IMPRS: Ultrafast Source Technologies

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Lectures: Tuesday 09.00-11.00 Geb 99, Sem Raum IV (1.OG)
Thursday 09.00-11.00 Geb 99, Sem Raum IV (1.OG)

Start: 18.02.2014

Course Secretary: Christine Berber
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Class website: http://desy.cfel.de/ultrafast_optics_and_x_rays_division/teaching/ imprs_summer_semester_2014/lecture_notes

General Information

Prerequisites: Basic course in electrodynamics

Text: Class notes will be distributed in class.

Grade: None

Recommended Texts:

Fundamentals of Photonics, B.E.A. Saleh and M.C. Teich, Wiley, 1991.
Ultrafast Optics, A. M. Weiner, Hoboken, NJ, Wiley 2009.
Ultrashort Laser Pulse Phenomena, Diels and Rudolph, Elsevier/Academic Press, 2006
Optics, Hecht and Zajac, Addison and Wesley Publishing Co., 1979.
Principles of Lasers, O. Svelto, Plenum Press, NY, 1998.
Waves and Fields in Optoelectronics, H. A. Haus, Prentice Hall, NJ, 1984.

Gratings, Mirrors and Slits: Beamline Design for Soft X-Ray Synchrotron Radiation Sources,
W. B. Peatman, Gordon and Breach Science Publishers, 1997.
Soft X-ray and Extreme Ultraviolet Radiation, David Attwood, Cambridge University Press, 1999

Schedule

Lecturer	Topic	Time	Location
Kärtner	Classical Optics	Tuesday Feb. 18, 9-11am	Geb. 99 Raum IV, 1.OG
Uphues	X-ray Optics	Thursday Feb. 20, 9-11am	Geb. 99 Raum IV, 1.OG
Kärtner	Ultrafast Lasers	Tuesday Feb. 25, 9-11am	Geb. 99 Raum IV, 1.OG
Uphues	Synchrotron Radiation	Thursday Feb. 27, 9-11am	Geb. 99 Raum IV, 1.OG
Kärtner	Electron Sources and Accelerators	Tuesday Mar. 04, 9-11am	Geb. 99 Raum IV, 1.OG
Uphues	High Order Harmonic Gen.	Thursday Mar. 06, 9-11am	Geb. 99 Raum IV, 1.OG

Classical Optics

2.1 Maxwell's Equations of Isotropic Media

Maxwell's Equations: Differential Form

Current due to free charges

Ampere's Law
$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}, \tag{2.1a}$$

Faraday's Law
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{2.1b}$$

Gauss's Law
$$\nabla \cdot \vec{D} = \rho, \tag{2.1c}$$

Free charge density

No magnetic charge
$$\nabla \cdot \vec{B} = 0. \tag{2.1d}$$

Material Equations: Bring Life into Maxwell's Equations

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}, \tag{2.2a}$$
Polarization

$$\vec{B} = \mu_0 \vec{H} + \vec{M}. \tag{2.2b}$$
Magnetization

Classical Optics

Vector Identity: $\nabla\times\left(\nabla\times\vec{E}\right)=\nabla\left(\nabla\cdot\vec{E}\right)-\Delta\vec{E},$

$$\nabla\times\left(\nabla\times\vec{E}\right)=-\nabla\times\frac{\partial\vec{B}}{\partial t}=-\frac{\partial}{\partial t}\left(\nabla\times\vec{B}\right)$$

$$=-\frac{\partial}{\partial t}\left(\nabla\times\left(\mu_0\vec{H}+\vec{M}\right)\right)=-\frac{\partial}{\partial t}\left(\mu_0\nabla\times\vec{H}+\nabla\times\vec{M}\right)$$

$$=-\frac{\partial}{\partial t}\left(\mu_0\left(\varepsilon_0\frac{\partial\vec{E}}{\partial t}+\frac{\partial\vec{P}}{\partial t}+\vec{J}\right)+\nabla\times\vec{M}\right)$$

$$\Delta\vec{E}-\mu_0\frac{\partial}{\partial t}\left(\vec{j}+\epsilon_0\frac{\partial\vec{E}}{\partial t}+\frac{\partial\vec{P}}{\partial t}\right)=\frac{\partial}{\partial t}\nabla\times\vec{M}+\nabla\left(\nabla\cdot\vec{E}\right)\qquad(2.3)$$

$$\left(\Delta-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}=\mu_0\left(\frac{\partial\vec{j}}{\partial t}+\frac{\partial^2}{\partial t^2}\vec{P}\right)+\frac{\partial}{\partial t}\nabla\times\vec{M}+\nabla\left(\nabla\cdot\vec{E}\right).\qquad(2.4)$$

Vacuum speed of light: $c_0=\sqrt{\frac{1}{\mu_0\epsilon_0}}.$

Classical Optics

No free charges, No currents from free charges, Non magnetic

$$\left(\Delta - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = \mu_0\left(\cancel{\frac{\partial \vec{j}}{\partial t}} + \frac{\partial^2}{\partial t^2}\vec{P}\right) + \frac{\partial}{\partial t}\cancel{\nabla\times\vec{M}} + \nabla\left(\cancel{\nabla\cdot\vec{E}}\right). \qquad (2.4)$$

Every field can be written as the sum of tansverse and longitudinal fields:

$$\vec{\nabla}\times\vec{E}_L = 0 \text{ and } \vec{\nabla}\cdot\vec{E}_T = 0$$

Only free charges create a longitudinal electric field:

$$\vec{E} = \vec{E}_T$$

Pure radiation field

Simplified wave equation:

$$\left(\Delta - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = \mu_0\frac{\partial^2}{\partial t^2}\vec{P}. \qquad (2.7)$$

Wave in vacuum **Source term**

2.1.1 Helmholtz Equation

$$\tilde{\vec{E}}(\vec{r},\omega) = \int_{-\infty}^{+\infty} \vec{E}(\vec{r},t)e^{-j\omega t}dt, \tag{2.13}$$

Linear, local medium

$$\tilde{\vec{P}}(\vec{r},\omega) = \epsilon_0 \tilde{\chi}(\omega)\tilde{\vec{E}}(\vec{r},\omega), \tag{2.14}$$

dielectric susceptibility

$$\left(\Delta + \frac{\omega^2}{c_0^2}\right)\tilde{\vec{E}}(\vec{r},\omega) = -\omega^2 \mu_0 \epsilon_0 \tilde{\chi}(\omega)\tilde{\vec{E}}(\vec{r},\omega), \tag{2.15}$$

$$\left(\Delta + \frac{\omega^2}{c_0^2}(1+\tilde{\chi}(\omega))\right)\tilde{\vec{E}}(\vec{r},\omega) = 0, \tag{2.16}$$

Medium speed of light: $c(\omega) = c_0/\tilde{n}(\omega)$ with $1+\tilde{\chi}(\omega) = \tilde{n}(\omega)^2$;

Refractive Index

2.1.2 Plane-Wave Solutions (TEM-Waves) and Compl. Notation

Real field: $$\vec{E}_{\vec{k}}(\vec{r},t) = \frac{1}{2}\left[\underline{\vec{E}}_{\vec{k}}(\vec{r},t) + \underline{\vec{E}}_{\vec{k}}(\vec{r},t)^*\right] = \Re e\left\{\underline{\vec{E}}_{\vec{k}}(\vec{r},t)\right\}, \qquad (2.18)$$

Artificial, complex field: $$\underline{\vec{E}}_{\vec{k}}(\vec{r},t) = \underline{E}_{\vec{k}}\; e^{\mathrm{j}(\omega t - \vec{k}\cdot\vec{r})}\; \vec{e}(\vec{k}). \qquad (2.19)$$

Into wave equation (2.16):

Dispersion relation: $$|\vec{k}|^2 = \frac{\omega^2}{c(\omega)^2} = k(\omega)^2. \qquad (2.20)$$

$$k = \left|\vec{k}\right| \qquad k(\omega) = \pm\frac{\omega}{c_0} n(\omega). \qquad (2.21)$$

$$k = 2\pi/\lambda, \qquad (2.22)$$ Wavelength

$$\nabla \cdot \vec{E} = 0 \longrightarrow \vec{k} \perp \vec{e}.$$

TEM-Waves

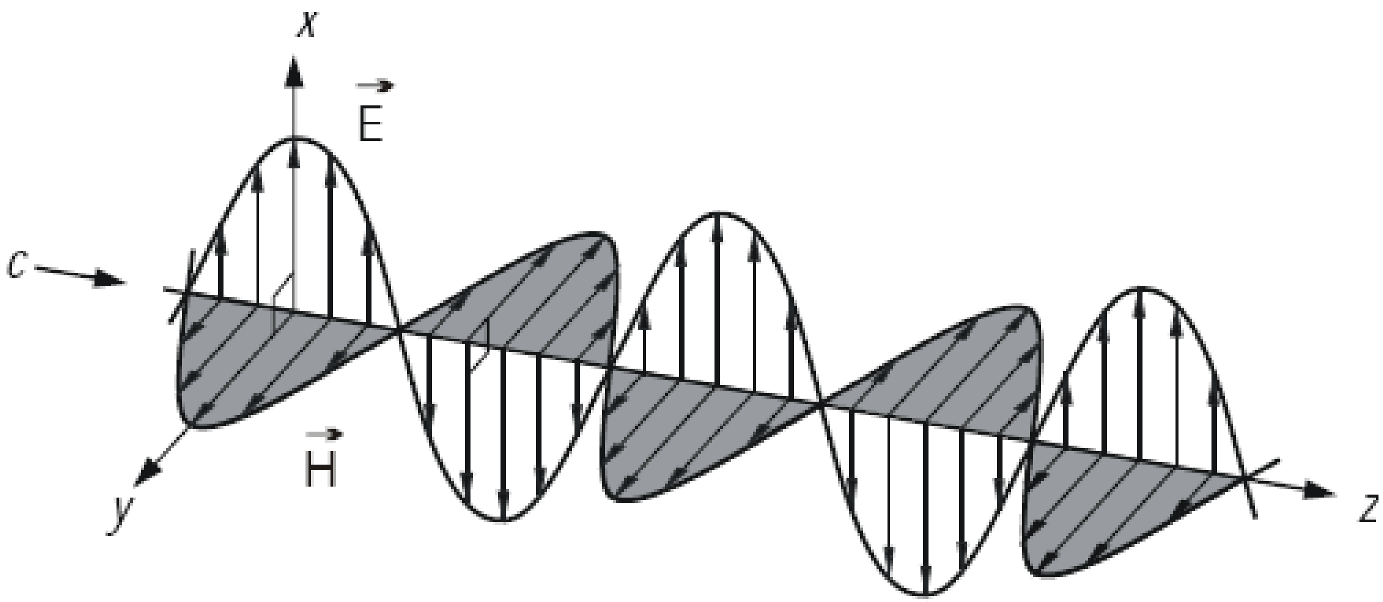


Figure 2.1: Transverse electromagnetic wave (TEM) [6]

TEM-Waves

What about the magnetic field?

$$\vec{H}_{\vec{k}}(\vec{r},t) = \frac{1}{2}\left[\underline{\vec{H}}_{\vec{k}}(\vec{r},t) + \underline{\vec{H}}_{\vec{k}}(\vec{r},t)^*\right] \tag{2.23}$$

$$\underline{\vec{H}}_{\vec{k}}(\vec{r},t) = \underline{H}_{\vec{k}}\; e^{\mathrm{j}(\omega t - \vec{k}\cdot\vec{r})}\; \vec{h}(\vec{k}). \tag{2.24}$$

Faraday's Law:

$$-\mathrm{j}\vec{k} \times \left(\underline{E}_{\vec{k}}\; e^{\mathrm{j}(\omega t - \vec{k}\cdot\vec{r})}\; \vec{e}(\vec{k})\right) = -\mathrm{j}\mu_0\omega\underline{\vec{H}}_{\vec{k}}(\vec{r},t), \tag{2.25}$$

$$\underline{\vec{H}}_{\vec{k}}(\vec{r},t) = \frac{\underline{E}_{\vec{k}}}{\mu_0\omega} e^{\mathrm{j}(\omega t - \vec{k}\cdot\vec{r})}\vec{k} \times \vec{e} = \underline{H}_{\vec{k}} e^{\mathrm{j}(\omega t - \vec{k}\cdot\vec{r})}\vec{h} \tag{2.26}$$

$$\longrightarrow \quad \vec{h}(\vec{k}) = \frac{\vec{k}}{|k|} \times \vec{e}(\vec{k}) \tag{2.27}$$

$$\longrightarrow \quad \underline{H}_{\vec{k}} = \frac{|k|}{\mu_0\omega}\underline{E}_{\vec{k}} = \frac{1}{Z_F}\underline{E}_{\vec{k}}. \tag{2.28}$$

Characteristic Impedance

$$Z_F = \mu_0 c = \sqrt{\frac{\mu_0}{\epsilon_0 \epsilon_r}} = \frac{1}{n} Z_{F_0} \qquad \varepsilon_r = 1 + \chi(\omega) \tag{2.29}$$

Vacuum Impedance:

$$Z_{F_0} = \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 377\,\Omega. \tag{2.30}$$

$\vec{e}$, $\vec{h}$ and $\vec{k}$ form an orthogonal trihedral,

$$\vec{e} \perp \vec{h}, \quad \vec{k} \perp \vec{e}, \quad \vec{k} \perp \vec{h}. \tag{2.31}$$

Example: EM-Wave polarized along x-axis and propagation along z-direction:

$$\vec{e}(\vec{k}) = \vec{e}_x. \qquad \frac{\vec{k}}{|k|} = \vec{e}_z,$$

$$\vec{E}(\vec{r}, t) = E_0 \cos(\omega t - kz)\ \vec{e}_x, \tag{2.32}$$

$$\vec{H}(\vec{r}, t) = \frac{E_0}{Z_{F_0}} \cos(\omega t - kz)\ \vec{e}_y, \tag{2.33}$$

Backwards Traveling Wave

Backwards

$$\underline{\vec{E}}(\vec{r},t) = \underline{E}\, e^{\mathrm{j}\omega t + \mathrm{j}\vec{k}\cdot\vec{r}}\, \vec{e}_x ,$$

$$\underline{\vec{H}}(\vec{r},t) = \underline{H}\, e^{\mathrm{j}(\omega t + \vec{k}\vec{r})}\, \vec{e}_y ,$$

$$\underline{H} = -\frac{|k|}{\mu_0 \omega}\underline{E},$$

2.1.3 Poynting Vector, Energy Density and Intensity

relative permittivity: $\varepsilon_r = 1 + \chi$

Quantity	Real fields	Complex fields
Electric and magnetic energy density	$w_e = \frac{1}{2}\vec{E}\cdot\vec{D} = \frac{1}{2}\epsilon_0\epsilon_r\vec{E}^2$ $w_m = \frac{1}{2}\vec{H}\cdot\vec{B} = \frac{1}{2}\mu_0\mu_r\vec{H}^2$ $w = w_e + w_m$	$\langle w_e\rangle = \frac{1}{4}\epsilon_0\epsilon_r\left\lvert\underline{\vec{E}}\right\rvert^2$ $\langle w_m\rangle = \frac{1}{4}\mu_0\mu_r\left\lvert\underline{\vec{H}}\right\rvert^2$ $\langle w\rangle = \langle w_e\rangle + \langle w_m\rangle$
Poynting vector	$\vec{S} = \vec{E}\times\vec{H}$	$\underline{\vec{T}} = \frac{1}{2}\underline{\vec{E}}\times\underline{\vec{H}}^*$
Poynting theorem	$\mathrm{div}\vec{S} + \vec{E}\cdot\vec{j} + \frac{\partial w}{\partial t} = 0$	$\mathrm{div}\underline{\vec{T}} + \frac{1}{2}\underline{\vec{E}}\cdot\underline{\vec{j}}^* +$ $+2j\omega(\langle w_m\rangle - \langle w_e\rangle) = 0$
Intensity	$I = \left\lvert\vec{S}\right\rvert = cw$	$I = \mathrm{Re}\{\underline{\vec{T}}\} = c\langle w\rangle$

Table 2.1: Poynting vector and energy density in EM-fields

Example: Plane Wave:

$$\langle w\rangle = \frac{1}{2}\epsilon_r\epsilon_0|\underline{E}|^2,$$

$$\underline{\vec{E}}(\vec{r},t) = \underline{E}e^{j(\omega t - kz)}\ \vec{e}_x$$

$$\boxed{I = \frac{1}{2Z_F}|\underline{E}|^2 = \frac{1}{2}Z_F|\underline{H}|^2.}$$

$$\underline{\vec{T}} = \frac{1}{2Z_F}|\underline{E}|^2\ \vec{e}_z,$$

2.2 Paraxial Wave Equation

A plane wave is described by:

$$\vec{\tilde{E}}(x,y,z) = \vec{e}_x \tilde{E}_0 \exp\left[-jk_x x - jk_y y - jk_z z\right]$$

$$k_0^2 = k_x^2 + k_y^2 + k_z^2 \qquad \text{free space wavenumber}$$

Although a plane wave is a useful model, strictly speaking it cannot be realized in practice, because it fills whole space and carries infinite energy.

