University of Hamburg, Department of Physics

Nonlinear Optics

Kärtner/Mücke, WiSe 2019/2020

Problem Set 9 (optional)

Issued: 10.01.'20 Due : 24.01.'20

Introduction to RK4 and split step Fourier method

For the computations in this problem set we will choose the 4th order Runge Kutta (RK4) method $O(h5)$. For detail please see the references given at the end and feel free to extend your reading.[2;5;6]

In order to solve the nonlinear Schödinger equation numerically, the split step Fourier method can be used. As an example we will take effects such as self phase modulation (SPM) and group velocity dispersion (GVD) into account.

$$\begin{aligned}\frac{\partial E(t,z)}{\partial z} &= \left(-i\frac{1}{2}\beta_2\frac{\partial^2}{\partial t^2} - i\frac{n_2\omega_0 I(t,z)}{c}\right)E(t,z)\\ &= (\mathcal{L}+\mathcal{N})E(t,z) \qquad (1)\end{aligned}$$

$$\begin{aligned}\mathcal{L} &= -i\frac{1}{2}\beta_2\frac{\partial^2}{\partial t^2}\\ \mathcal{N} &= -i\frac{n_2\omega_0 I(t,z)}{c}\end{aligned}$$

where $\mathcal{N}$ is a nonlinear operator in which n_2 is the nonlinear coefficient of SPM and $\mathcal{L}$ is a linear operator in which β_2 is the coefficient associated with GVD. The numerical method we want to use to solve equation (1) is the split step method.[1;4] The idea of the split step model is to calculate the linear operator in the frequency domain. The non linear operator in contrast is calculated in the time domain separately.[3] As such we will neglect the linear operator in the time domain and only consider the nonlinear operator. We get:

$$\begin{aligned}\frac{\partial E(t,z)}{\partial z} &= \mathcal{N}E(t,z) \qquad (2)\\ E(t,z+\frac{\triangle z}{2}) &= E(t,z)e^{\frac{\triangle z}{2}\mathcal{N}}\end{aligned}$$

In the frequency domain, neglecting the nonlinear operator and only considering the linear operator, we obtain

$$\begin{aligned}\frac{\partial E(t,z)}{\partial z} &= \mathcal{L}E(t,z)\\ \frac{\partial \tilde{E}(\omega,Z)}{\partial Z} &= i\left(\frac{1}{2}\beta_2\omega^2\right)\tilde{E}(\omega,Z)\\ \tilde{E}(\omega,z+\frac{\triangle z}{2}) &= \tilde{E}(\omega,z)e^{\left(i\frac{1}{2}\beta_2\omega^2\right)\frac{\triangle z}{2}}\end{aligned}$$

After calculating the $\frac{\triangle z}{2}$ segment in both time and frequency domain separately, the total final numerical solution of equation (1) is[3]

$$E(t,z+\triangle z) = \mathcal{F}^{-1}\left[e^{\left(i\frac{1}{2}\beta_2\omega^2\right)\frac{\triangle Z}{2}}\mathcal{F}[e^{\frac{\triangle z}{2}\mathcal{N}}E(t,z)]\right] \qquad (3)$$

where $\mathcal{F}$ represents the Fourier transform. However, equation (3) is usually not sufficiently precise. A better solution is to implement the so called Runge Kutta (RK) method than rather merely using exponentials of the operators as solutions. As an example, equation (2) is solved using the RK4 method here:

$$\frac{\partial E(t,z)}{\partial z} = -i\frac{n_2\omega_0 I(t,z)}{c}E(t,z)$$
$$f(E,t,z) = -i\frac{n_2\omega_0 I(t,z)}{c}E(t,z)$$

where the right hand side is written as $f(E,t,z)$ for convenience. The idea is to use the known data value at grid point z_i to calculate the next grid point z_{i+1} via[2]

$$\begin{aligned}
k_1 &= f(E_i, t, z_i) \triangle z \\
k_2 &= f(E_i + \frac{k_1}{2}, t, z_i + \frac{\triangle z}{2}) \\
k_3 &= f(E_i + \frac{k_2}{2}, t, z_i + \frac{\triangle z}{2}) \\
k_4 &= f(E_i + k_3, t, z_i + \triangle z) \\
E(t, z_{i+1}) &= E(t, z_i) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\end{aligned}$$

1 Simulation of SPM using RK4 - Due Date 17.01.'20

With the techniques described above, write a code to simulate the evolution of an optical field under the influence of self-phase modulation in BBO. The BBO crystal shall have an effective length of $L_{eff} =$ 2.5mm and you can assume the initial pulse to have a waist size of $w =$ 1mm, a fluence of $F = \frac{0.5}{\pi * w^2/2}\mathrm{mJ/m^2}$ and a FWHM duration of 150fs.
You will be provided with a code example via email where the part of the code (namely the SPM code for the frequency domain) is given as a reference and starting point. Please write your own code for the pulse propagation including SPM using the RK4 method for the time domain.

