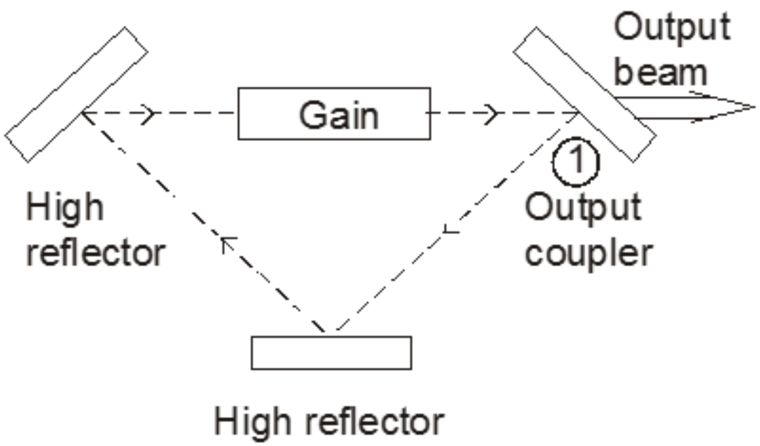
Ultrafast Optical Physics II (SoSe 2019)
Lecture 4, May 3, 2019

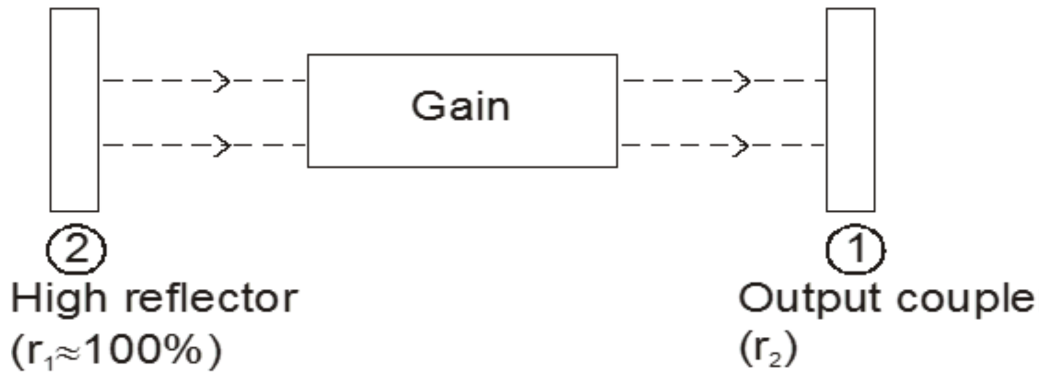
(1) Laser rate equations

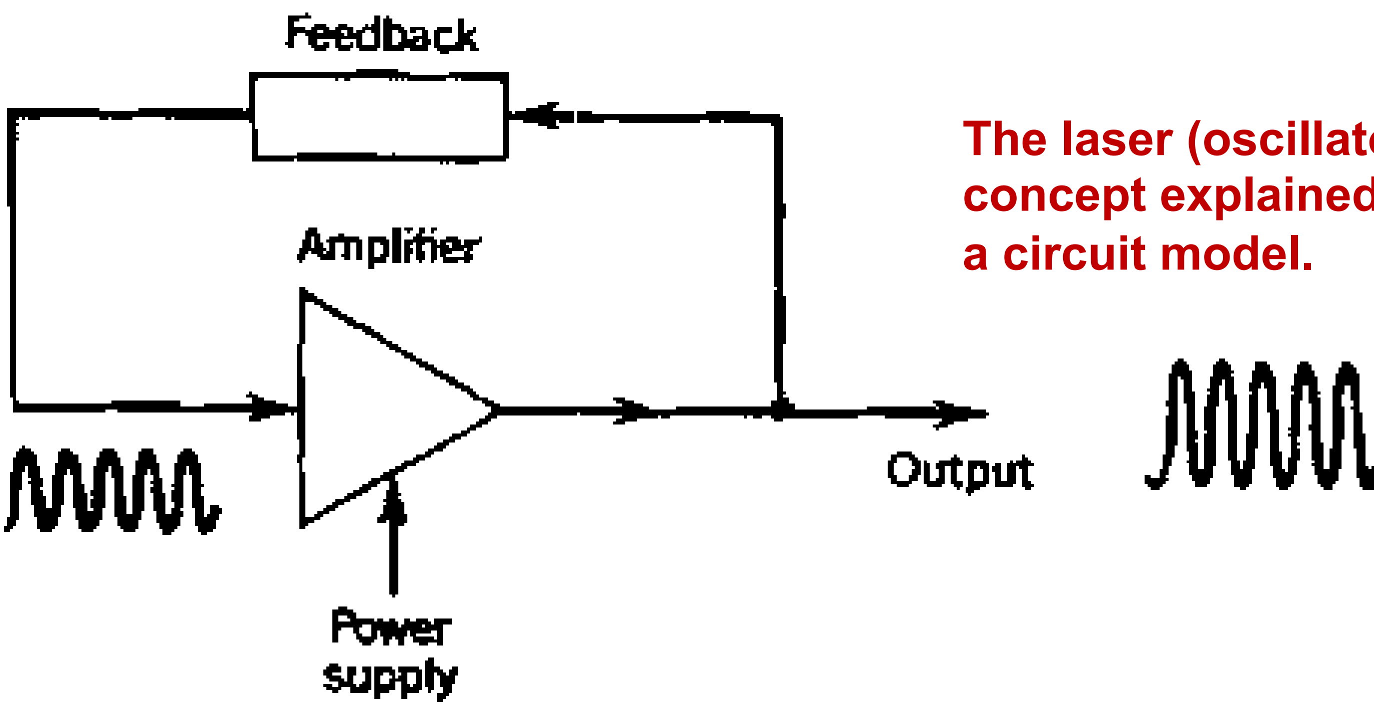
(2) Laser CW operation: stability
and relaxation oscillation