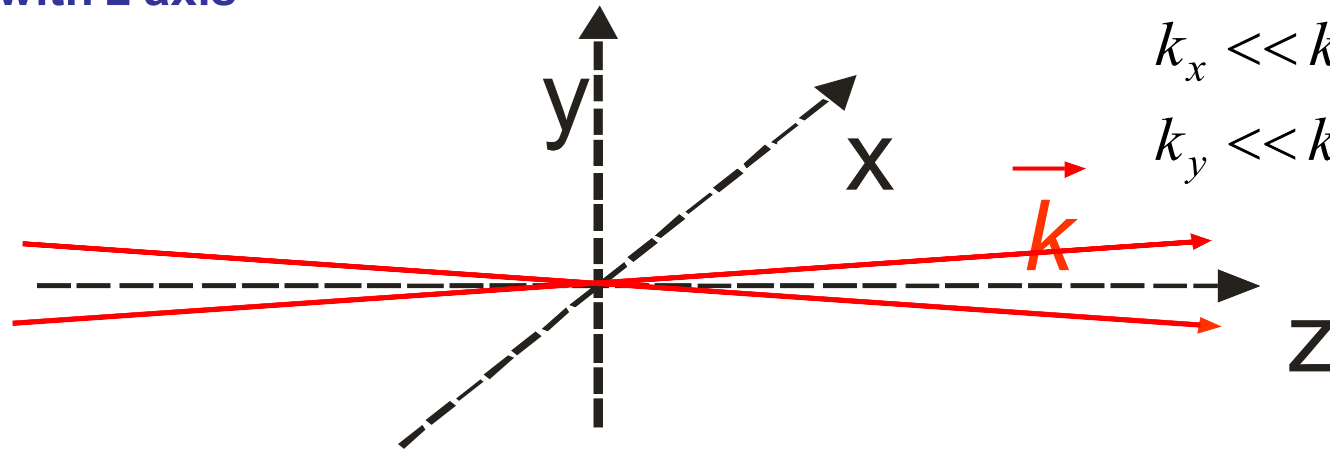
Maxwell's equations are linear, so a sum of solutions is also a solution. An arbitrary beam can be can be formed as a superposition of multiple plane waves:

$$\tilde{E}(x,y,z) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \tilde{E}_0 \exp\left[-jk_x x - jk_y y - jk_z z\right] dk_x dk_y$$

there is no integral over k_z because once k_x and k_y are fixed, k_z is constrained by dispersion relation $k_0^2 = k_x^2 + k_y^2 + k_z^2$

Paraxial Beam

Consider a beam which consists of plane waves propagating at small angles with z axis



$$k_x << k_z$$

$$k_y << k_z$$

such a beam is called a <u>paraxial beam</u>

Figure 2.2: Construction of paraxial beam by superimposing many plane waves with a dominant k-component in z-direction

k_z can be approximated as

$$\begin{aligned} k_z &= \sqrt{k_0^2 - k_x^2 - k_y^2}, \\ &\approx k_0 \left(1 - \frac{k_x^2 + k_y^2}{2k_0^2}\right). \end{aligned}$$

(paraxial approximation)

Paraxial Diffraction Integral

The beam

$$\tilde{E}(x,y,z) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \tilde{E}_0(k_x,k_y)\exp\left[-jk_x x - jk_y y - jk_z z\right] dk_x dk_y$$

can then be expressed as

$$\tilde{E}(x,y,z) = \underbrace{e^{-jk_0 z}}_{\text{quickly varying carrier wave}} \underbrace{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \tilde{E}_0(k_x,k_y)\exp\left[-j\left(\frac{k_x^2+k_y^2}{2k_0}\right)z - jk_x x - jk_y y\right] dk_x dk_y}_{\text{slowly varying envelope (changing slowly along z)}}$$

Allows to find field distribution at any point in space.

The beam profile is changing as it propagates in free space.
This is called diffraction.

Paraxial Diffraction Integral – Finding Field at Arbitrary z

Given the field at some plane (e.g. z=0), how to find the field at any z?

From the previous slide

$$\widetilde{E}(x,y,z) = e^{-jk_0 z}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\widetilde{E}_0(k_x,k_y)\exp\left[-j\left(\frac{k_x^2+k_y^2}{2k_0}\right)z - jk_x x - jk_y y\right]dk_x dk_y$$

At z=0

$$\widetilde{E}(x,y,z=0) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\widetilde{E}_0(k_x,k_y)\exp\left[-jk_x x - jk_y y\right]dk_x dk_y$$

This is Fourier integral. Using Fourier transforms:

$$\widetilde{E}_0(k_x,k_y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\widetilde{E}(x,y,z=0)\exp\left[jk_x x + jk_y y\right]dx dy$$

Knowing $\widetilde{E}(x,y,z=0)$ **can find** $\widetilde{E}_0(k_x,k_y)$ **and then** $\widetilde{E}(x,y,z)$ **at any z.**

2.3 Gaussian Beam

Choose transverse Gaussian distribution of plane wave amplitudes:

$$\widetilde{E}_0(k_x, k_y) \sim \exp\left[-\frac{k_x^2 + k_y^2}{2k_T^2}\right],$$

(Via Fourier transforms, this is equivalent to choosing Gaussian transversal field distribution at z=0)

Substitution into paraxial diffraction integral gives

$$\widetilde{E}_0(x, y, z) \sim \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp\left[j\left(\frac{k_x^2 + k_y^2}{2k_0}\right)(z + jz_R) - jk_x x - jk_y y\right] dk_x dk_y,$$

with $z_R = k_0/k_T^2$ called Rayleigh range.

Performing the integration (i.e. taking Fourier transforms of a Gaussian)

$$\widetilde{E}_0(x, y, z) \sim \frac{j}{z + jz_R} \exp\left[-jk_0\left(\frac{x^2 + y^2}{2(z + jz_R)}\right)\right].$$

Gaussian Beam

$$\widetilde{E}_0(x,y,z) \sim \frac{j}{z+jz_R}\exp\left[-jk_0\left(\frac{x^2+y^2}{2(z+jz_R)}\right)\right].$$

The quantity $1/(z+jz_R)$ **can be expressed as**

$$\frac{1}{z+jz_R} = \frac{z-jz_R}{z^2+z_R^2} \equiv \frac{1}{R(z)} - j\frac{\lambda}{\pi w^2(z)}$$

where *R*(*z*) and *w*(*z*) are introduced according to

$$\frac{1}{R(z)} \equiv \frac{z}{z^2+z_R^2} \qquad \frac{\lambda}{\pi w^2(z)} \equiv \frac{z_R}{z^2+z_R^2}$$

The pre-factor $j/(z+jz_R)$ **can be expressed as**

$$\frac{j}{z+jz_R} = \frac{\exp(j\zeta(z))}{\sqrt{z^2+z_R^2}} = \sqrt{\frac{\lambda}{\pi z_R}}\frac{1}{w(z)}\exp(j\zeta(z)) \quad \textbf{with} \quad \tan\zeta(z) = z/z_R$$

We get

$$\widetilde{E}_0(r,z) \sim \frac{1}{w(z)}\exp\left[-\frac{r^2}{w^2(z)} - jk_0\frac{r^2}{2R(z)} + j\zeta(z)\right]$$

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Final Expression for Gaussian Beam Field

Finally, the field is normalized, giving

$$\tilde{E}_0(r,z) = \sqrt{\frac{4Z_{F0}P}{\pi}}\,\frac{1}{w(z)}\exp\left[-\frac{r^2}{w^2(z)} - jk_0\frac{r^2}{2R(z)} + j\zeta(z)\right]$$

Field of a Gaussian beam

Normalization means that the total power carried by the beam is *P*:

$$I(r,z) = \frac{2P}{\pi w^2(z)}\exp\left[-\frac{2r^2}{w^2(z)}\right],$$

$$\text{i.e. } P = \int_0^\infty \int_0^{2\pi} I(r,z)\, r dr\, d\varphi.$$

Gaussian Beams: Spot Size and Rayleigh Range

We derived

$$\tilde{E}_0(r,z) = \sqrt{\frac{4Z_{F0}P}{\pi}}\frac{1}{w(z)}\exp\left[\underbrace{-\frac{r^2}{w^2(z)}}-jk_0\frac{r^2}{2R(z)}+j\zeta(z)\right]$$

Field of a Gaussian beam

Defines transversal intensity distribution

$$I(r,z) \sim \exp\left[-\frac{2r^2}{w^2(z)}\right]$$

$w(z)$ **spot size**

We have

$$\frac{\lambda}{\pi\, w^2(z)} \equiv \frac{z_R}{z^2+z_R^2} \Rightarrow w^2(z) = \frac{\lambda}{\pi} z_R\left[1+\left(\frac{z}{z_R}\right)^2\right]$$

Spot size at z=0 is min

$$w^2(0) = \frac{\lambda}{\pi} z_R \equiv w_0^2 \Rightarrow w(z) = w_0\sqrt{1+\left(\frac{z}{z_R}\right)^2}$$

$$z_R = \frac{\pi\, w_0^2}{\lambda}$$

Rayleigh range

spot size as a function of z

Gaussian Beams: Phase Distribution

We derived

$$\tilde{E}_0(r,z) = \sqrt{\frac{4Z_{F0}P}{\pi}}\frac{1}{w(z)}\exp\left[-\frac{r^2}{w^2(z)} \underbrace{- jk_0\frac{r^2}{2R(z)} + j\zeta(z)}_{\text{defines phase distribution}}\right]$$

Field of a Gaussian beam

We have

$$\frac{1}{R(z)} \equiv \frac{z}{z^2 + z_R^2} \Rightarrow R(z) = z\left[1 + \left(\frac{z_R}{z}\right)^2\right]$$

Radius of wavefront curvature

Phase fronts (surfaces of constant phase) are parabolic, can be approximated as spherical for small *r*. *R*(z) turns out to be the radius of the sphere.

$$\zeta(z) = \arctan(z / z_R)$$

Guoy phase shift

Defines extra phase shift as compared to plane wave

Gaussian Beams: All Equations in One Slide

$$\tilde{E}_0(r,z) = \sqrt{\frac{4Z_{F0}P}{\pi}}\frac{1}{w(z)}\exp\left[-\frac{r^2}{w^2(z)} - jk_0\frac{r^2}{2R(z)} + j\zeta(z)\right]$$

Gaussian Beam Field

$$w(z) = w_0\sqrt{1+\left(\frac{z}{z_R}\right)^2}$$

Spot size w_0 **min spot size (z=0)** (2.54)

$$R(z) = z\left[1+\left(\frac{z_R}{z}\right)^2\right]$$

Radius of wavefront curvature (2.55)

$$\zeta(z) = \arctan\left(\frac{z}{z_R}\right)$$

Guoy phase shift (2.65)

$$z_R = \frac{\pi w_0^2}{\lambda}$$

Rayleigh range (2.56)

Intensity Distribution

$$I(r,z) = I_0 \frac{w_0^2}{w^2(z)} \exp\left[-\frac{2r^2}{w^2(z)}\right], \quad \text{with } I_0 = \frac{2P}{\pi w_0^2}$$

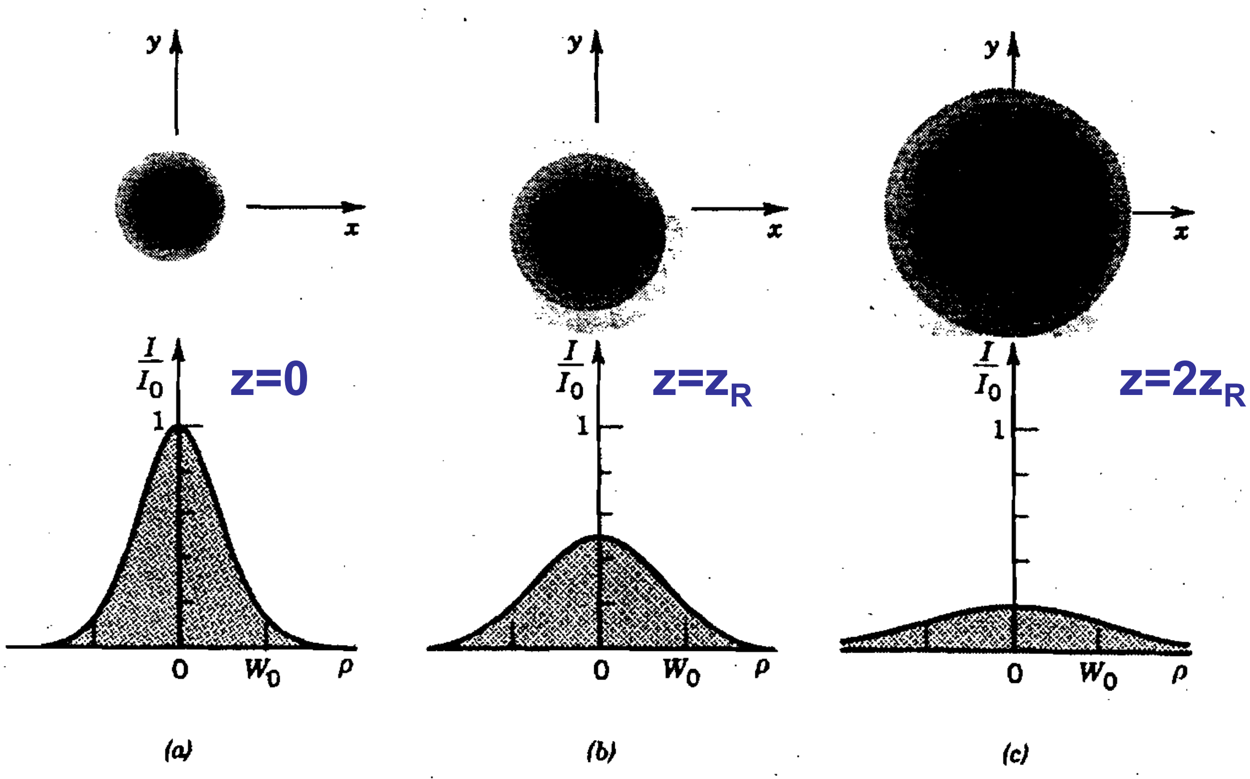


Figure 2.3: The normalized beam intensity I/I_0 as a function of the radial distance r at different axial distances: (a) z=0, (b) $z=z_R$ (c) $z=2z_R$.

