic perturbations

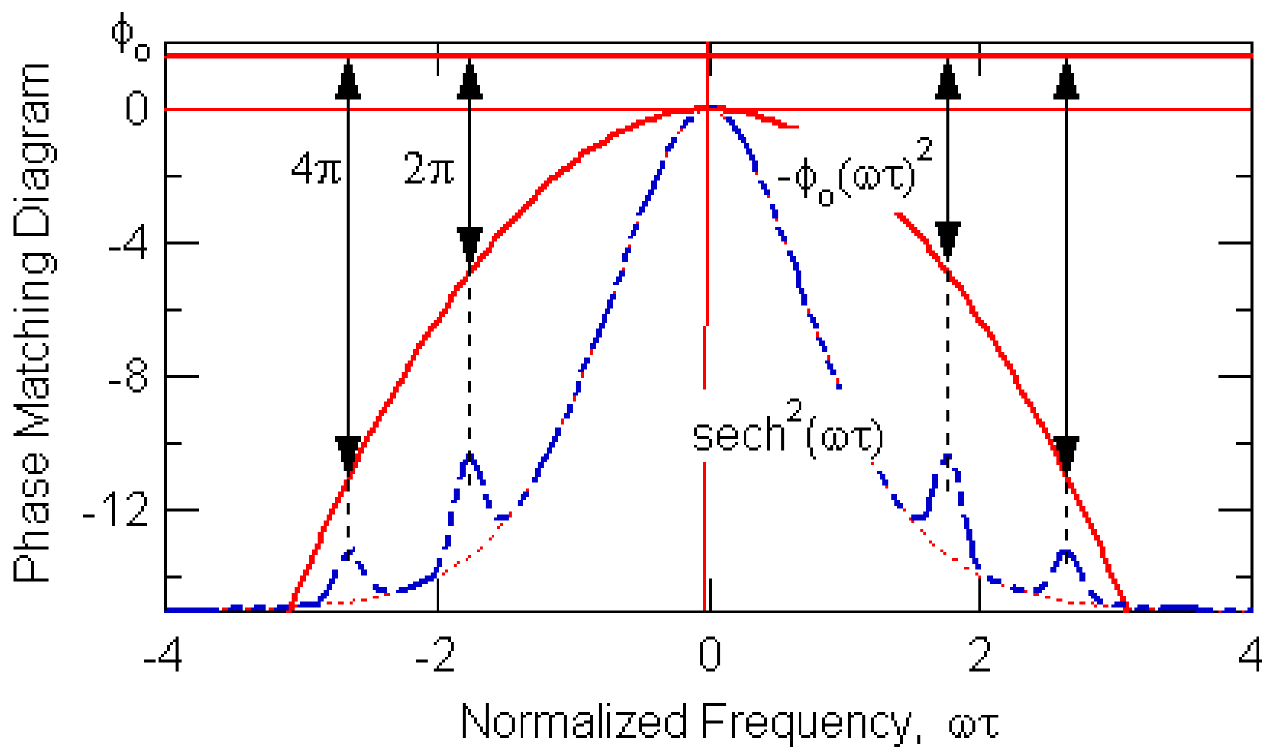
Long haul opt. communication link



Modelocked fiber laser



$$F(A, A^*, z) = \xi \sum^{\infty} \delta(z - nz_A)A(z,t).$$



Rogue wave

Find more information from New York times:
http://www.nytimes.com/2006/07/11/science/11wave.html

One more Rogue wave