(3) Q-switching: active and passive

Possible laser cavity configurations

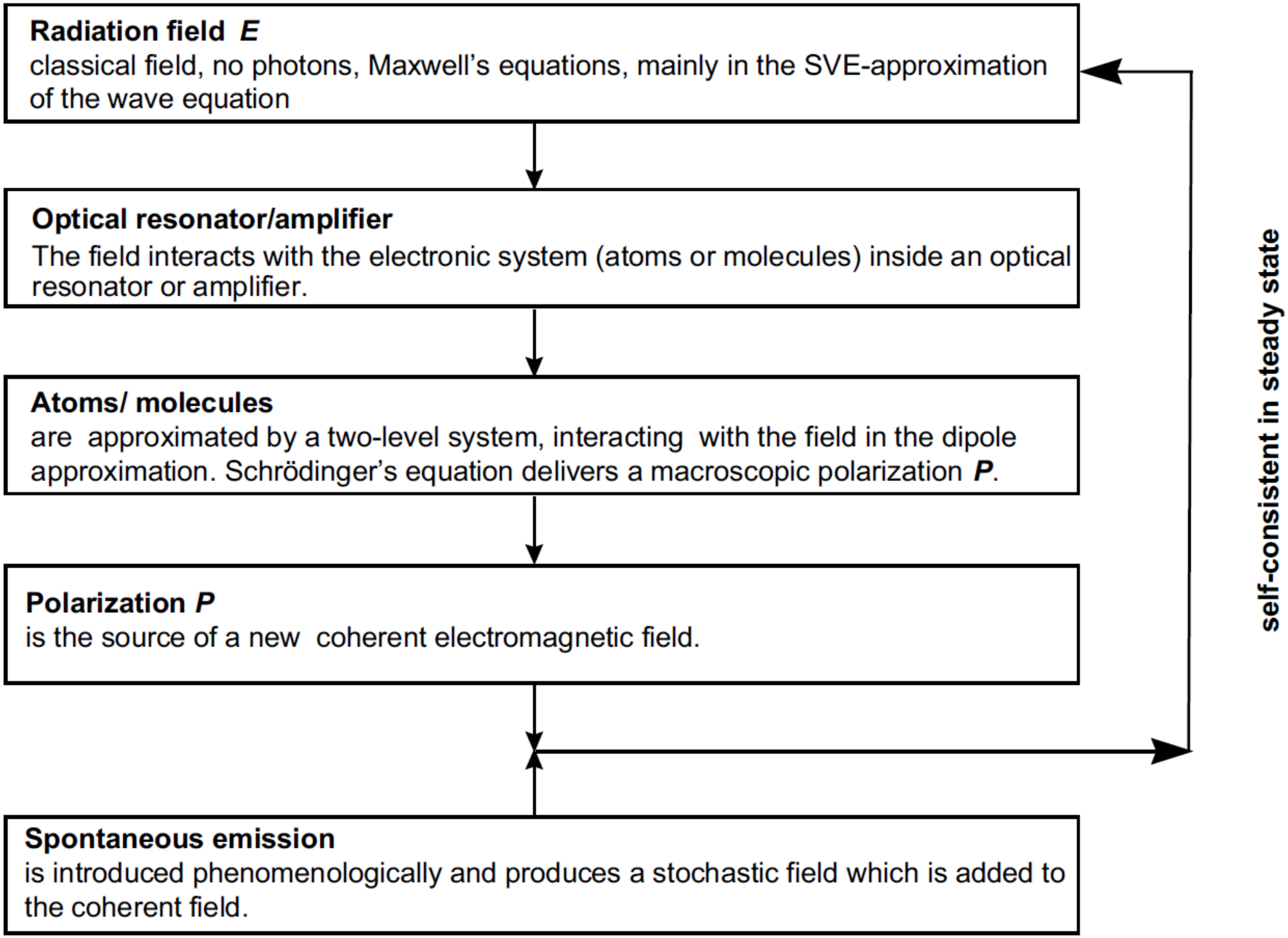






The laser (oscillator) concept explained using a circuit model.

Self-consistent in steady state



V.A. Lopota and H. Weber, "fundamentals of the semiclassical laser theory"

Laser rate equations

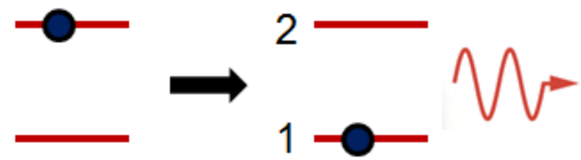
Interaction cross section: [Unit: cm^2]

$$\sigma = \frac{\hbar\omega_{eg}}{T_1 I_s} = \frac{2\omega_{eg} T_2 Z_F}{\hbar} |\vec{M}^*_{eg} \cdot \vec{e}|^2$$

$$\dot{w}\big|_{induced} = -\frac{w}{T_1 I_s} I = -\sigma w I_{ph}$$

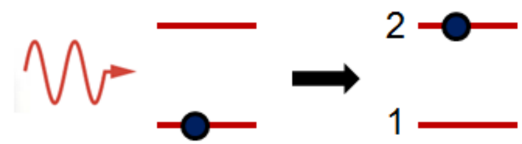
- Interaction cross section is the probability that an interaction will occur between EM field and the atomic system.
- Interaction cross section only depends on the dipole matrix element and the linewidth of the transition

$$\dot{w} = -\frac{w(t) - w_0}{T_1} - \sigma w I_{ph}$$



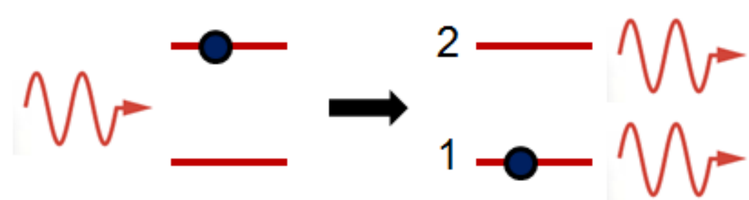
$$\frac{dN_2}{dt} = -AN_2 = -\frac{N_2}{\tau_{21}}$$