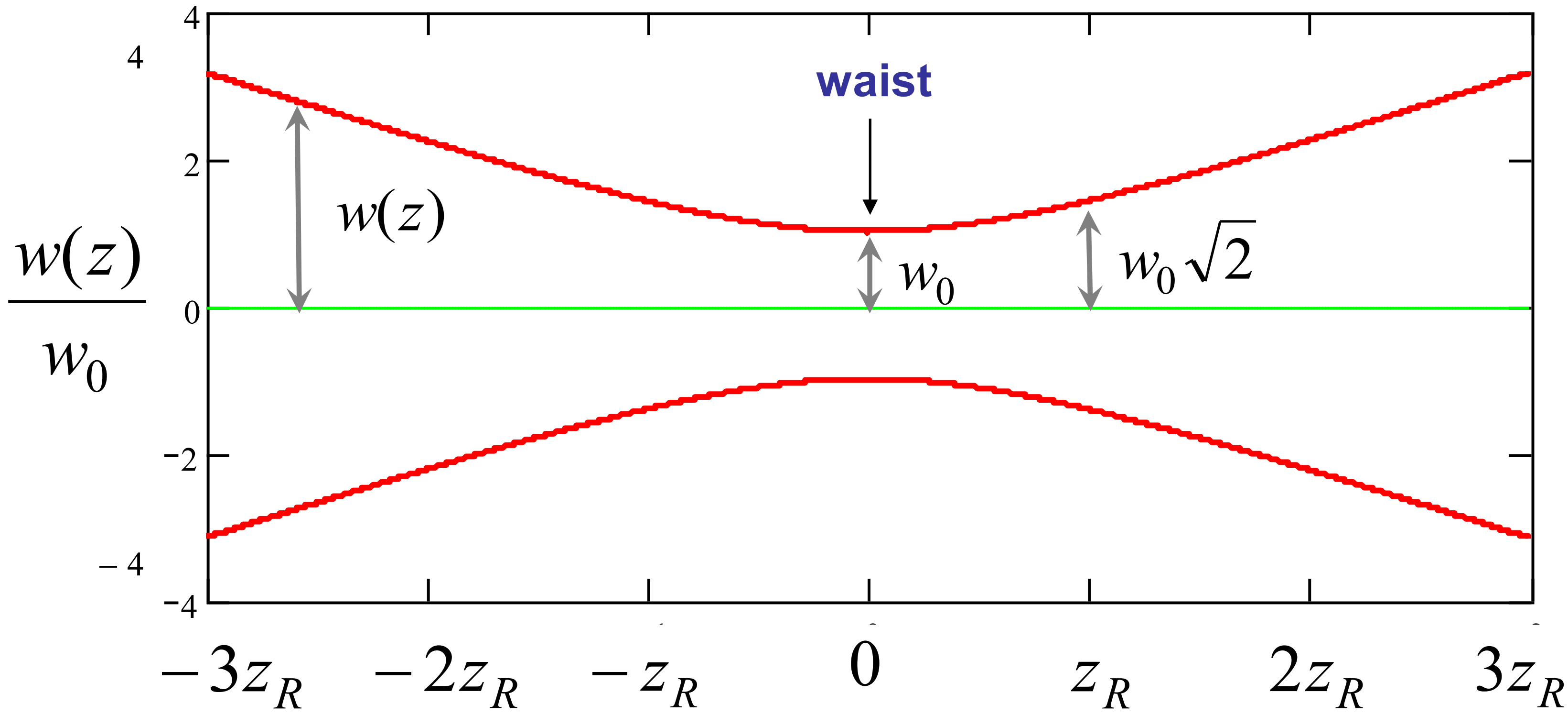
Spot Size as a Function of z

Spot size $w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_R}\right)^2}$

At z = z$_R$ $w(z_R) = w_0\sqrt{2}$

Waist: minimum radius

Rayleigh range z_R: distance from the waist at which the spot area doubles



Divergence Angle and Confocal Parameter

For large z $$w(z) = w_0\sqrt{1+\left(\frac{z}{z_R}\right)^2} \approx w_0\frac{z}{z_R}$$

Divergence angle $$\theta \equiv \frac{w(z)}{z} \approx \frac{w_0}{z_R}$$

Beam Divergence: $$\theta = \frac{\lambda}{\pi w_o},$$

Smaller spot size → larger divergence

Confocal parameter and depth of focus:

$2\pi w^2$

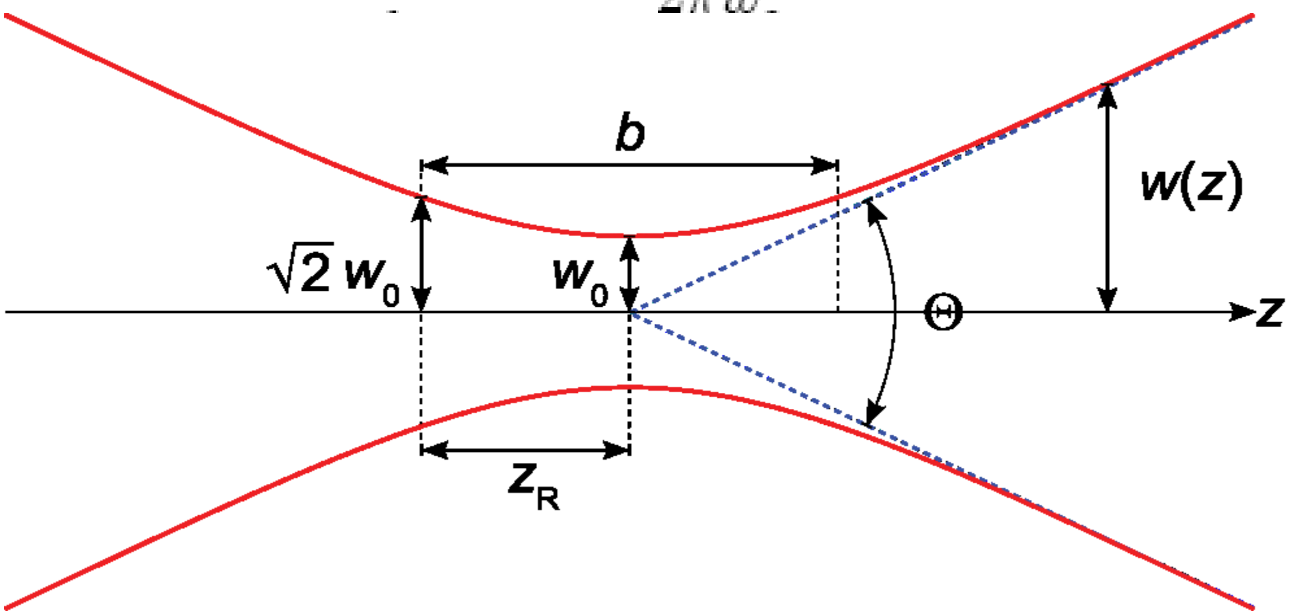


Figure 2.5: Gaussian beam and its characteristics

Axial Intensity Distribution

r=0

$$I(r,z) = I_0 \frac{w_0^2}{w^2(z)} = \frac{I_0}{1+\left(\frac{z}{z_R}\right)^2}$$

Intensity distribution along Z axis (r=0)

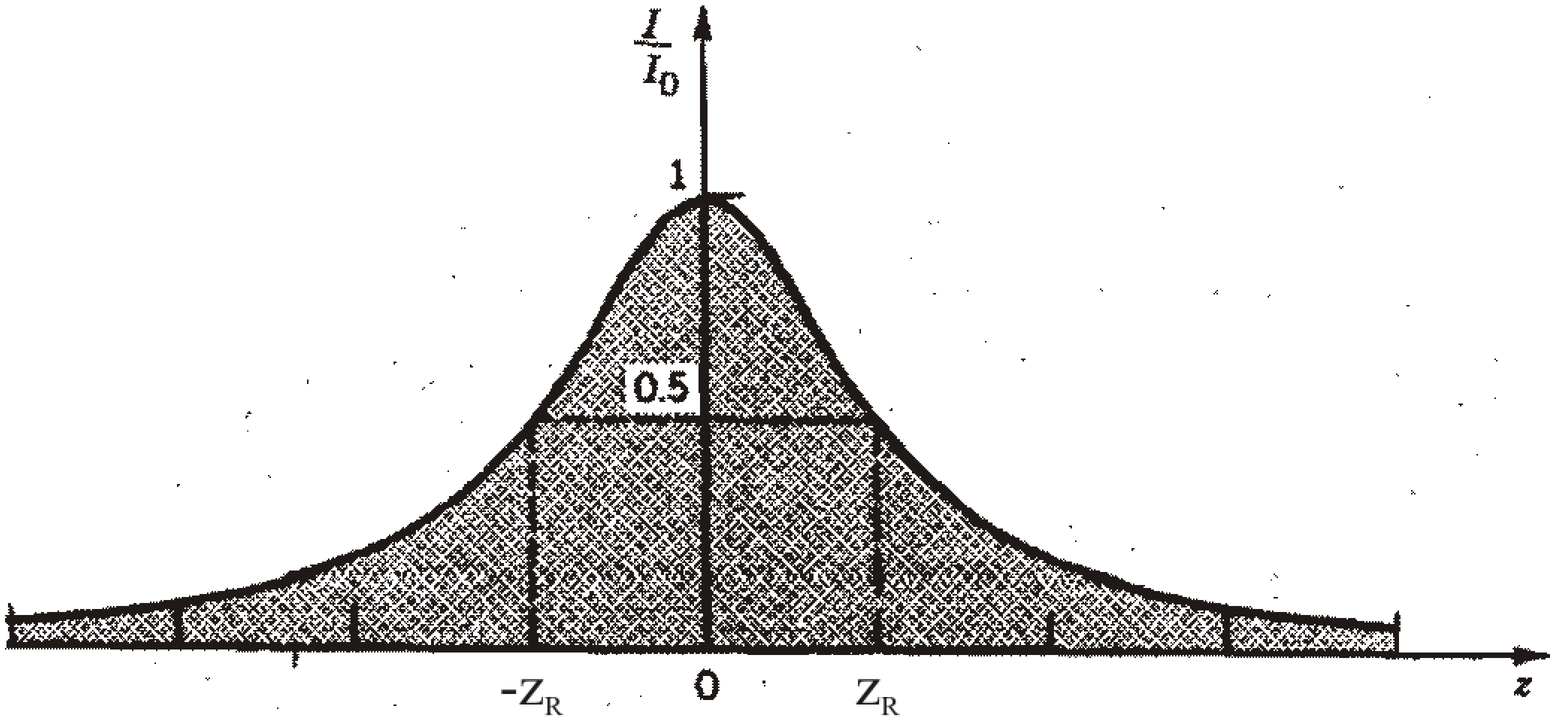


Figure 2.4: Normalized beam intensity $I(r=0)/I_0$ on the beam axis as a function of propagation distance z.

Power Confinement

Which fraction of the total power is confined within radius r_0 from the axis?

$$
\begin{aligned}
\frac{P(r < r_0)}{P} &= \frac{2\pi}{P}\int_0^{r_0} I(r,z)\,r\,dr \\
&= \frac{4}{w^2(z)}\int_0^{r_0} \exp\left[-\frac{2r^2}{w^2(z)}\right] r\,dr \qquad (2.283)\\
&= 1 - \exp\left[-\frac{2r_0^2}{w^2(z)}\right].
\end{aligned}
$$

Dependence is exponential

$$
\frac{P(r < w(z))}{P} = 0.86, \qquad (2.284)
$$

$$
\frac{P(r < 1.5w(z))}{P} = 0.99. \qquad (2.285)
$$

99% of the power is within radius of 1.5 w(z) from the axis

Wavefront

Wavefronts, i.e. surfaces of constant phase, are parabolic

$$k_0 z - \zeta(z) + k_0 \frac{r^2}{2R(z)} = \textit{constant}$$

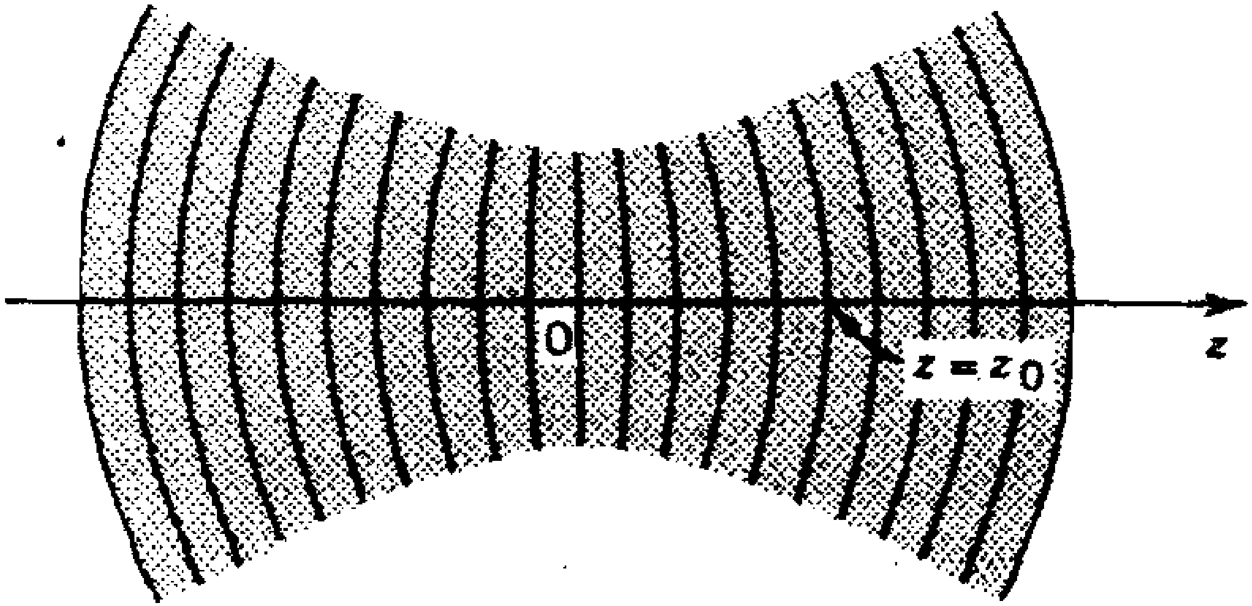


Figure 2.8: Wavefronts of Gaussian beam

Wavefront and Radius of Curvature

Wavefront radius of curvature:

$$R(z) = z\left[1+\left(\frac{z_R}{z}\right)^2\right]$$

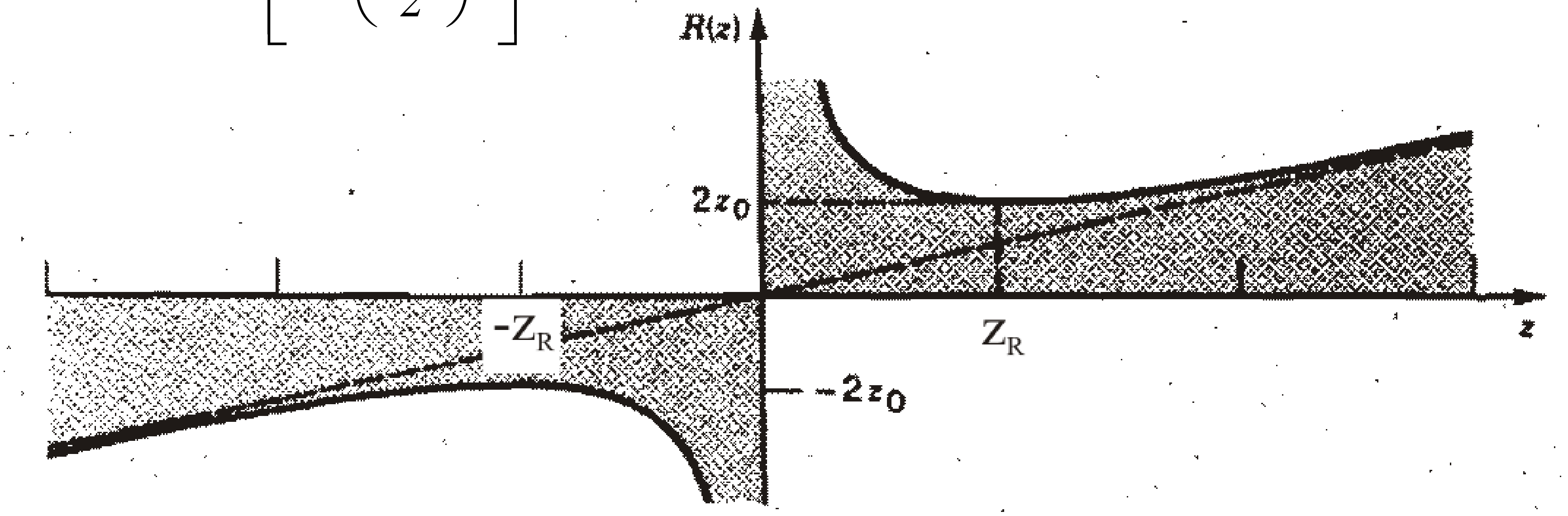


Figure 2.7: Radius of curvature R(z)