2 Simulation of an OPA - Due Date 24.01.'20

For simplicity, let's consider the situation when the nonlinear Schödinger equation only contains two term for the pump—SPM and second order polarization of difference frequency generation (DFG); one term for signal and idler (DFG). In the following, everything is converted to frequency domain. It is the same concept as in a split step Fourier model, but the difference is that here we don't Fourier transform the field, but we transform the terms on the right hand side. Please write a code to solve equation (4,5,6) using the same method as used in homework 1. Note: you are advised to solve everything in the frequency domain, but of course you could also do that in the time domain if you wished. We will provide you with a Matlab code template, that will serve as a good starting point and requires you to add certain sections to the code in order to make it work.

$$\frac{\partial E_p(\omega,z)}{\partial z} = P_p^{(NL)}(\omega) - \mathcal{F}[i\frac{n_2\omega_0 I_p(t,z)}{c}E_p(t,z)] \tag{4}$$

$$\frac{\partial E_s(\omega,z)}{\partial z} = P_s^{(NL)}(\omega) \tag{5}$$
$$\frac{\partial E_i(\omega,z)}{\partial z} = P_i^{(NL)}(\omega) \tag{6}$$

It is VERY important to know that the polarization terms in all the equations about are the results of all the possible combinations of frequencies. For instance, this is how to calculate the polarization term for the signal:

$$
\begin{aligned}
P_s^{(2)}(f_s) \quad &= \chi^{(2)} \int_{-\infty}^{\infty} E^*(f_p - f_s) E(f_p) df_p \\
&= \chi^{(2)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (E(t_1) e^{i2\pi t_1 (f_p - f_s)})^* dt_1 E(t_2) e^{i2\pi t_2 f_p} dt_2 df_p \\
&= \chi^{(2)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(t_1)^* E(t_2) e^{i2\pi (t_2 - t_1) f_p} e^{i2\pi t_1 f_s} dt_1 dt_2 df_p \\
&= \chi^{(2)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(t_1)^* E(t_2) \delta(t_2 - t_1) e^{i2\pi t_1 f_s} dt_1 dt_2 \\
&= \chi^{(2)} \int_{-\infty}^{\infty} |E(t_1)|^2 e^{i2\pi t_1 f_s} dt_1
\end{aligned} \tag{7}
$$

To help you out a little bit, a code example for Matlab is given here:

```
function [f_ir] =P_NL(chi_2,u_p,u_s,u_i,k_p,k_s,k_i,f_s,n_s,z0)
%note that u_s u_i u_p are the envelop of the field , e_i,e_s,e_p are the field
    c=3e8;
    e_i=u_i.*exp(-1i*k_i*z0);
    e_s=u_s.*exp(-1i*k_s*z0);
    e_p=u_p.*exp(-1i*k_p*z0);


    df=f_s(2)-f_s(1);
    N=length(f_s);
%inverse Fourier transform field of pump E_p and idler E_i* to time domain
    buff1 = fftshift(ifft(ifftshift(e_p))).*N*df;
    buff2 = conj(fftshift(ifft(ifftshift(e_i))).*N*df);
%  Fourier transform the result back to frequency domain. the exp(1i*k_s*z0) comes ...
     from the phase matching
    p_s =chi_2*fftshift(fft(ifftshift(buff2.*buff1))).*exp(1i*k_s*z0)./(N*df);

    f_ir =-0.5i*(2*pi)^2*f_s.^2.*p_s./(c^2.*k_s);

end
```

Please note: Feel free to use whatever language you want. Our suggestions are Matlab (available via the university's Rechenzentrum for free) or Python. In the case of Python, the suggestion is to download the latest version of Anaconda as this includes all the necessary scientific computation packages as well as IDEs such as Spyder. Also note: the example code you will be given for Homework 1 will be written in Matlab.

We hope you have fun tackling this challenge! If you have any questions along the way, feel free to contact us and we will try to solve your problems together. ;)

References

[1] Thorsten Hohage and Frank Schmidt. *On the numerical solution of nonlinear Schrödinger type equations in fiber optics*. ZIB, 2002.

[2] Mathematics. Help with using the Runge-Kutta 4th order method on a system of 2 first order ode's., 2014. URL http://math.stackexchange.com/questions/721076/help-with-using-the-runge-kutta-4th-order-method-on-a-system-of-2-first-order-od. [Online; accessed 6-Dec-2016].

[3] Pablo Suarez. An introduction to the split step Fourier method using matlab, 2013. [https://www.researchgate.net/publication/281441538_An_introduction_to_the_Split_Step_Fourier_Method_using_MATLAB; accessed 23-April-2016].

[4] Hanquan Wang. Numerical studies on the split-step finite difference method for nonlinear Schrödinger equations. *Applied mathematics and computation*, 170(1):17–35, 2005.

[5] Wikipedia. Runge–Kutta methods — wikipedia, the free encyclopedia, 2016. URL https://en.wikipedia.org/w/index.php?title=Runge%E2%80%93Kutta_methods&oldid=752035966. [Online; accessed 6-Dec-2016].

[6] Michael Zeltkevic. Runge-Kutta methods, 1998-04-15. URL http://web.mit.edu/10.001/Web/Course_Notes/Differential_Equations_Notes/node5.html. [Online; accessed 6-Dec-2016].