Spontaneous emission



$$\frac{dN_1}{dt} = -BN_1 I$$

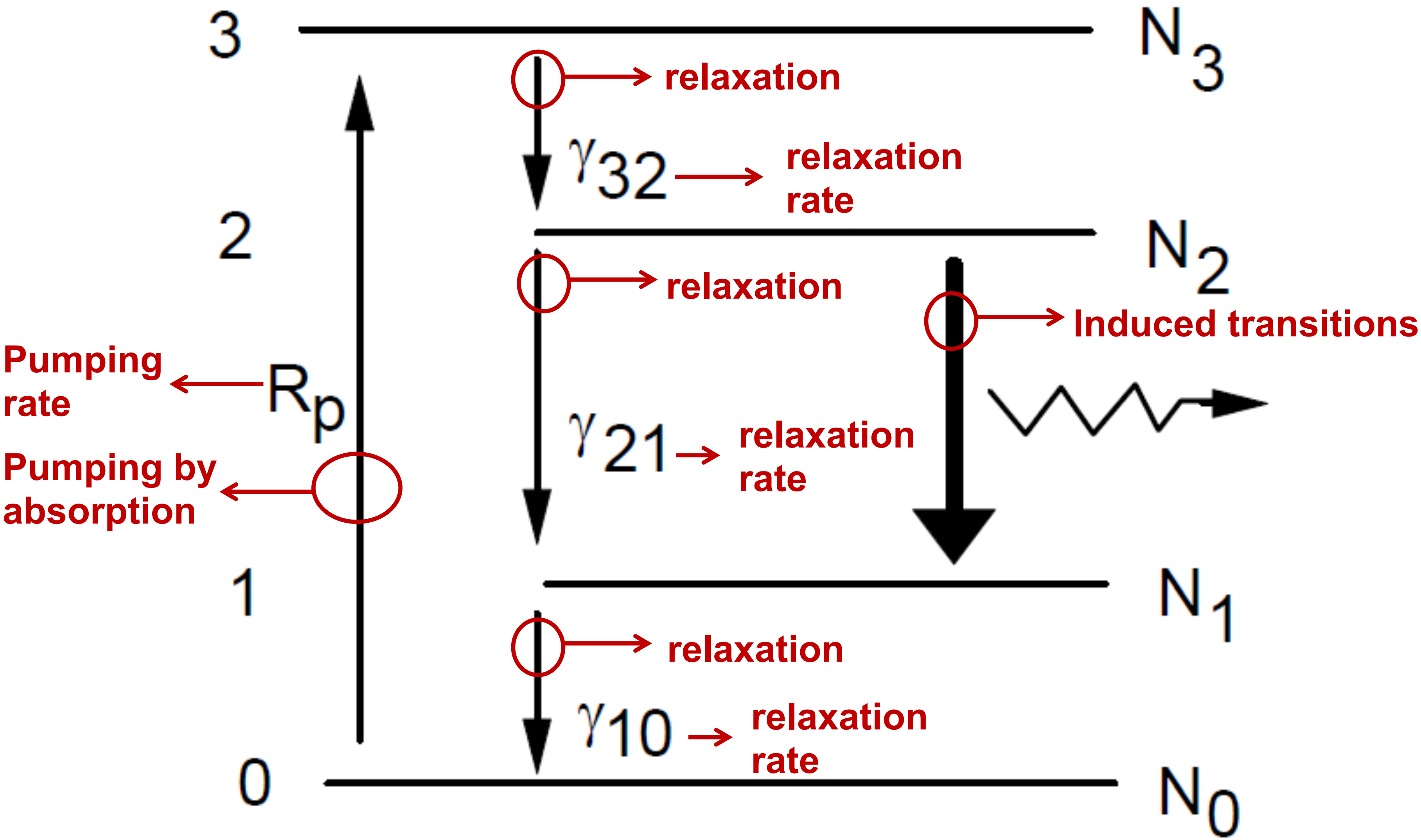
Absorption



$$\frac{dN_2}{dt} = -BN_2 I$$

Stimulated emission

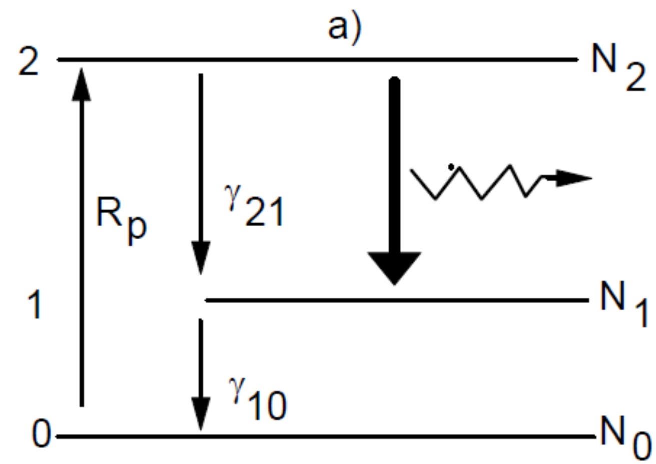
How to achieve population inversion?



Four-level gain medium

Laser rate equations for three-level laser medium

If the relaxation rate γ_{10} is much faster than γ_{21} and the number of possible stimulated emission events that can occur $\sigma_{21}\left(N_2 - N_1\right) I_{ph}$, we can set $N_1 = 0$ and obtain only a rate equation for the upper laser level:



$$\frac{d}{dt} N_2 = -\gamma_{21}\left(N_2 - \frac{R_p}{\gamma_{21}}\right) - \sigma_{21} N_2 \cdot I_{ph}$$

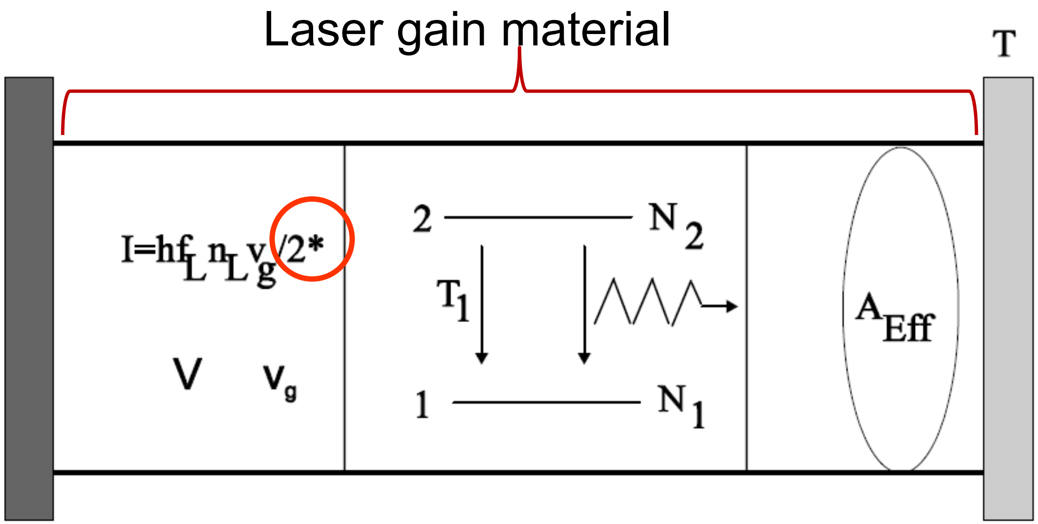
This equation is identical to the equation for the inversion of the two-level system:

$$\dot{w} = -\frac{w(t) - w_0}{T_1} - \sigma w I_{ph} \qquad I_{ph} = I / \hbar\omega_{eg}$$

$\dfrac{R_p}{\gamma_{21}}$ $\longrightarrow$ **equilibrium upper level population w/o photons present**

$\gamma_{21} = \dfrac{1}{\tau_L}$ $\longrightarrow$ **upper level lifetime due to radiative and non-radiative processes**

More on laser rate equations



***V*:= *A*$_{eff}$ *L* Mode volume**
***f_L*: laser frequency**
***I*: Intensity**
***v_g*: group velocity at laser frequency**
***N_L*: number of photons in mode**
***n_L*: photon density in mode**

Intensity I in a mode propagating at group velocity v_g in one direction with a mode volume V is related to the number of photons N_L stored in the mode with volume V by

$$I = hf_L \frac{N_L}{2^* V} v_g = \frac{1}{2^*} hf_L n_L v_g$$

$$I_{ph} = I / \hbar\omega_{eg} = \frac{N_L}{2^* V} v_g$$

2^* = 2 for a linear laser resonator (then only half of the photons are going in one direction)

2^* = 1 for a ring laser

σ: interaction cross section $\sigma = hf_L / (I_s \tau_L)$

More on laser rate equations

Number of atoms in upper level:

$$\frac{d}{dt}N_2 = -\frac{N_2}{T_1} - \frac{\sigma v_g}{V}N_2N_L + R_p$$

upper level lifetime

Number of photons in mode:

$$\frac{d}{dt}N_L = -\frac{N_L}{\tau_p} + \frac{\sigma v_g}{V}N_2\left(N_L + 1\right)$$

Photon lifetime in the cavity or cavity decay time

Number of photons spontaneously emitted into laser mode

Laser cavity with a semi-transparent mirror with transmission T produces a small power loss 2l = − ln(1−T) ≈ T (for small T) per roundtrip in the cavity.

Cavity round trip time: $T_R = 2^* L/v_g$

Photon lifetime: $\tau_p = T_R\,/2l$

***l*: amplitude loss per roundtrip**

***2l*: power loss per roundtrip**

Rewrite rate equations using measurable quantities

$$\frac{d}{dt}N_2 = -\frac{N_2}{\tau_L} - \sigma v_g N_2 n_L + R_p$$
$$\frac{d}{dt}N_L = -\frac{N_L}{\tau_p} + \frac{\sigma v_g}{V} N_2 (N_L + 1)$$

Circulating intracavity power

$$P = I \cdot A_{eff} = h f_L \frac{N_L}{T_R}$$

Round trip amplitude gain

$$g = \frac{\sigma v_g}{2V} N_2 T_R$$

$$\frac{d}{dt}g = -\frac{g - g_0}{\tau_L} - \frac{gP}{E_{sat}}$$
$$\frac{d}{dt}P = -\frac{1}{\tau_p}P + \frac{2g}{T_R}(P + P_{vac})$$

$$E_{sat} = \frac{h f_L V}{\sigma v_g T_R} = \frac{1}{2^*} I_s A_{eff} \tau_L$$
$$P_{sat} = E_{sat}/\tau_L$$
$$P_{vac} = h f_L / T_R$$
$$g_0 = 2^* \frac{R_p}{2A_{eff}} \sigma \tau_L,$$

Output power: $P_{out} = T \cdot P.$

small signal gain ~ $\sigma\tau_L$ - product

Buildup of laser oscillation

$$\frac{d}{dt}g = -\frac{g-g_0}{\tau_L} - \frac{gP}{E_{sat}} \qquad \frac{d}{dt}P = -\frac{1}{\tau_p}P + \frac{2g}{T_R}(P+P_{vac})$$