Comparison to Plane and Spherical Waves

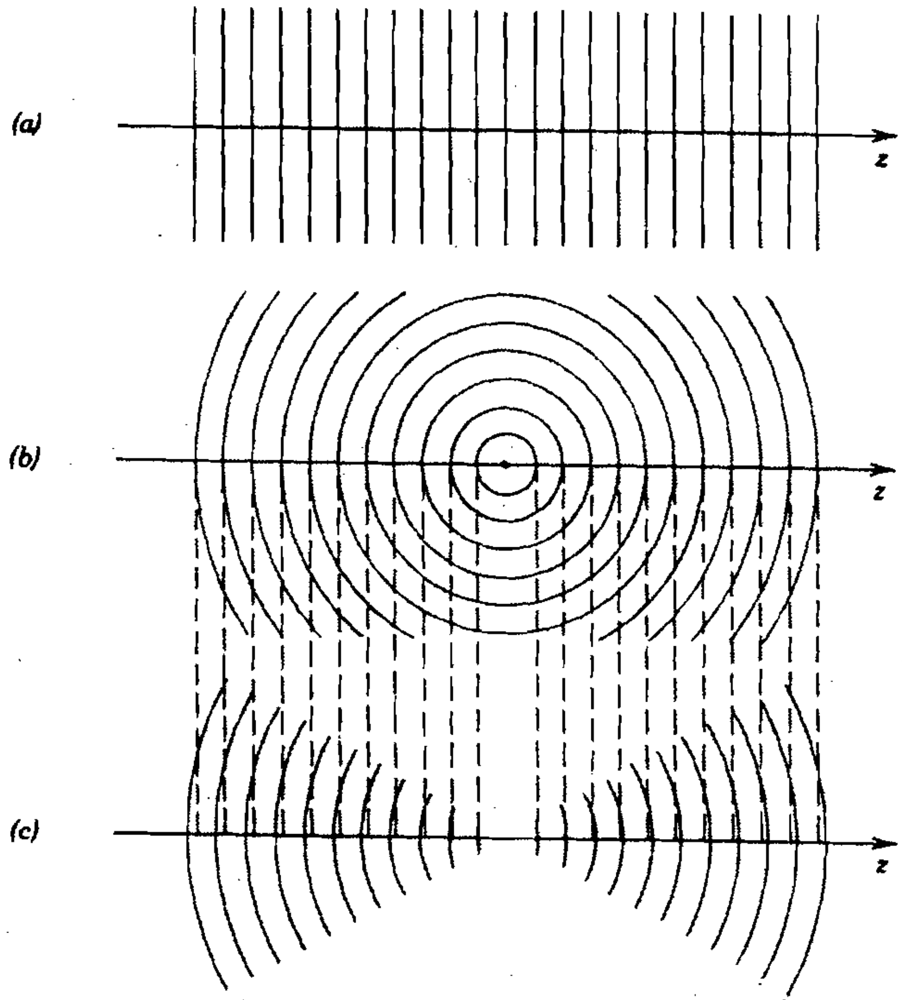


Figure 2.9: Wave fronts of (a) a uniform plane wave, (b) a spherical wave; (c) a Gaussian beam.

Guoy Phase Shift

Phase delay of Gaussian Beam, Guoy-Phase Shift:

$$\Phi(r,z) = k_0 z - \zeta(z) + k_0 \frac{r^2}{2R(z)} \tag{2.288}$$

$$\zeta(z) = \arctan\left(\frac{z}{z_R}\right). \tag{2.289}$$

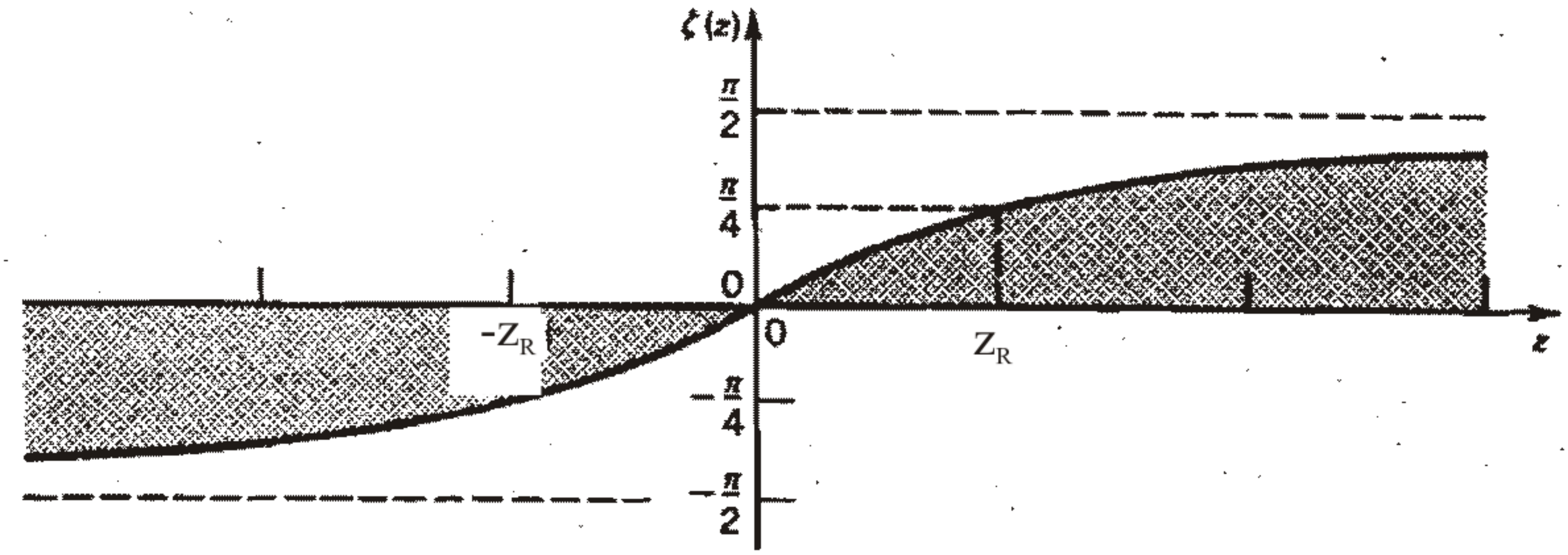
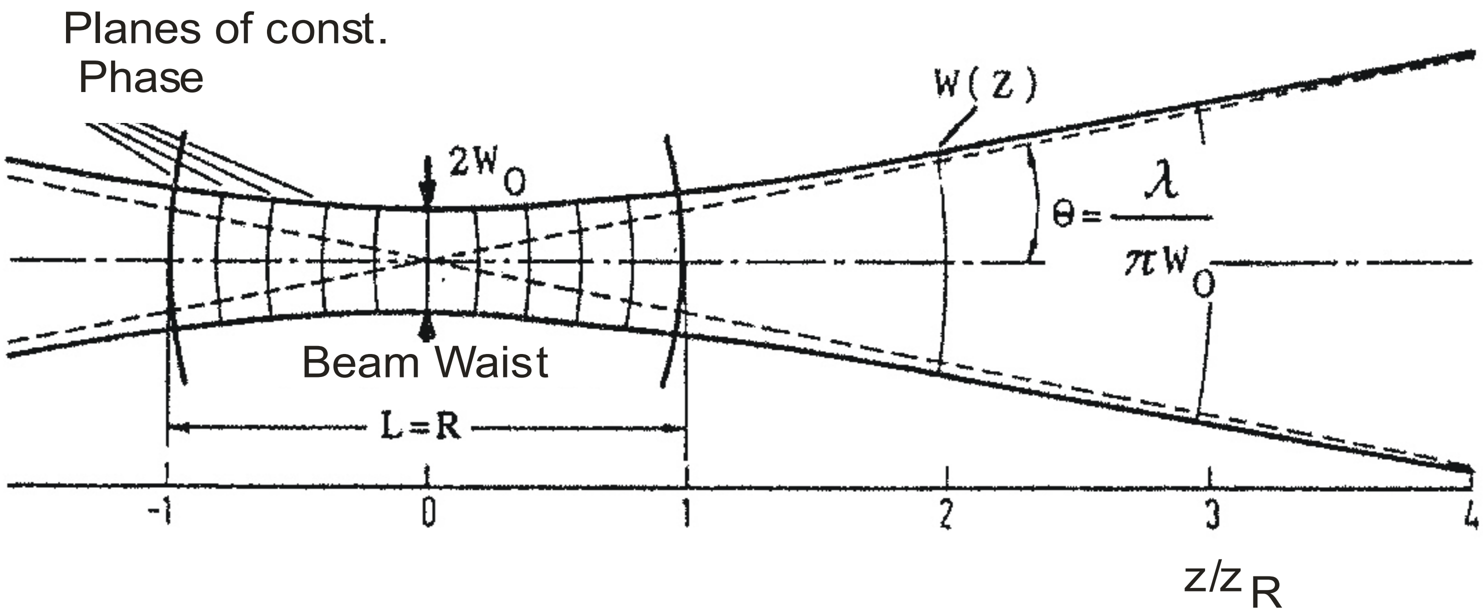


Figure 2.6: Phase delay of Gaussian beam, Guoy-Phase-Shift

Gaussian Beams: Summary

- Solution of wave equation in paraxial approximation
- Wave confined in space and with finite amount of power
- Intensity distribution in any cross-section has the same shape (Gaussian), only size and magnitude is scaled
- At the waist, the spot size is minimum and wave fronts are flat
- Lasers are usually built to generate Gaussian beams



2.4 Ray Propagation

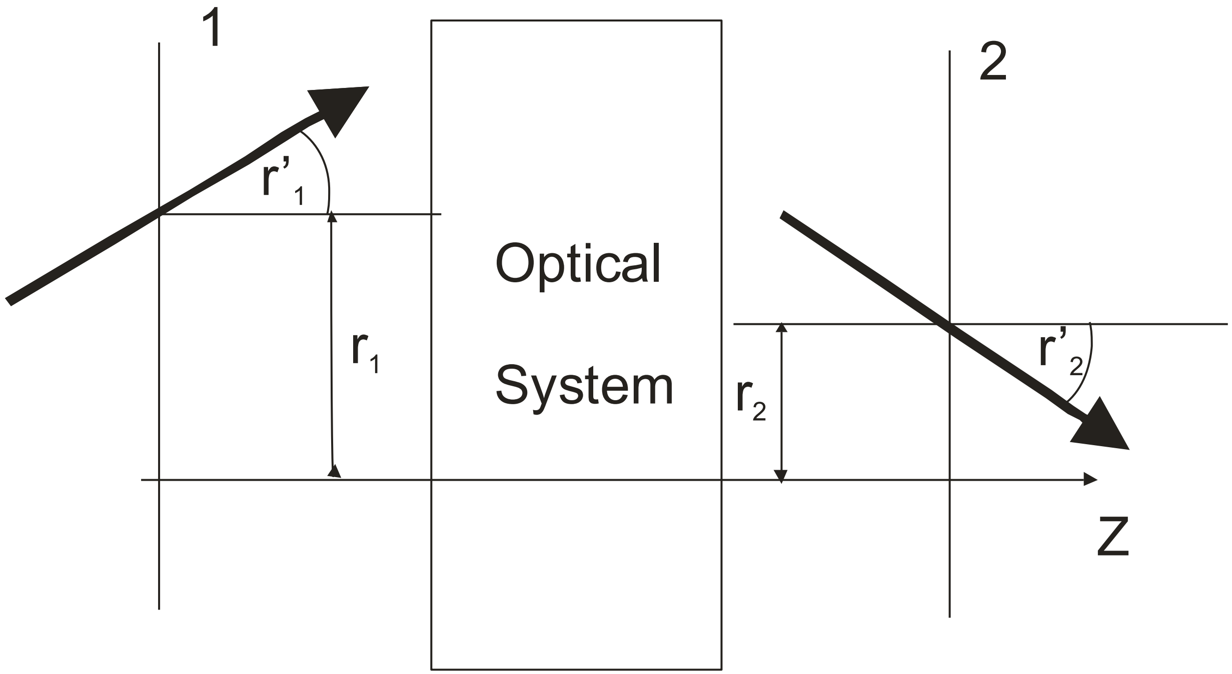


Figure 2.10: Description of optical ray propagation by its distance and inclination from the optical axis.

$$\begin{pmatrix} r_2 \\ n_2 r_2' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} r_1 \\ n_1 r_1' \end{pmatrix}.$$

Ray Matrix: Transition medium 1 to 2

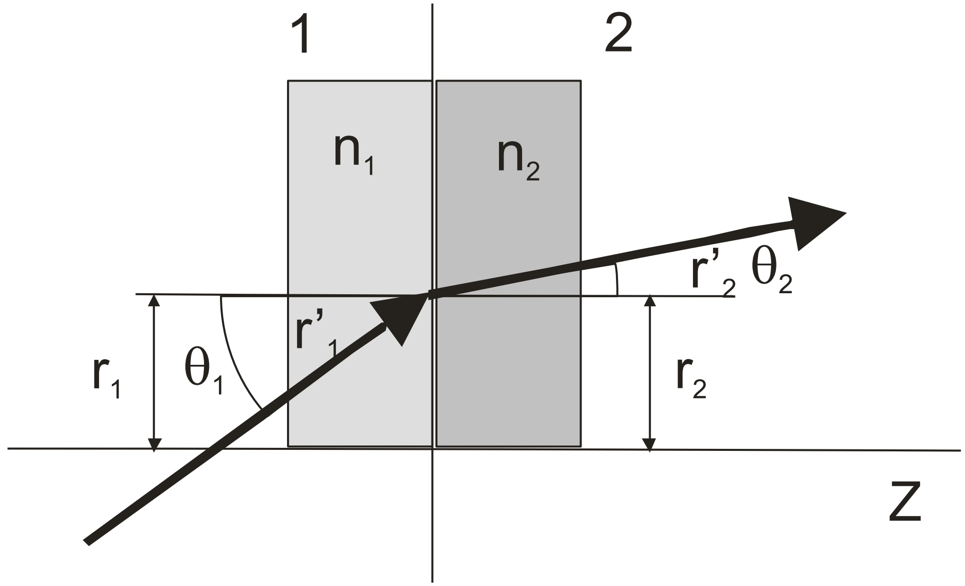


Figure 2.11: Snell's law for paraxial rays.

$$r_1' = \tan\theta_1 \approx \sin\theta_1 \approx \theta_1, \text{ and } r_2' = \tan\theta_2 \approx \sin\theta_2 \approx \theta_2.$$

Then Snell's law is

$$n_1\, r_1' = n_2\, r_2'.$$

$$\begin{aligned} r_2 &= r_1 \\ n_2\, r_2' &= n_1\, r_1'. \end{aligned}$$

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Ray Matrix: Propagation over Distance L