During laser buildup,

$$P_{vac} \ll P \ll P_{sat} = E_{sat}/\tau_L$$

we can neglect the spontaneous emission P_{vac}, and the gain is unsaturated: $g = g_0$

$$\frac{d}{dt}P = -\frac{1}{\tau_p}P + \frac{2g}{T_R}(P+P_{vac})$$

$$\downarrow \quad \tau_p = T_R/2l$$

$$\frac{dP}{P} = 2(g_0 - l)\frac{dt}{T_R}$$

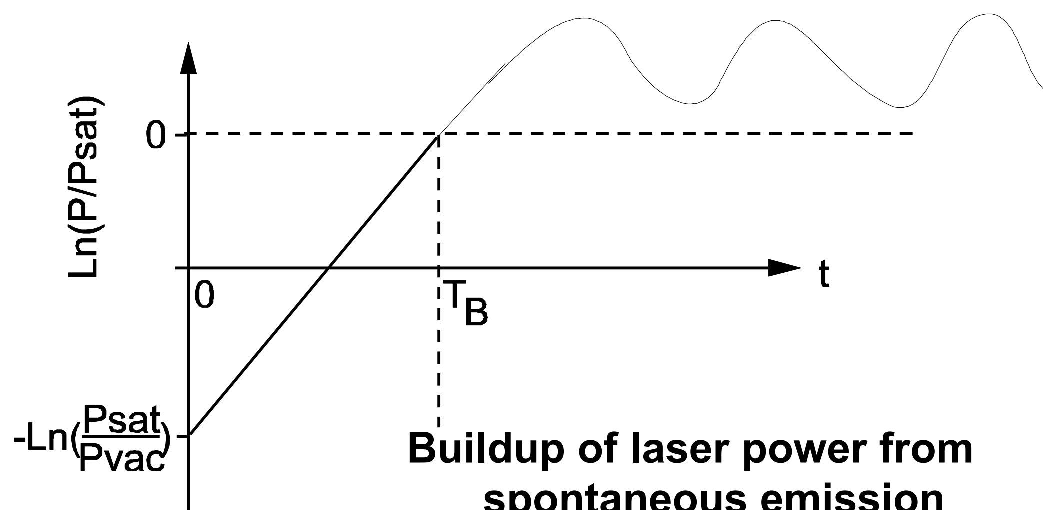
$$\downarrow$$

$$P(t) = P(0)e^{2(g_0-l)\frac{t}{T_R}}$$

The laser power builds up from vacuum fluctuations once the small signal gain surpasses the laser losses: $g_0 > g_{th} = l$

Saturation sets in within the built-up time

$$T_B = \frac{T_R}{2(g_0-l)}\ln\frac{P_{sat}}{P_{vac}} = \frac{T_R}{2(g_0-l)}\ln\frac{A_{eff}T_R}{\sigma\tau_L}$$



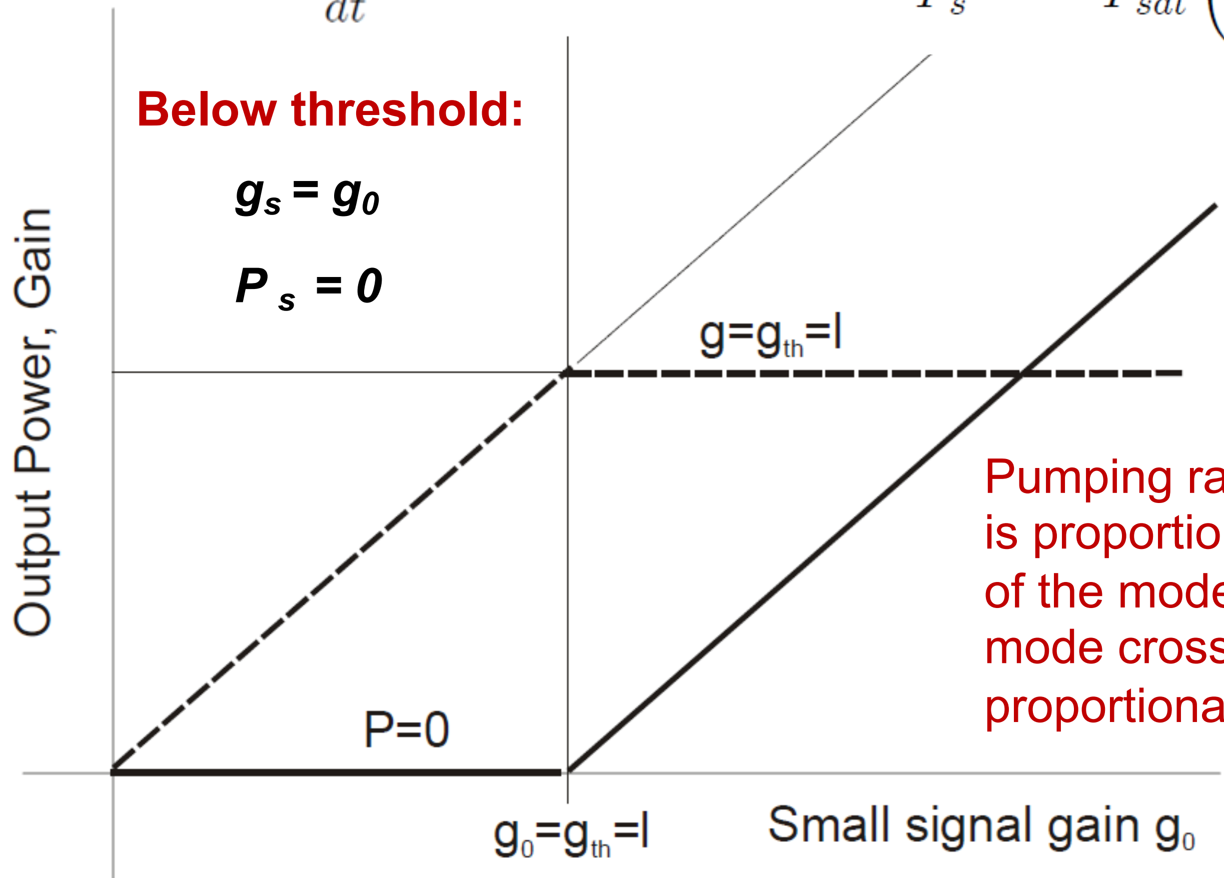
Buildup of laser power from spontaneous emission

Steady state operation: output power vs small signal gain

Beyond the gain threshold, some time after the buildup phase, the laser reaches steady state. Neglecting the spontaneous emission, saturated gain and steady state power can be calculated:

$$\frac{d}{dt}g = 0 \qquad \frac{d}{dt}P = 0$$

$$\longrightarrow \quad g_s = \frac{g_0}{1+\frac{P_s}{P_{sat}}} = l \qquad P_s = P_{sat}\left(\frac{g_0}{l}-1\right)$$



$$g_{th} = l, \qquad R_{p,th} = \frac{2lA_{eff}}{2^{*}\sigma\tau_L}$$

Pumping rate to reach threshold is proportional to the optical loss of the mode per roundtrip, the mode cross section, and inversely proportional to the $\sigma \cdot \tau_L$.

General description of laser operation

- Pumping begins when a laser is turned on, and the population inversion eventually reaches a steady-state value.
- This steady-state population inversion is determined by the pumping rate and the upper level lifetime, $R_p \tau_L$.
- This steady-state population inversion corresponds to the small signal gain g_0.
- As the gain exceeds the cavity losses, the laser intra-cavity power begins to grow until it eventually reaches the saturation power and begins to extract energy from the medium.
- As the intra-cavity power grows, stimulated emission reduces the population inversion, and consequently the inversion reaches a new, lower steady-state value such that the reduced gain equals the losses in the cavity: $$g_s = \frac{g_0}{1 + \frac{P_s}{P_{sat}}} = l$$

Stability and relaxation oscillations

How does the laser reach steady state, once a perturbation occurs?

$$\begin{aligned} P &= P_s + \Delta P \\ g &= g_s + \Delta g \end{aligned} \Rightarrow \begin{aligned} \frac{d\Delta P}{dt} &= +2\frac{P_s}{T_R}\Delta g \\ \frac{d\Delta g}{dt} &= -\frac{g_s}{E_{sat}}\Delta P - \frac{1}{\tau_{stim}}\Delta g \end{aligned}$$

Stimulated lifetime

$$\frac{1}{\tau_{stim}} = \frac{1}{\tau_L}\left(1 + \frac{P_s}{Psat}\right)$$

The perturbations decay or grow like

$$\begin{pmatrix} \Delta P \\ \Delta g \end{pmatrix} = \begin{pmatrix} \Delta P_0 \\ \Delta g_0 \end{pmatrix} e^{st} \Rightarrow A \begin{pmatrix} \Delta P_0 \\ \Delta g_0 \end{pmatrix} = \begin{pmatrix} -s & 2\frac{P_s}{T_R} \\ -\frac{T_R}{E_{sat}2\tau_p} & -\frac{1}{\tau_{stim}} - s \end{pmatrix} \begin{pmatrix} \Delta P_0 \\ \Delta g_0 \end{pmatrix} = 0$$

Non-zero solution exists only if the determinant of the coefficient matrix is 0:

$$s\left(\frac{1}{\tau_{stim}} + s\right) + \frac{P_s}{E_{sat}\tau_p} = 0$$

$$\boxed{s_{1/2} = -\frac{1}{2\tau_{stim}} \pm \sqrt{\left(\frac{1}{2\tau_{stim}}\right)^2 - \frac{P_s}{E_{sat}\tau_p}}}$$

Stability and relaxation oscillations

Introducing the pump parameter $r = 1 + \frac{P_s}{P_{sat}} = \frac{g_0}{l}$, which tells us how much we pump the laser over threshold, the eigen frequencies can be rewritten as

$$\boxed{s_{1/2} = -\frac{1}{2\tau_{stim}}\left(1 \pm j\sqrt{\frac{4(r-1)}{r}\frac{\tau_{stim}}{\tau_p} - 1}\right) = -\frac{r}{2\tau_L} \pm j\sqrt{\frac{(r-1)}{\tau_L \tau_p} - \left(\frac{r}{2\tau_L}\right)^2}}$$

(i): The stationary state $(0, g_0)$ for $g_0 < l$ and (P_s, g_s) for $g_0 > l$ are always stable, i.e. $\mathrm{Re}\{s_i\} < 0$.

(ii): For lasers pumped above threshold, $r > 1$, and long upper state lifetimes, i.e. $\frac{r}{4\tau_L} < \frac{1}{\tau_p}$, the relaxation rate becomes complex, i.e. there are relaxation oscillations

$$s_{1/2} = -\frac{1}{2\tau_{stim}} \pm j\omega_R \qquad \omega_R = \sqrt{\frac{1}{\tau_{stim}\tau_p}} \qquad \tau_{stim} = \frac{\tau_L}{r}$$

If the laser can be pumped strong enough, i.e. r can be made large enough so that the stimulated lifetime becomes as short as the cavity decay time, relaxation oscillations vanish.

Relaxation oscillations: a case study

Diode-pumped Nd:YAG-Laser:

$$\lambda_0 = 1064\ \mathrm{nm},\ \sigma = 4 \cdot 10^{-20} cm^2,\ A_{eff} = \pi\,(100\mu m \times 150\mu m)$$

$$r = 50 \quad \tau_L = 1.2\ ms,\ l = 1\%,\ T_R = 10ns$$

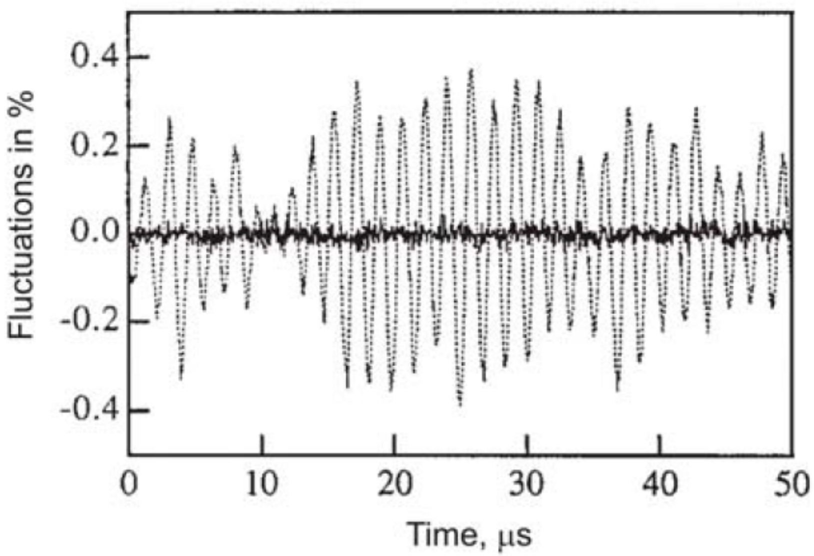
$$I_{sat} = \frac{hf_L}{\sigma\tau_L} = 3.9\frac{kW}{cm^2},\ P_s = 91.5W$$

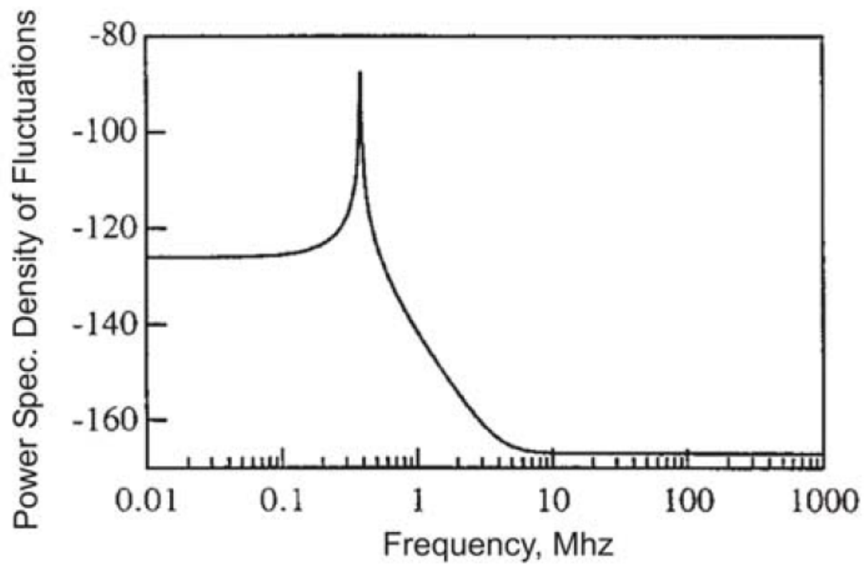
$$P_{sat} = I_{sat}A_{eff} = 1.8\ W,$$

$$\tau_{stim} = \frac{\tau_L}{r} = 24\mu s,\ \tau_p = 1\mu s$$

$$\omega_R = \sqrt{\frac{1}{\tau_{stim}\tau_p}} = 2 \cdot 10^5 s^{-1}$$

The physical reason for relaxation oscillations and later instabilities is, that the gain reacts too slow on the light field, i.e. the stimulated lifetime is long in comparison with the cavity decay time.





Typically observed relaxation oscillations in time and frequency domain.