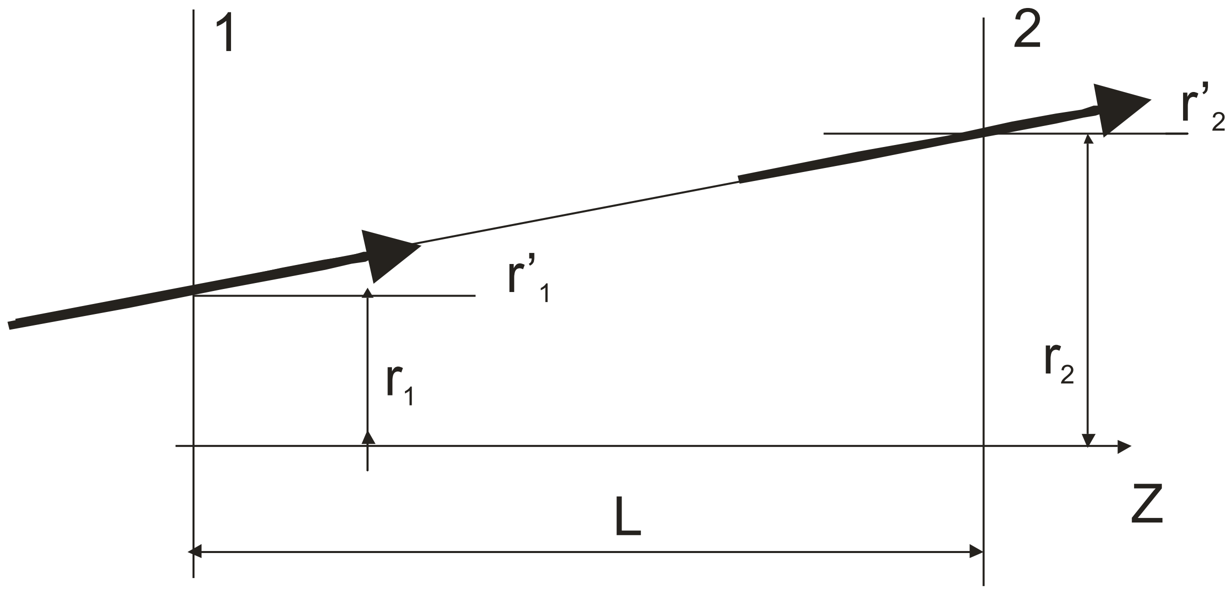


Figure 2.12: Free space propagation

$$
\begin{aligned}
r_2 &= r_1 + r_1' \cdot L \\
r_2' &= r_1'
\end{aligned}
\qquad
\mathbf{M} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}
$$

Prop. in Medium with Index n over Distance L

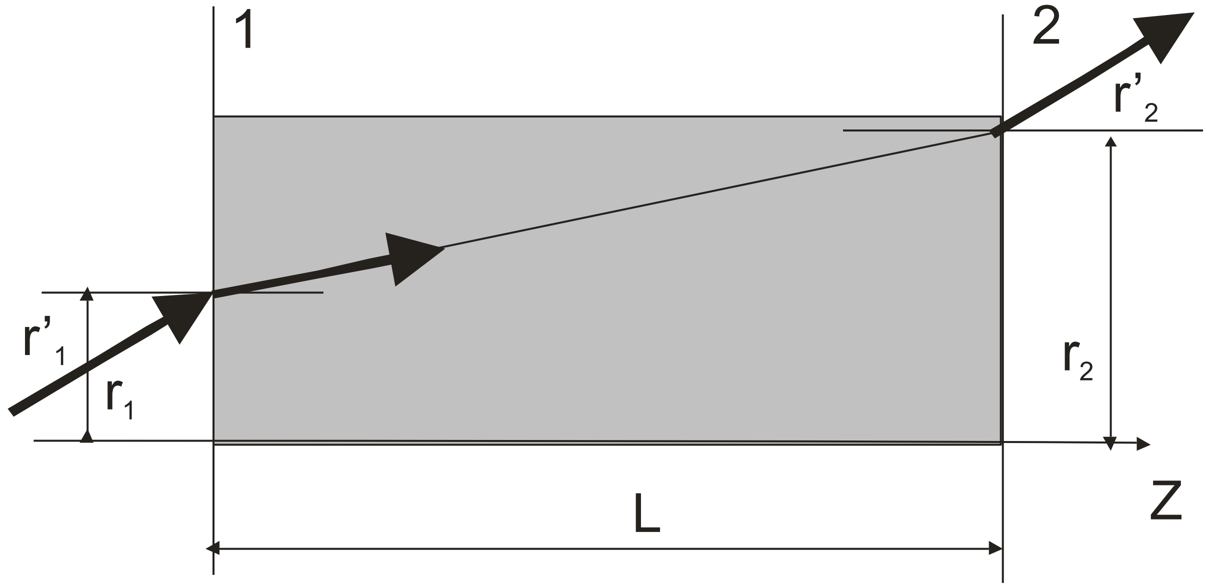
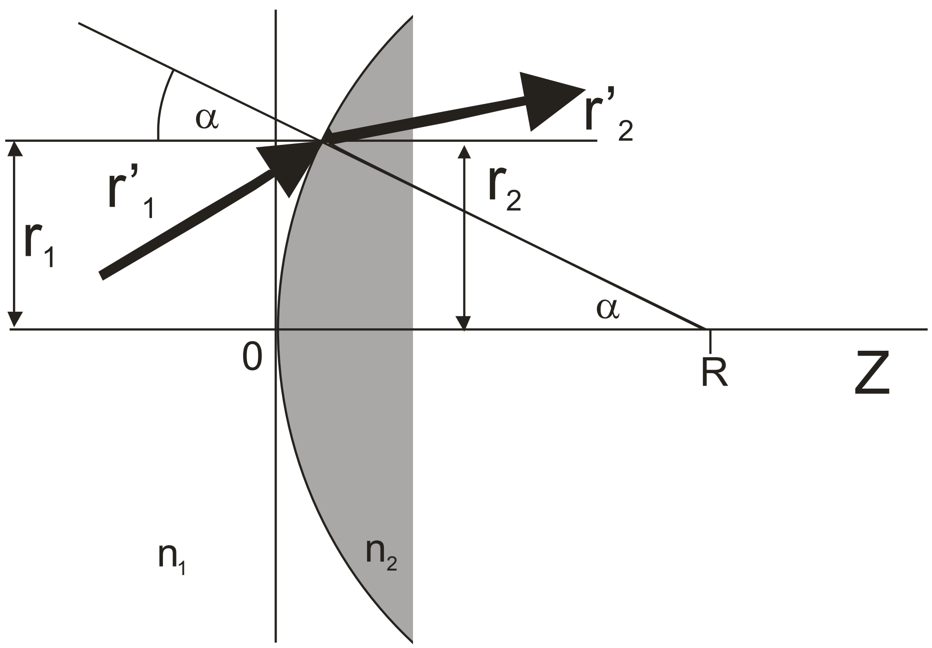


Figure 2.13: Ray propagation through a medium with refractive index n.

$$\mathbf{M} = \begin{pmatrix} 1 & L/n \\ 0 & 1 \end{pmatrix}$$

Thin Plano-convex Lens



$$n_1 \ (r_1' + \alpha) = n_2 \ (r_2' + \alpha)$$

for $\alpha \approx r_1/R$

$$\begin{aligned} r_2 &= r_1 \\ n_2 \ r_2' &= n_1 \ r_1' + \frac{n_1 - n_2}{R} r_1 \end{aligned}$$

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ \frac{n_1 - n_2}{R} & 1 \end{pmatrix}.$$

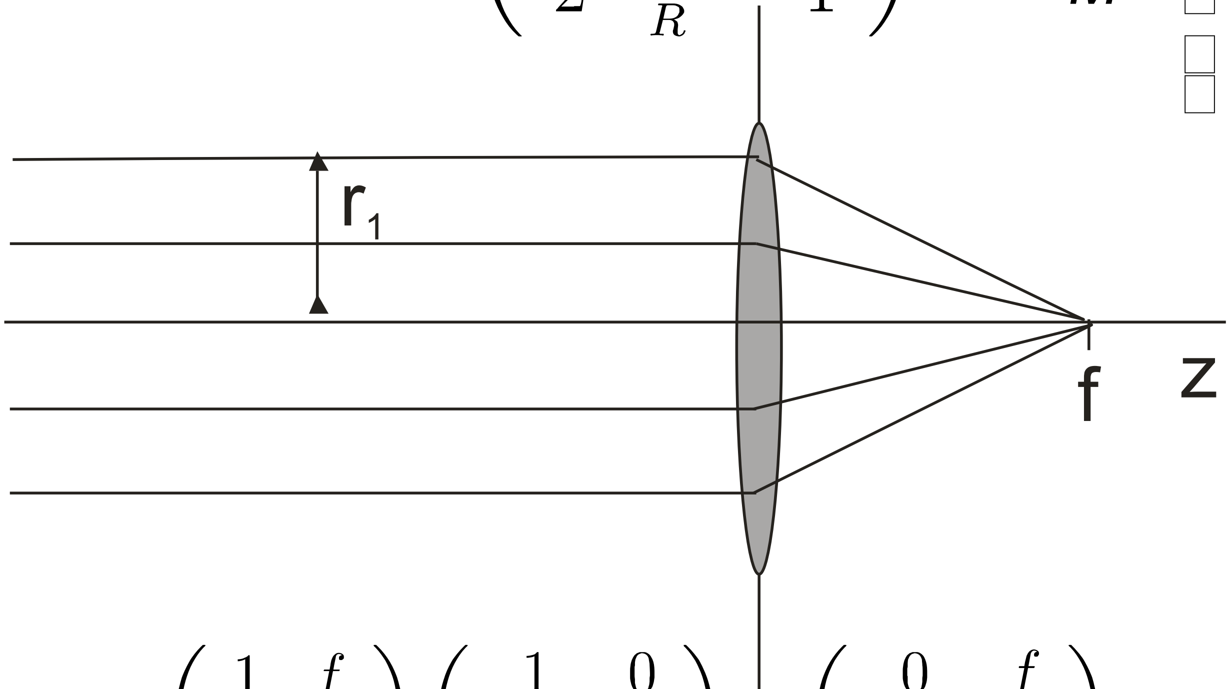
Figure 2.14: Derivation of ABCD-matrix of a thin plano-convex lens.

Biconvex Lens and Focussing

Biconvex Lens

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 2\frac{n_1 - n_2}{R} & 1 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$



$$\mathbf{M}_{tot} = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} = \begin{pmatrix} 0 & f \\ -\frac{1}{f} & 1 \end{pmatrix}$$

Figure 2.15: Imaging of parallel rays through a lens with focal length f

Concave Mirror with ROC R

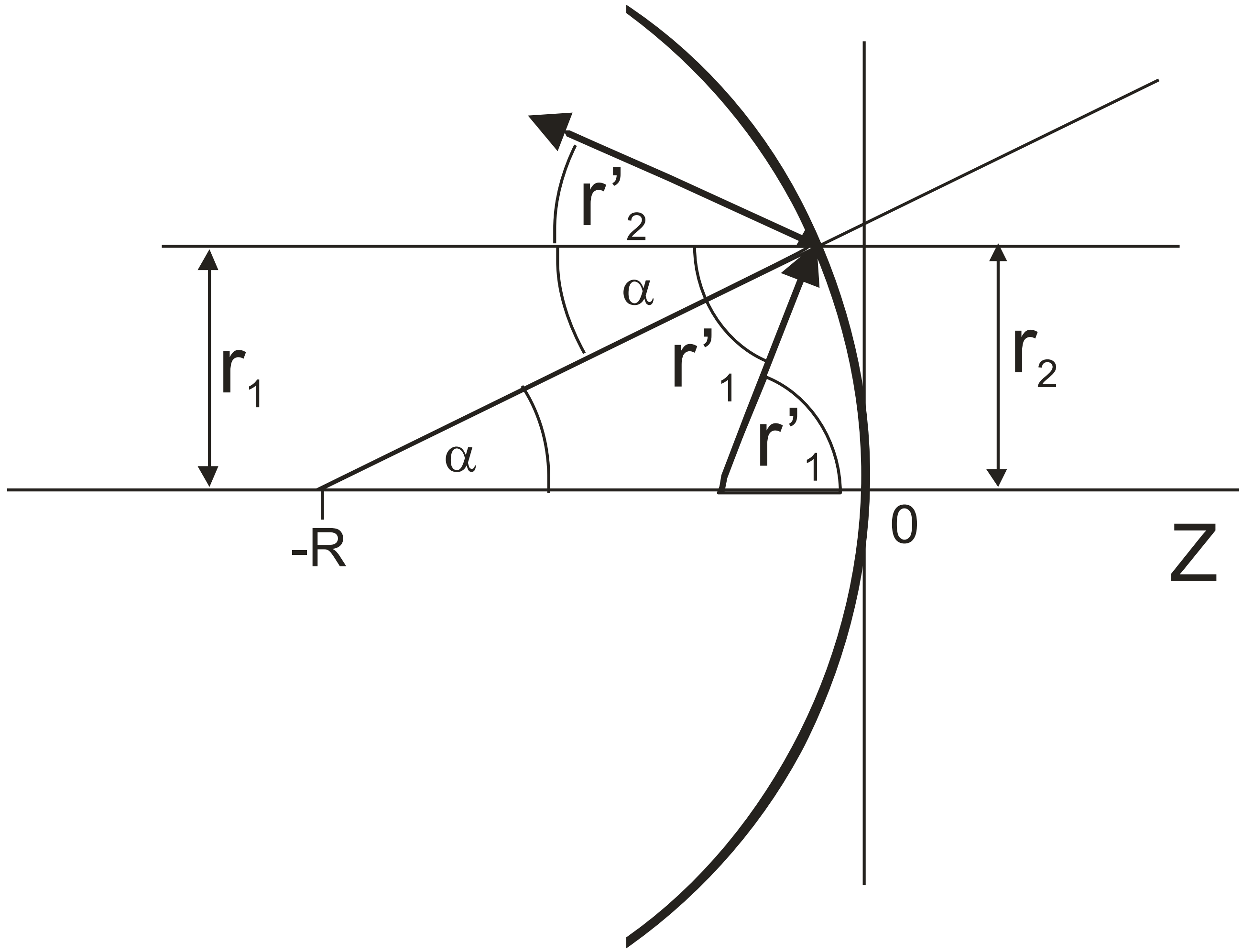


Figure 2.16: Derivation of Ray matrix for concave mirror with Radius R.

Table of Ray Matrices

Optical Element	ABCD-Matrix
Propagation in Medium with index n and length L	$\begin{pmatrix} 1 & L/n \\ 0 & 1 \end{pmatrix}$
Thin Lens with focal length f	$\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$
Mirror under Angle θ to Axis and Radius R Sagittal Plane	$\begin{pmatrix} 1 & 0 \\ \frac{-2\cos\theta}{R} & 1 \end{pmatrix}$
Mirror under Angle θ to Axis and Radius R Tangential Plane	$\begin{pmatrix} 1 & 0 \\ \frac{-2}{R\cos\theta} & 1 \end{pmatrix}$
Brewster Plate under Angle θ to Axis and Thickness d, Sagittal Plane	$\begin{pmatrix} 1 & \frac{d}{n} \\ 0 & 1 \end{pmatrix}$
Brewster Plate under Angle θ to Axis and Thickness d, Tangential Plane	$\begin{pmatrix} 1 & \frac{d}{n^3} \\ 0 & 1 \end{pmatrix}$

Table 2.6: ABCD matrices for commonly used optical elements.