Laser efficiency: how much pump power converted to laser output power

Steady-state intracavity power:

$$P_s = P_{sat}\left(\frac{2g_0}{2l} - 1\right)$$

$$2g_0 = 2^* \frac{R_p}{A_{eff}}\sigma\tau_L,$$

$$P_{sat} = \frac{hf_L}{2^*\sigma\tau_L}A_{eff}$$

Laser power losses include the internal losses $2l_{int}$ and the transmission T through the output coupling mirror: $2l = 2l_{int} + T$

Pump photon energy

Laser output power: $P_{out} = T \cdot P_{sat}\left(\frac{2g_0}{2l_{int}+T} - 1\right)$

Pump power: $P_p = R_p h f_P$

Efficiency: $\eta = \frac{P_{out}}{P_p}$

Differential Efficiency: $\eta_D = \frac{\partial P_{out}}{\partial P_p}$

If the laser is pumped many times over threshold: $r = 2g_0/2l \to \infty$

$$\eta_D = \eta = \frac{T}{2l_{int}+T}P_{sat}\frac{2^*}{A_{eff}hf_P}\sigma\tau_L = \frac{T}{2l_{int}+T}\cdot\frac{hf_L}{hf_P}$$

Laser efficiency is fundamentally limited by the ratio of output coupling to total losses and the quantum defect in pumping.

How to efficiently use the energy storage capability?

- Typical cavity length of a solid-state laser: 0.1-10 m
- Cavity round-trip time: ~1-100 ns
- Photon lifetime in the cavity $\tau_p = T_R / 2l$: ~0.1-1 us
- Upper level lifetime of solid-state gain materials: 10-1000 us

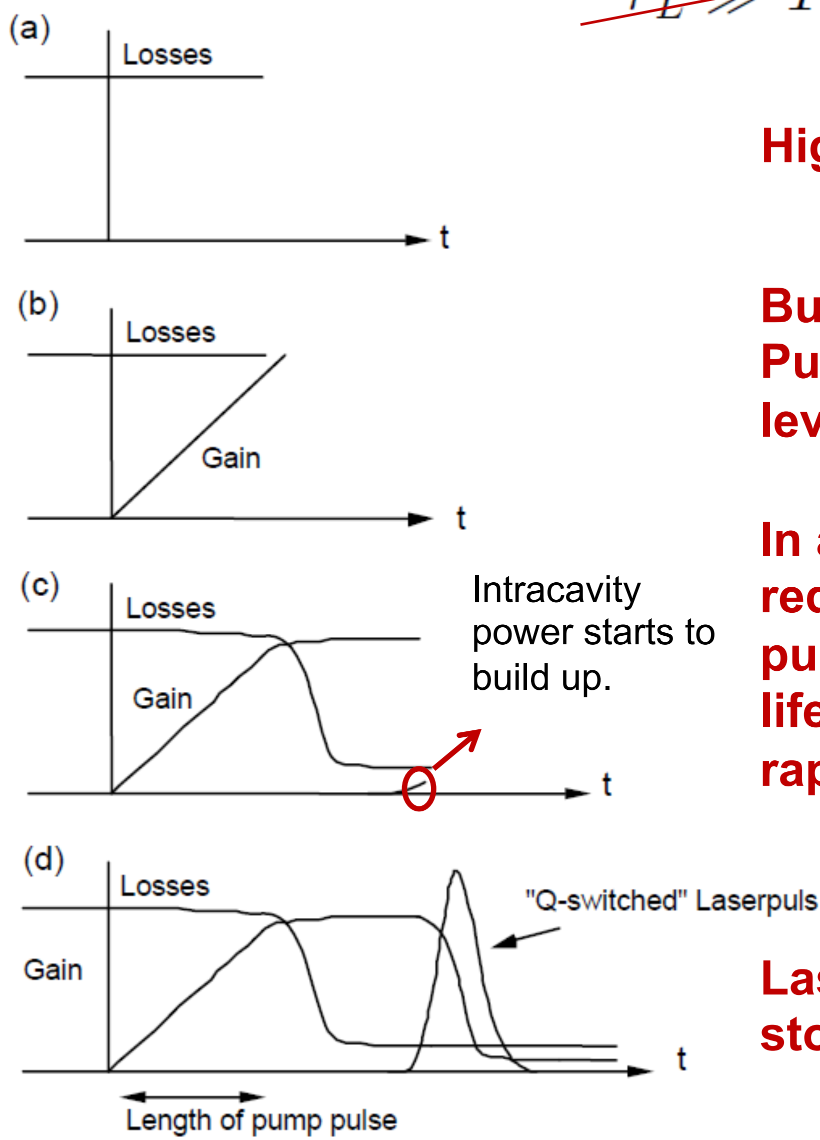
$$\tau_L >> \tau_p >> T_R$$

- Given pumping rate, τ_L time is needed such that the full energy storage capability is reached.
- However, the intra-cavity power starts to build up as the gain exceeds loss and it grows much faster such that it reaches the saturation power and starts to saturate population inversion. So the full energy storage capability cannot be reached.

How to solve this dilemma?

Q-Switching: manage cavity loss to obtain giant pulses

$$\cancel{\tau_L \gg T_R \gg \tau_p} \qquad \tau_L >> \tau_p >> T_R \qquad (4.137)$$



High losses, laser is below threshold.

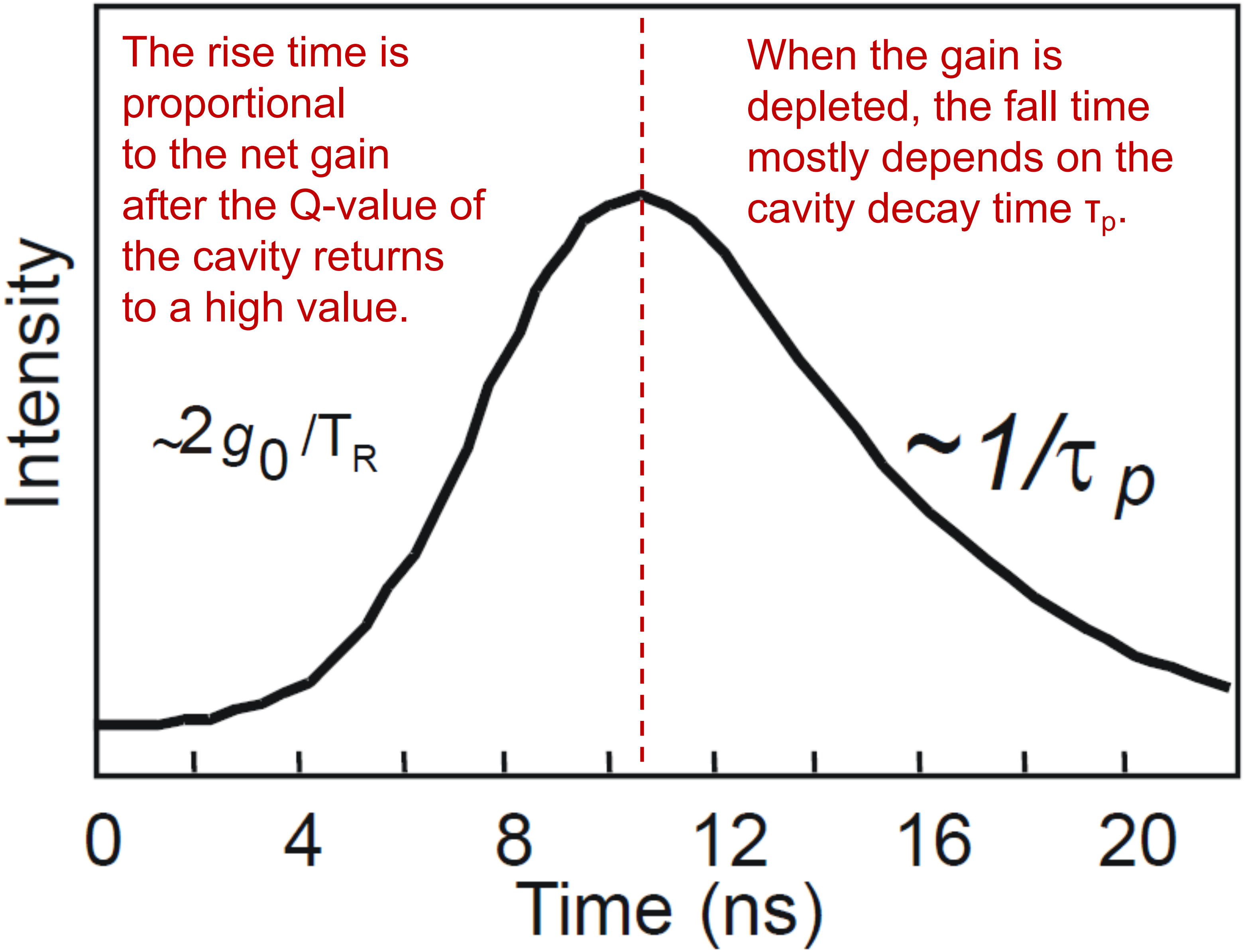
Build-up of inversion by pumping. Pump population to upper laser level until gain approaches loss.

In active Q-switching, the losses are reduced, after the laser medium is pumped for as long as the upper state lifetime. Then the loss is reduced rapidly and laser oscillation starts.

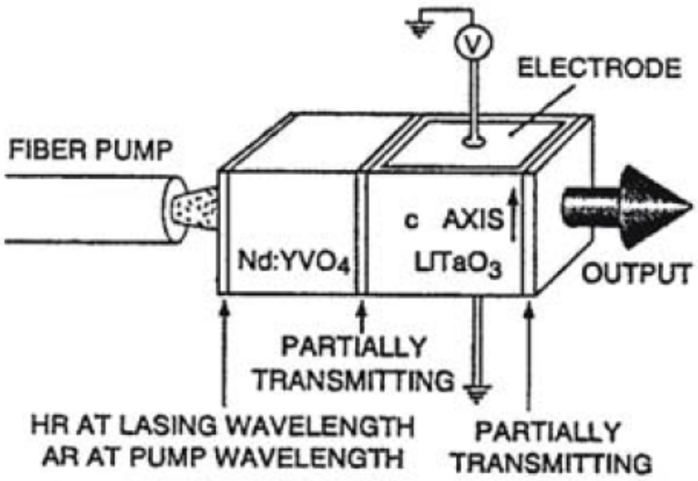
Laser emission stops after the energy stored in the gain medium is extracted.

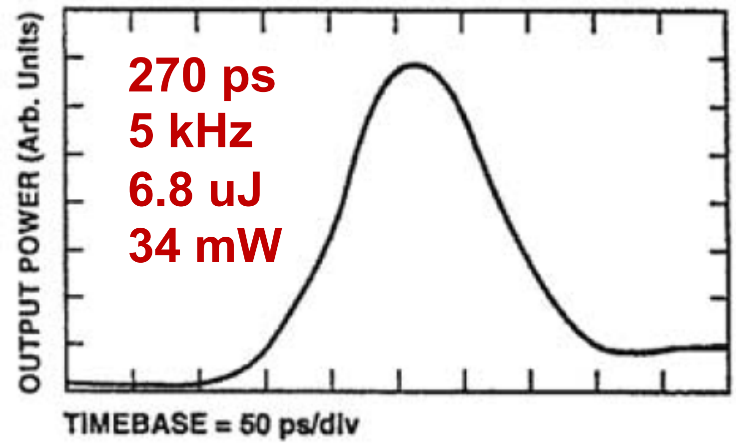
Gain and Loss Dynamics of Q-switched Laser