Application: Gauss' Lens Formula

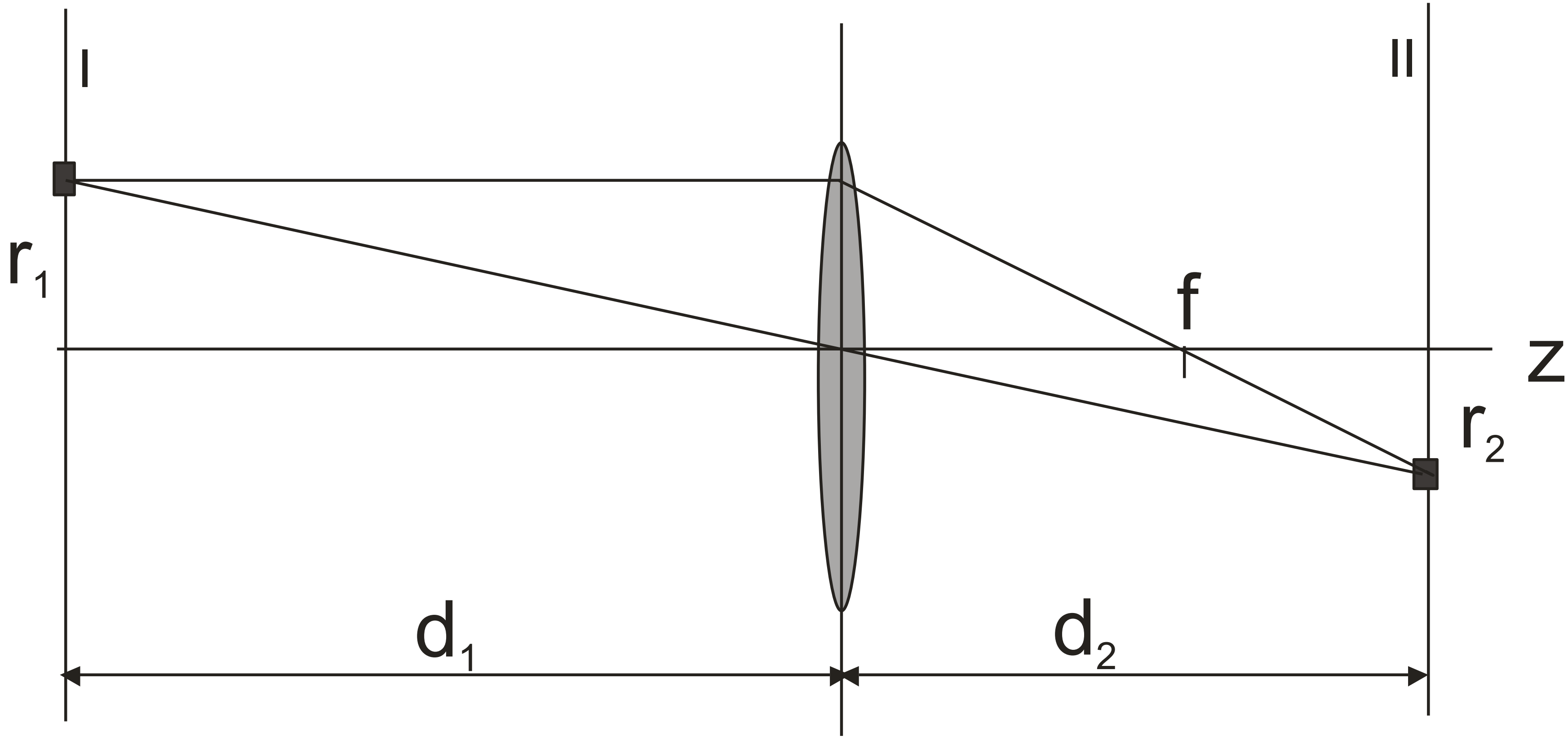


Figure 2.17: Gauss' lens formula.

$$\frac{1}{d_1} + \frac{1}{d_2} = \frac{1}{f} \qquad (2.80)$$

An Image is formed?

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & d_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & d_1 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 - \frac{d_2}{f} & \left(1 - \frac{d_2}{f}\right) d_1 + d_2 \\ -\frac{1}{f} & 1 - \frac{d_1}{f} \end{pmatrix}. \quad (2.81)$$

In order that the distance r_2 only depends on r_1, but not on r_1', B must be 0, which is Eq. (2.80). Thus in total we have

Magnification	$M_r =$	$\frac{f}{d_1 - f}$	
Distance to focus	$d_2 - f = M_r^2 (d_1 - f)$		

(2.82)

Gaussian Beam Propagation

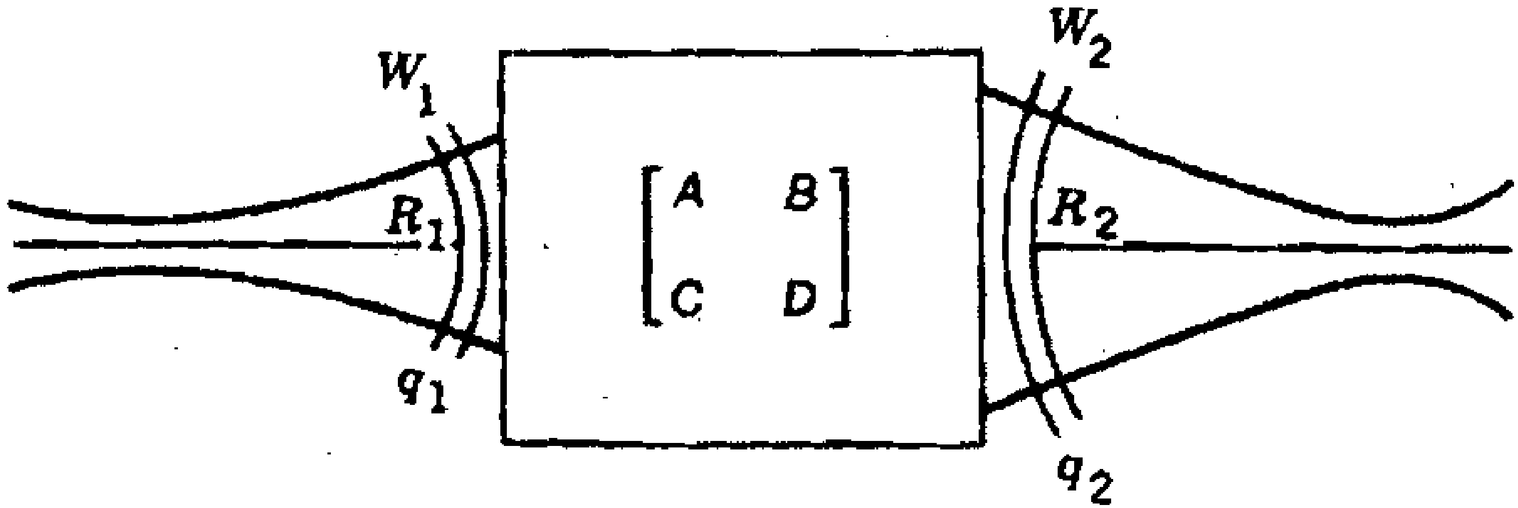


Figure 2.18: Gaussian beam transformation by ABCD law.

$$\widetilde{E}_0(x,y,z) = \frac{j}{q(z)} \exp\left[-jk_0\left(\frac{x^2+y^2}{2q(z)}\right)\right]. \quad (2.84)$$

$$\widetilde{E}_0(k_z,k_y,z) = 2\pi j \exp\left[-jq(z)\left(\frac{k_z^2+k_y^2}{2k_0}\right)\right] \quad (2.85)$$

$$q_2 = \frac{Aq_1+B}{Cq_1+D}$$

Telescope

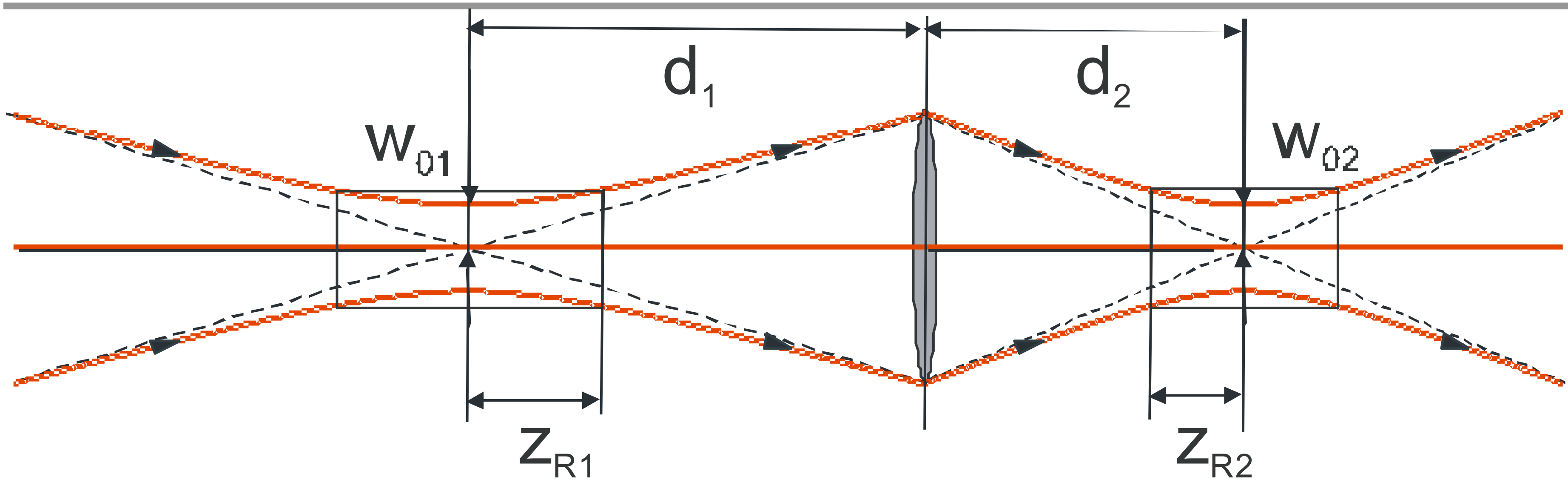


Figure 2.19: Focussing of a Gaussian beam by a lens.

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1-\frac{d_2}{f} & \left(1-\frac{d_2}{f}\right)d_1+d_2 \\ -\frac{1}{f} & 1-\frac{d_1}{f} \end{pmatrix}. \qquad (2.89)$$

The q-parameter of the Gaussian beam at the position of minimum waist is purely imaginary $q_1 = jz_{R1} = j\frac{\pi w_{01}^2}{\lambda}$ and $q_2 = jz_{R2} = j\frac{\pi w_{02}^2}{\lambda}$, where

$$q_2 = \frac{A\ q_1 + B}{C\ q_1 + D} = \frac{jz_{R1}A + B}{jz_{R1}C + D} = \frac{jz_{R1}A + B}{jz_{R1}C + D} = jz_{R2}. \qquad (2.90)$$

Gaussian Beam Optics: Physical Optics

Ray Optics: *B = 0* **Gauss' lens formula:** $\frac{1}{d_1} + \frac{1}{d_2} = \frac{1}{f}$

Physical Optics: *Real part of Eq. (2.90) = 0* $BD - z_{R1}^2 AC = 0$

$$\frac{1}{d_2} = \frac{1}{f} - \frac{1}{d_1 + \frac{z_{R1}^2}{d_1 - f}}. \qquad \frac{1}{w_{02}^2} = \frac{1}{w_{01}^2}\left(1 - \frac{d_1}{f}\right)^2 \left[1 + \left(\frac{z_{R1}}{d_1 - f}\right)^2\right]$$

Magnification	$M = M_r/\sqrt{1+\xi^2}$, with $\xi = \frac{z_{R1}}{d_1 - f}$ and $M_r = \left\lvert\frac{f}{d_1 - f}\right\rvert$
Beam waist	$w_{02} = M \cdot w_{01}$
Confocal parameter	$2z_{R2} = M^2\ 2z_{R2}$
Distance to focus	$d_2 - f = M^2\,(d_1 - f)$
Divergence	$\theta_{02} = \theta_{01}/M$

(2.271)

2.6 Optical Resonators

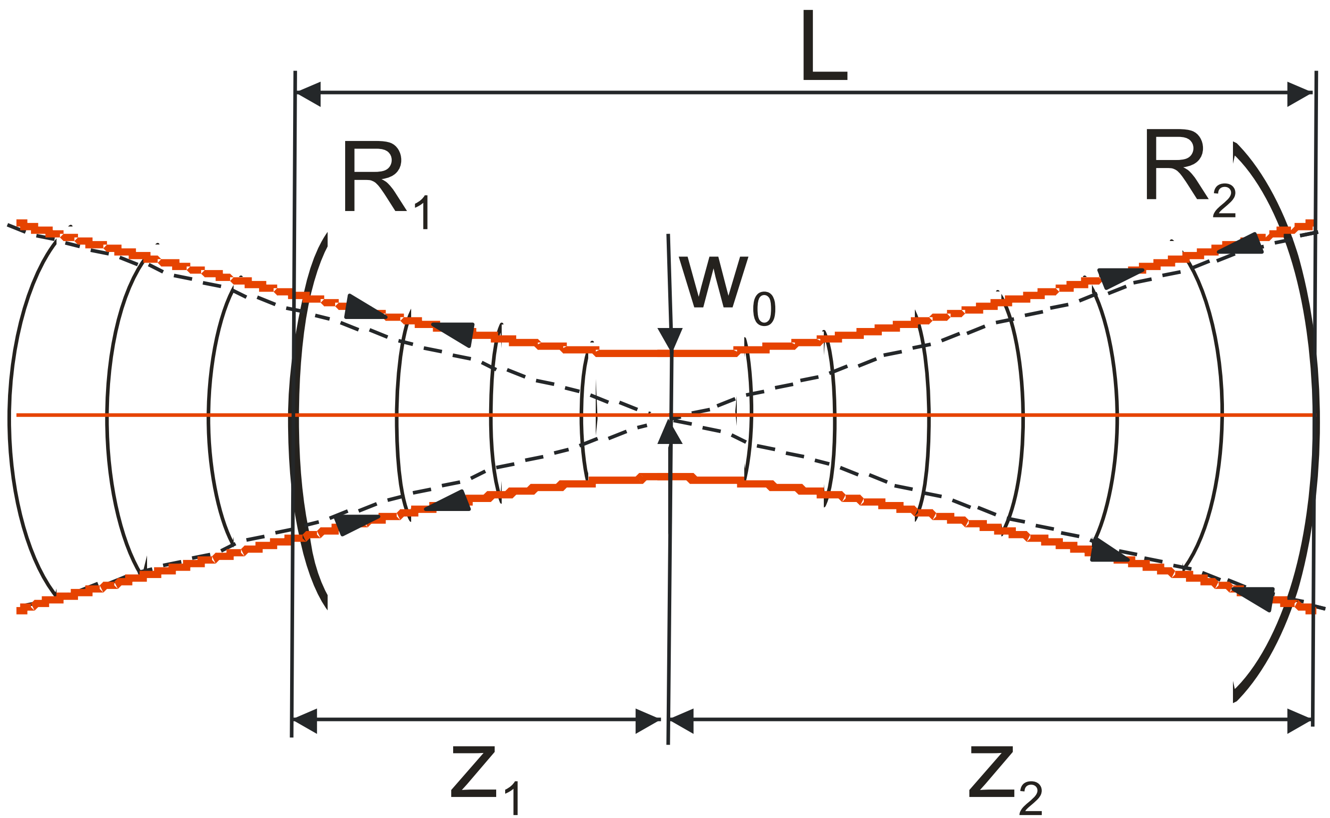


Figure 2.20: Fabry-Perot Resonator with finite beam size.

Curved - Flat Mirror Resonator

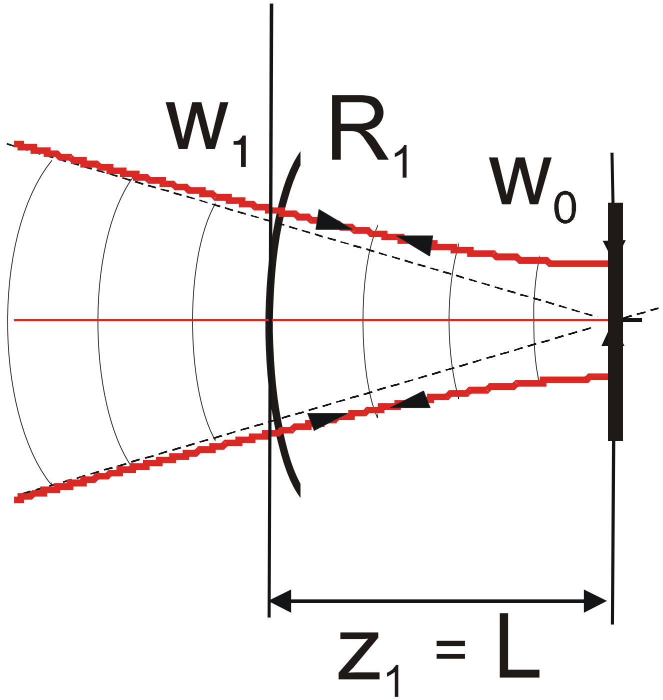


Figure 2.21: Curved-Flat Mirror Resonator

$$w_1 = w_o \left[1 + \left(\frac{L}{z_R} \right)^2 \right]^{1/2}$$

with $$z_R = \frac{\pi w_o^2}{\lambda}$$

$$R_1 = L \left[1 + \left(\frac{z_R}{L} \right)^2 \right]$$

$$w_o = \sqrt{\frac{\lambda R_1}{\pi}} \sqrt[4]{\frac{L}{R_1} \left(1 - \frac{L}{R_1} \right)},$$

$$w_1 = \sqrt{\frac{\lambda R_1}{\pi}} \sqrt[4]{\frac{\frac{L}{R_1}}{1 - \frac{L}{R_1}}}.$$

Curved - Flat Mirror Resonator

For given R_1

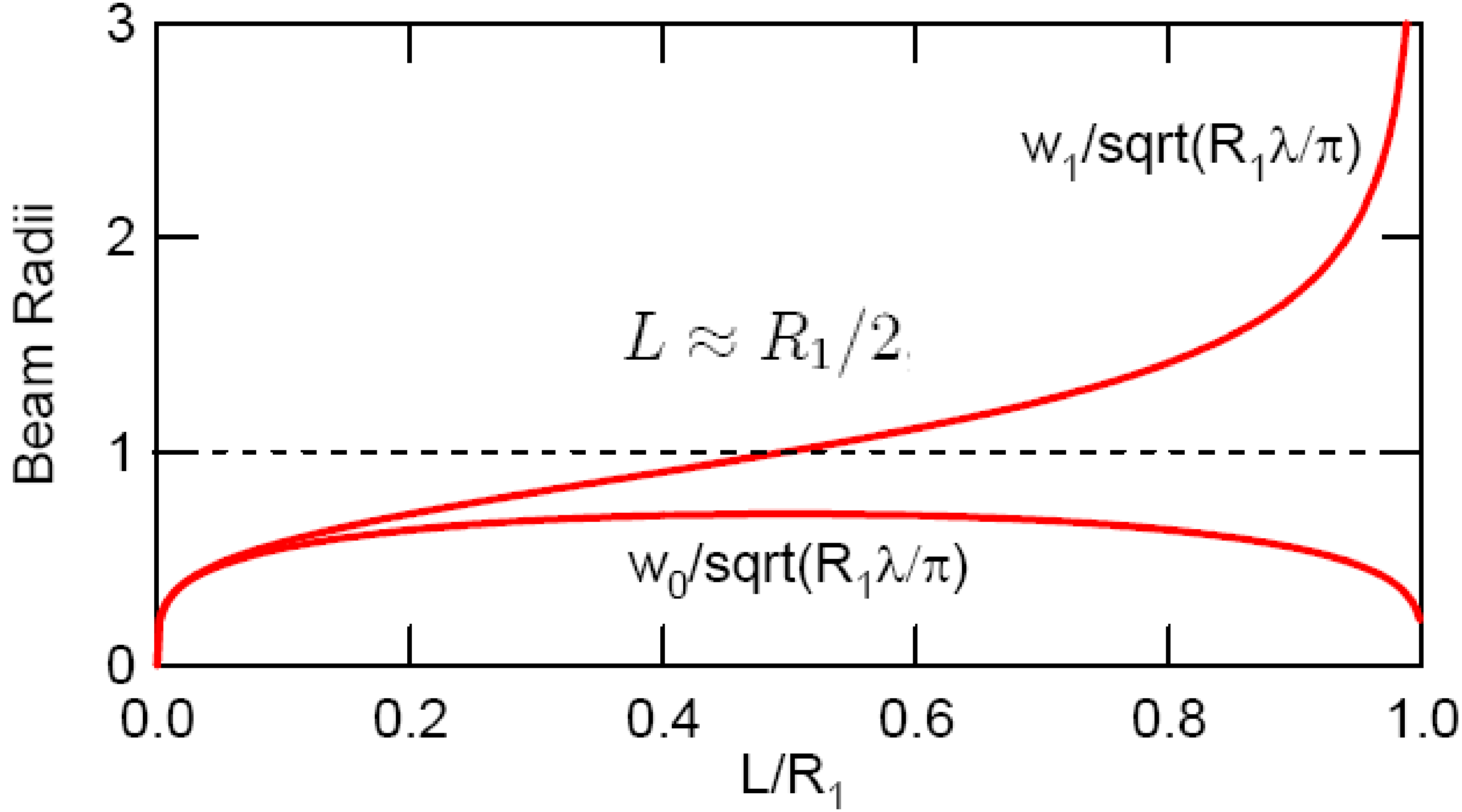


Figure 2.22: Beam waists of the curved-flat mirror resonator as a function of L/R_1.

Two Curved Mirror Resonator

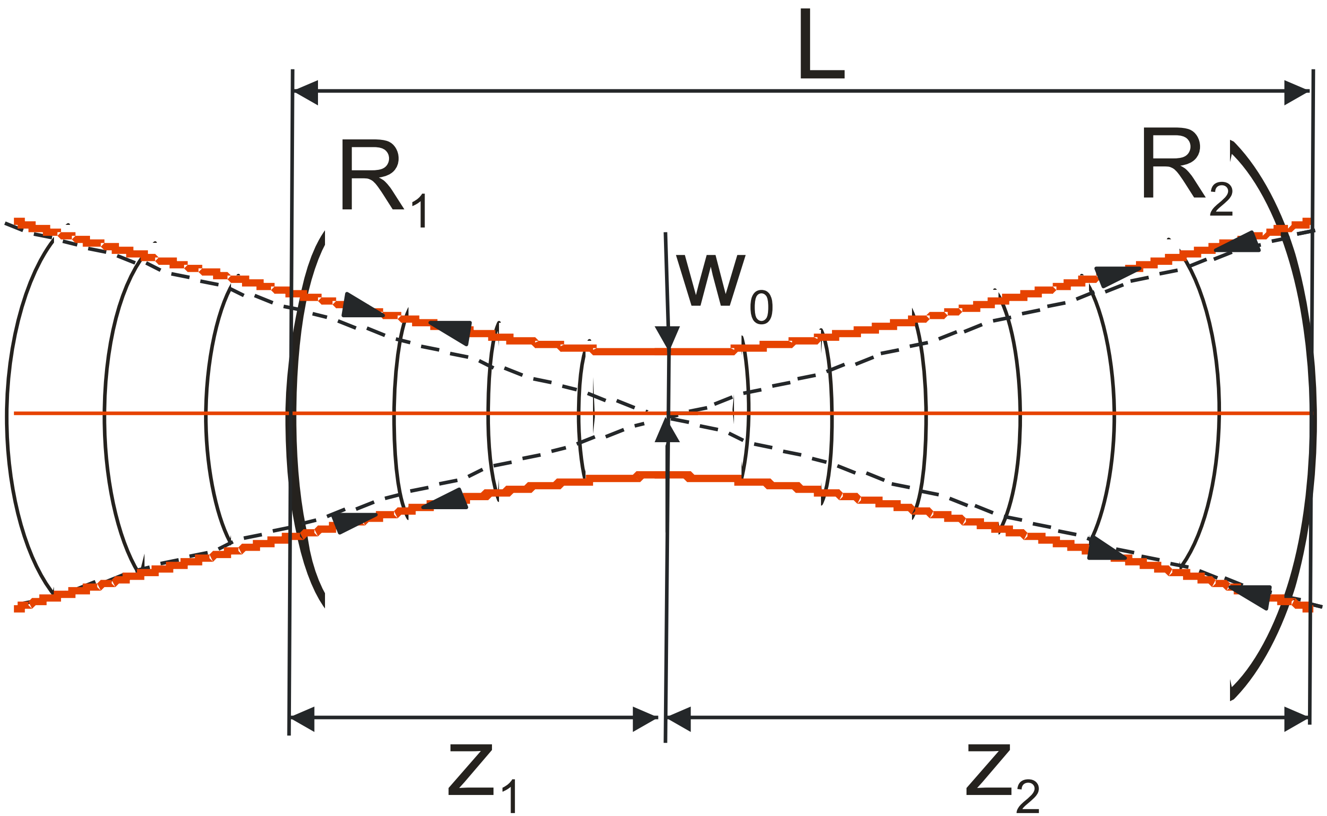


Figure 2.20: Fabry-Perot Resonator with finite beam size.

Two Curved Mirror Resonator

$$R_1 = z_1\left[1+\left(\frac{z_R}{z_1}\right)^2\right]$$

$$R_2 = z_2\left[1+\left(\frac{z_R}{z_2}\right)^2\right]$$

$$L = z_1 + z_2.$$

$$z_1 = \frac{L(R_2 - L)}{R_1 + R_2 - 2L},$$

by symmetry:

$$z_2 = \frac{L(R_1 - L)}{R_1 + R_2 - 2L} = L - z_1$$

and:

$$z_R^2 = L\frac{(R_1 - L)(R_2 - L)(R_1 + R_2 - L)}{(R_1 + R_2 - 2L)^2}$$

Or:

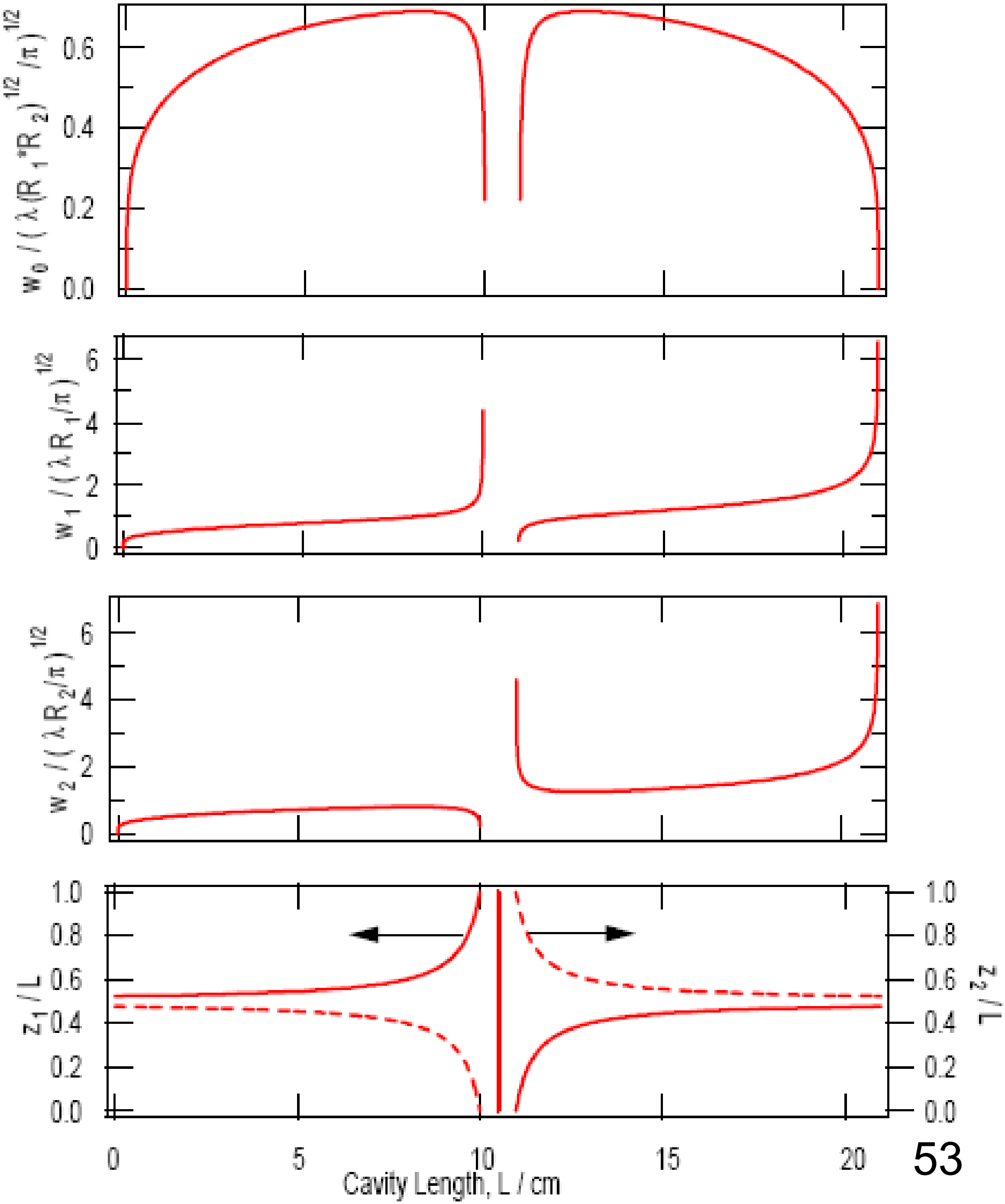
$$\begin{aligned} w_o^4 &= \left(\frac{\lambda L}{\pi}\right)^2 \frac{(R_1 - L)(R_2 - L)(R_1 + R_2 - L)}{L(R_1 + R_2 - 2L)^2} \\ &= \left(\frac{\lambda\sqrt{R_1 R_2}}{\pi}\right)^2 \frac{L^2}{R_1 R_2}\frac{(1-\frac{L}{R_1})(1-\frac{L}{R_2})(\frac{L}{R_1}+\frac{L}{R_2}-\frac{L}{R_1}\frac{L}{R_2})}{(\frac{L}{R_1}+\frac{L}{R_2}-2\frac{L}{R_1}\frac{L}{R_2})^2}. \end{aligned}$$

Figure 2.23: Two curved mirror resonator.

$$R_1 = 10\text{cm and } R_2 = 11\text{cm}$$

$$w_1 = w_o \left[1 + \left(\frac{z_1}{z_R} \right)^2 \right]^{1/2},$$

$$w_2 = w_o \left[1 + \left(\frac{z_2}{z_R} \right)^2 \right]^{1/2},$$



Resonator Stability

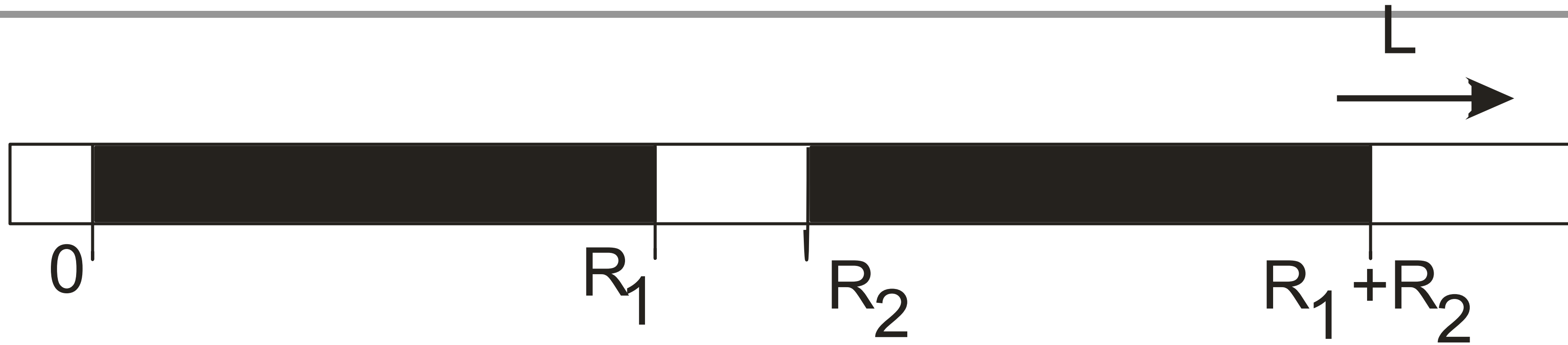


Figure 2.75: Stable regions (black) for the two-mirror resonator

$$0 \leq L \leq R_1 \text{ and } R_2 \leq L \leq R_1 + R_2$$

Or introduce cavity parameters:

$$g_i = (R_i - L)/R_i, \text{ for } i = 1, 2$$

$$\text{stable}: 0 \leq g_1 \cdot g_2 = S \leq 1 \tag{2.308}$$

$$\text{unstable}: g_1 g_2 \leq 0; \text{ or } g_1 g_2 \geq 1. \tag{2.309}$$

Geometrical Interpretation

$$g_i = (R_i - L)/R_i = -S_i/R_i \qquad \text{stable} : 0 \leq \frac{S_1 S_2}{R_1 R_2} \leq 1.$$

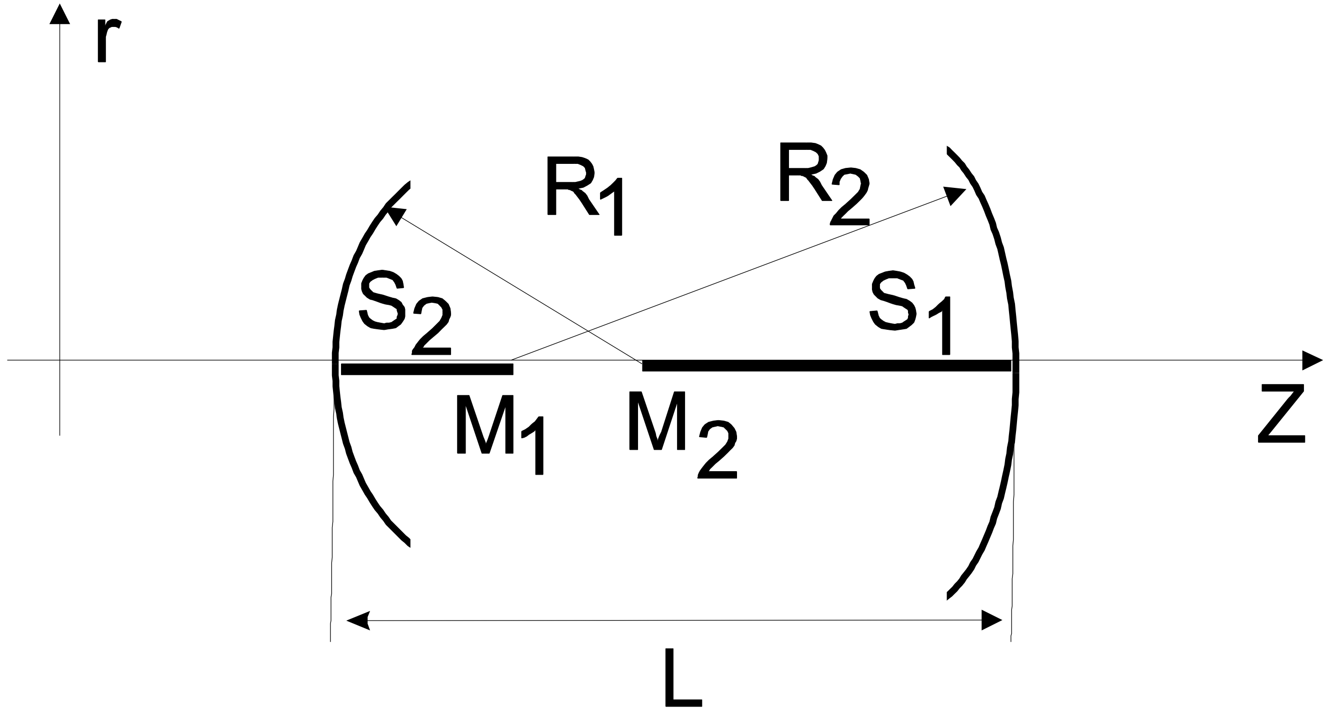


Figure 2.24: Stability Criterion

- A resonator is stable if the mirror radii, laid out along the optical axis, overlap.
- A resonator is unstable if the radii do not overlap or one lies within the other.

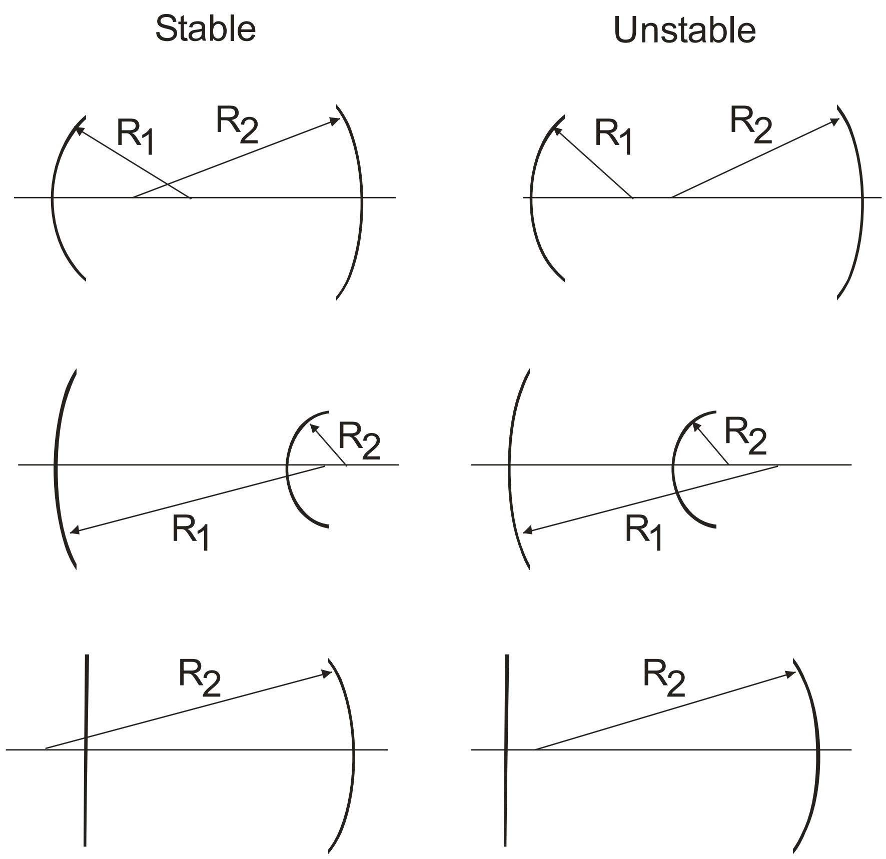


Figure 2.26: Stable and unstable resonators

Hermite Gaussian Beams

Other solutions to the paraxial wave equation:

$$\widetilde{E}_{m,n}(x,y,z) = A_{m,n}\left[\frac{w_0}{w(z)}\right] G_m\left[\frac{\sqrt{2x}}{w(z)}\right] G_n\left[\frac{\sqrt{2y}}{w(z)}\right] \cdot \exp\left[-jk_0\left(\frac{x^2+y^2}{2R(z)}\right)+j(m+n+1)\zeta(z)\right]$$

$$G_m[u] = H_m[u]\exp\left[-\frac{u^2}{2}\right], \text{ for } m = 0,1,2,\ldots$$

Hermite Polynomials:

$$\begin{aligned} H_0[u] &= 1, \\ H_1[u] &= 2u, \\ H_2[u] &= 4u^2-1, \\ H_3[u] &= 8u^3-12u, \end{aligned}$$

Hermite Gaussian Beams

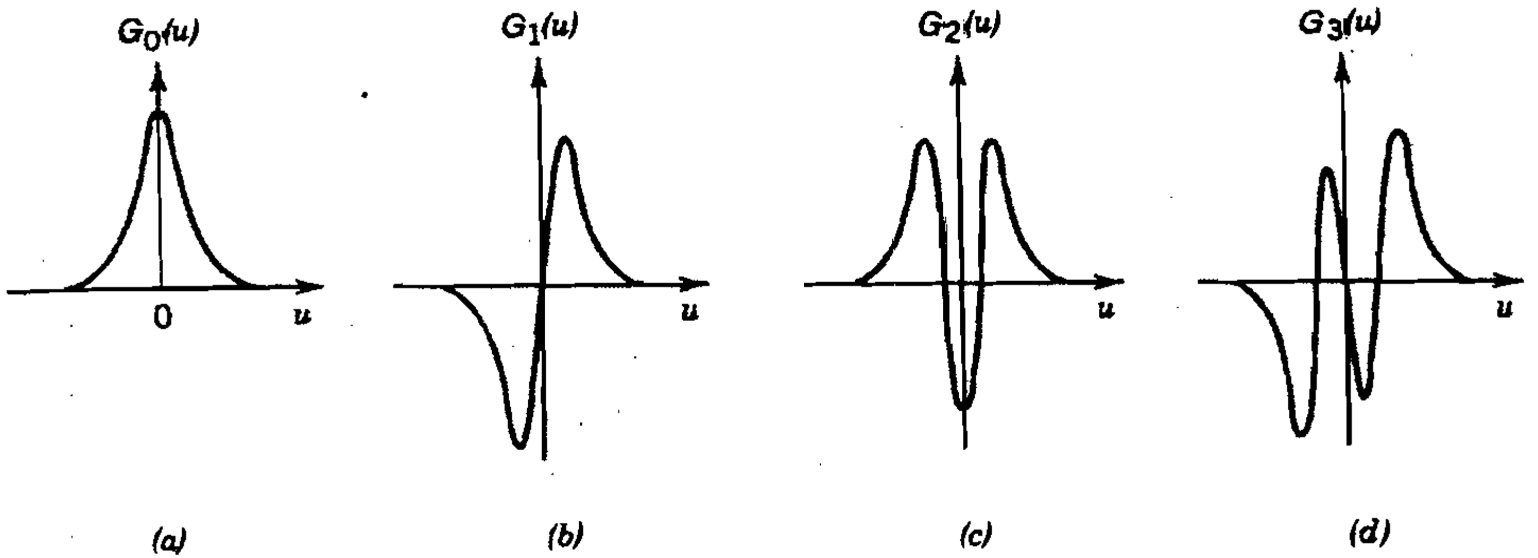


Figure 2.27: Hermite Gaussians $G_l(u)$

Hermite Gaussian Beams

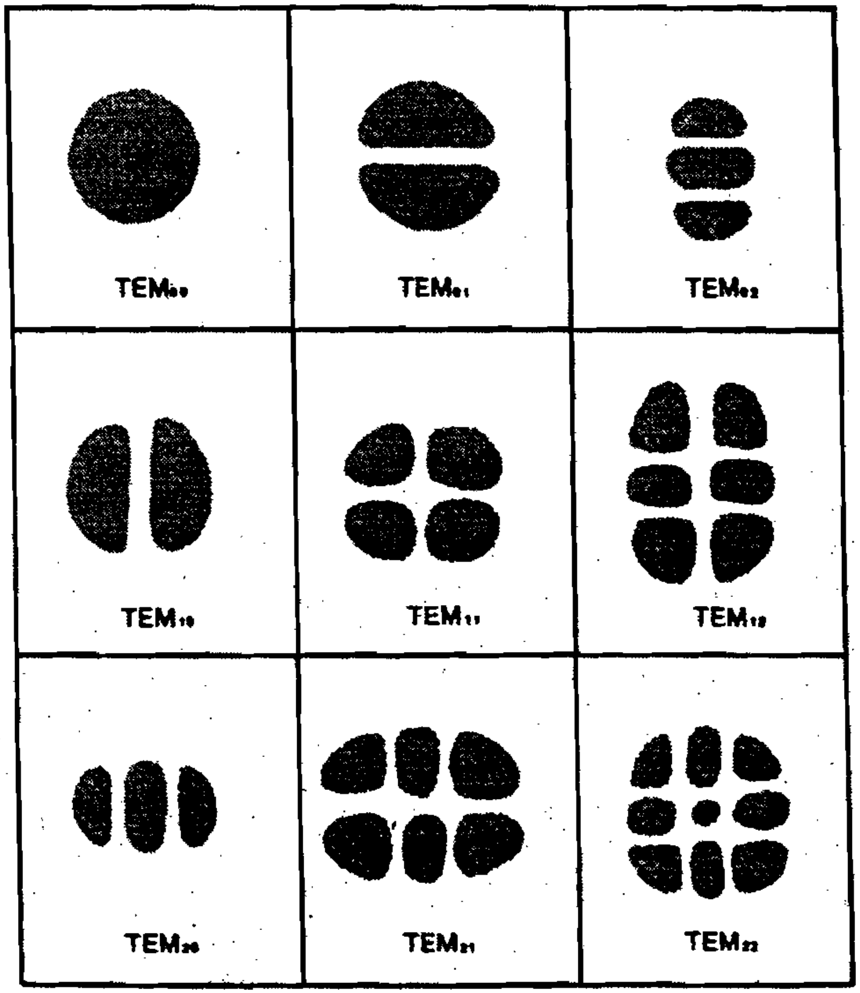


Figure 2.28: Intensity profile of TEM_{lm}-beams. by ABCD law.

Axial Mode Structure:

Roundtrip Phase = 2 p π:

$$\phi_{pmn} = 2p\pi, \text{ for } p = 0, \pm 1, \pm 2, \ldots$$

$$\phi_{pmn} = 2kL - 2(m+n+1)\left(\zeta(z_2) - \zeta(z_1)\right),$$

Resonance Frequencies:

$$\omega_{pmn} = \frac{c}{L}\left[\pi p + (m+n+1)\left(\zeta(z_2) - \zeta(z_1)\right)\right]$$

Special Case: Confocal Resonator: L = R → $\zeta(z_2) - \zeta(z_1) = \frac{\pi}{2}$

$$f_{pmn} = \frac{c}{2L}\left[p + \frac{1}{2}(m+n+1)\right].$$

Resonance Frequencies

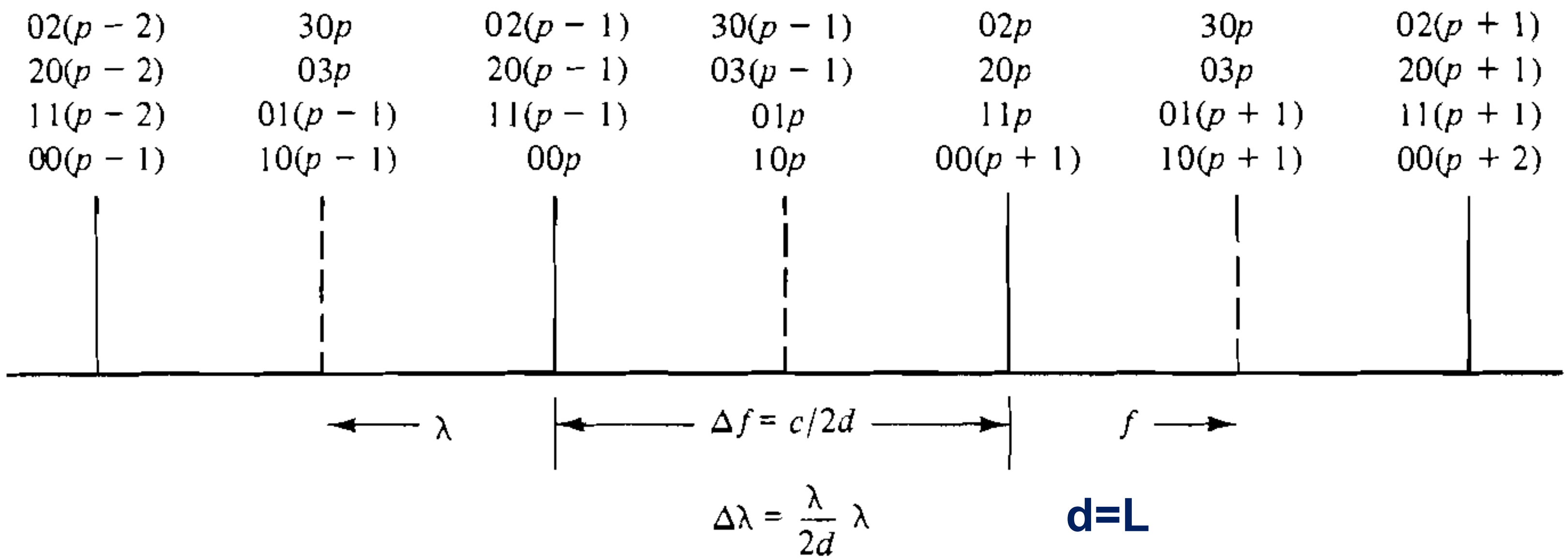


Figure 2.29: Resonance frequencies of the confocal Fabry-Perot resonator,