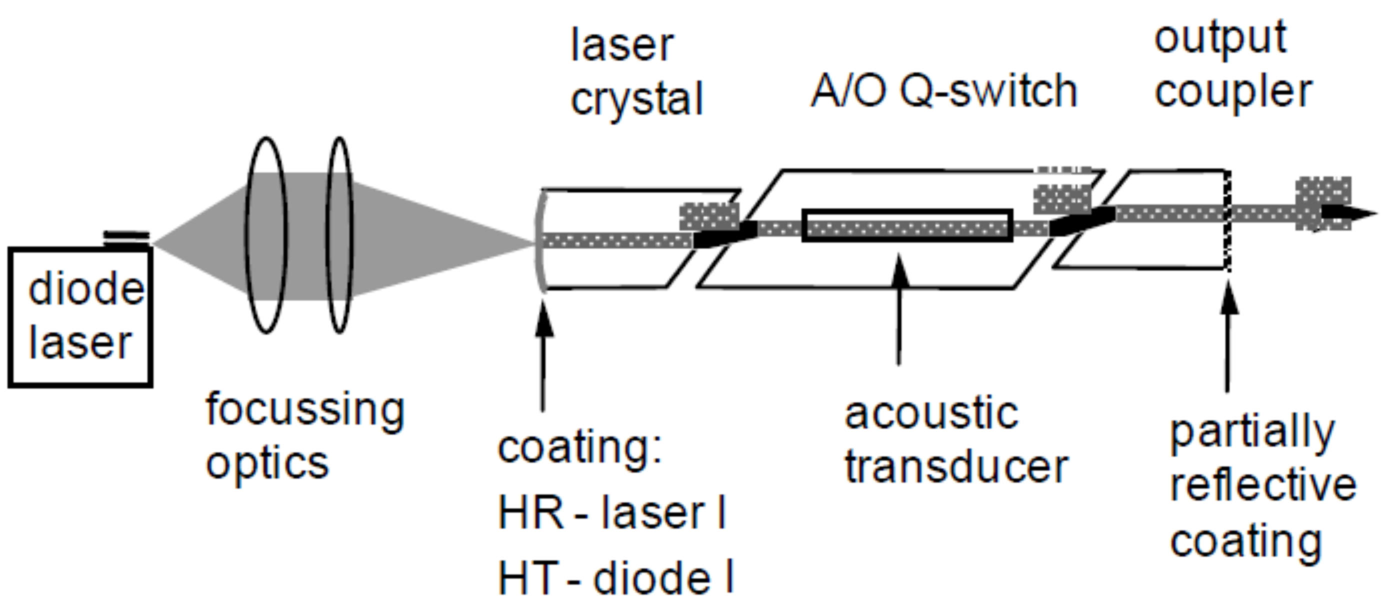
Asymmetric actively Q-switched pulse



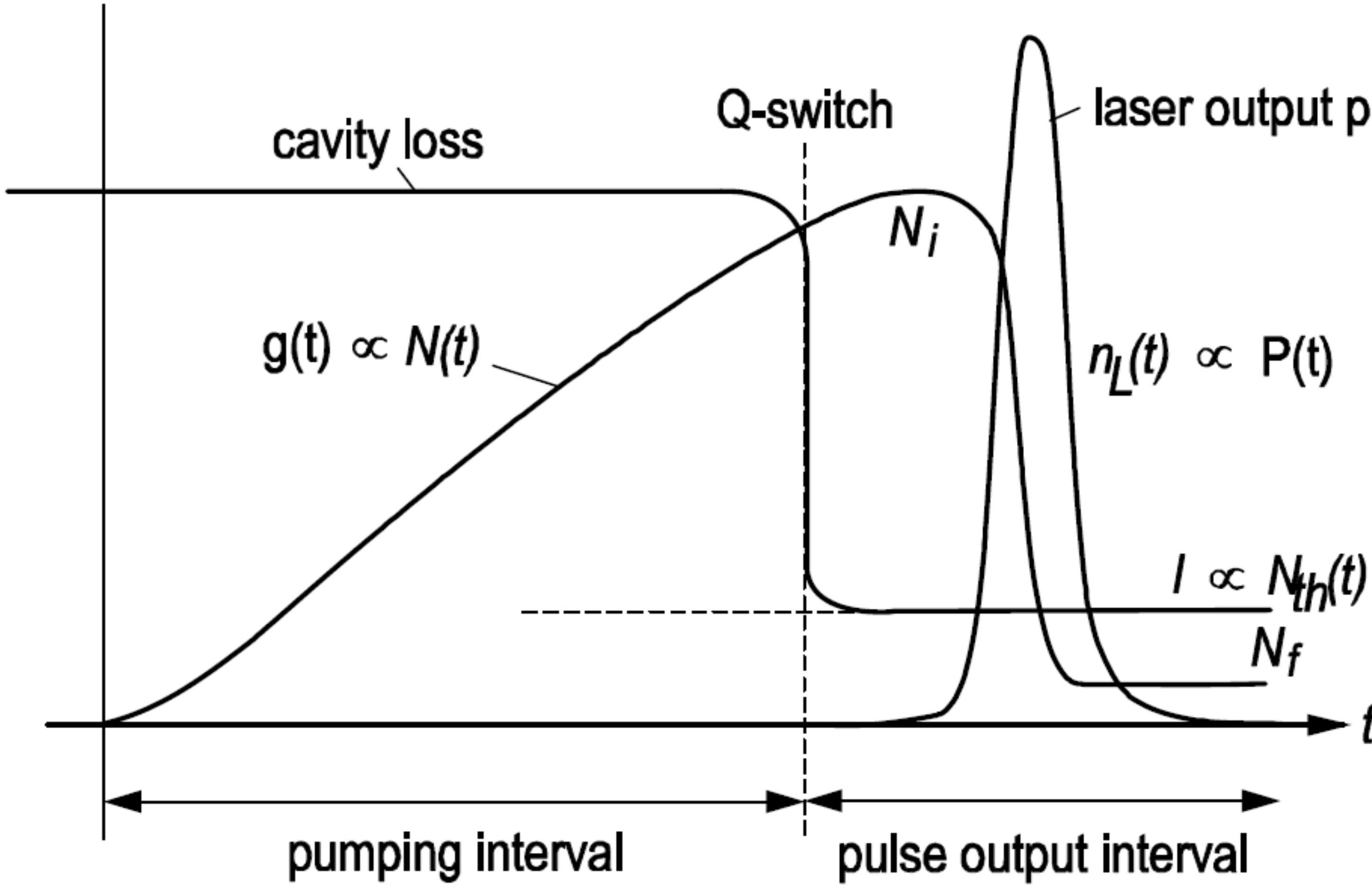
Active Q-switching lasers: EO switch and AO switch







Theory on active Q-switching



Active Q-switching dynamics assuming an instantaneous switching

Theory on active Q-switching

Rate equations:

$$\frac{d}{dt}g = -\frac{g-g_0}{\tau_L} - \frac{gP}{E_{sat}} \qquad \frac{d}{dt}P = -\frac{1}{\tau_p}P + \frac{2g}{T_R}(P+P_{vac})$$

Pump interval with constant *R_p*:

$$\frac{d}{dt}g = -\frac{g-g_0}{\tau_L} \longrightarrow g(t) = g_0(1-e^{-t/\tau_L})$$

Pulse built-up phase:

Index is left away since there is only an upper state population.

From Eqn 4.101 and 4.105, we know the initial gain:

$$g_i = hf_L N_{2i}/(2E_{sat}) = hf_L N_i/(2E_{sat})$$

Assume that during pulse buildup, stimulated emission rate is the dominant term changing the inversion:

$$\begin{aligned} \frac{d}{dt}g &= -\frac{gP}{E_{sat}} \\ \frac{d}{dt}P &= \frac{2(g-l)}{T_R}P \end{aligned} \longrightarrow \frac{dP}{dg} = \frac{2E_{sat}}{T_R}\left(\frac{l}{g}-1\right)$$

Theory on active Q-switching

Initial conditions: $g(t = 0) = g_i = r \cdot l$ $\qquad P(t=0)=0$

How many times the laser is pumped above the threshold after the Q-switch is operated.

Intra-cavity loss after the Q-switch is operated.

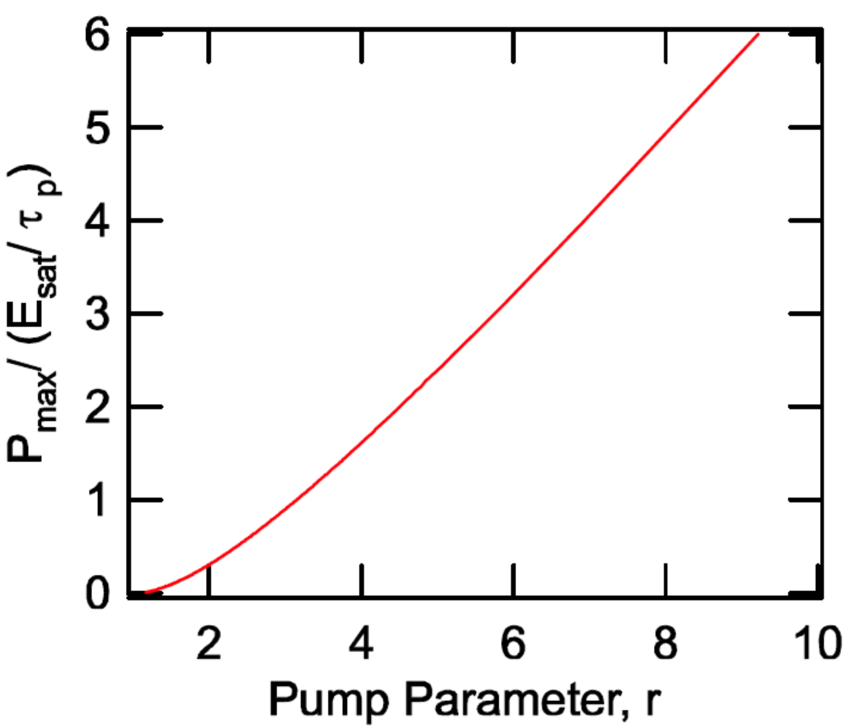
Intra-cavity power evolution:

$$P(t) = \frac{2E_{sat}}{T_R}\left(g_i - g(t) + l\ln\frac{g(t)}{g_i}\right)$$

Maximum power as gain equals loss:

$$P_{\max} = \frac{2lE_{sat}}{T_R}(r - 1 - \ln r)$$

$$= \frac{E_{sat}}{\tau_p}(r - 1 - \ln r)$$



Energy extraction

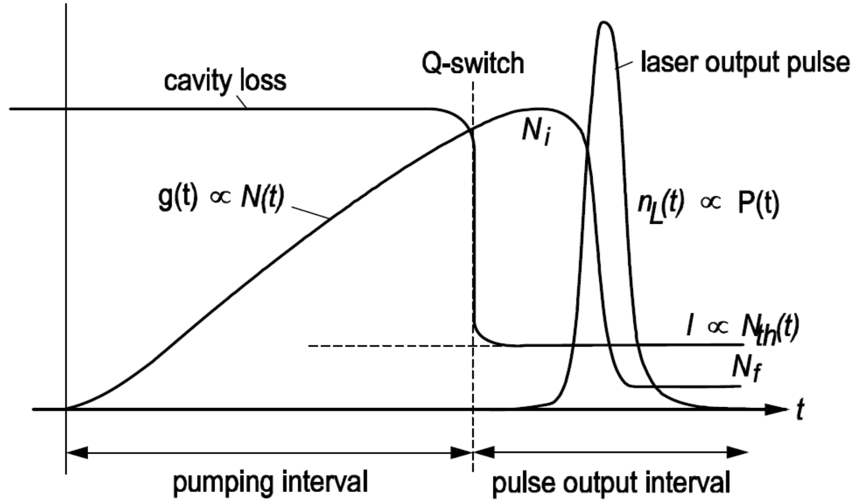
Final gain when power vanishes:

$$\left(g_i - g_f + l \ln\left(\frac{g_f}{g_i}\right)\right) = 0 \qquad r = g_i/l$$

$$\downarrow$$

$$1 - \frac{g_f}{g_i} + \frac{1}{r}\ln\left(\frac{g_f}{g_i}\right) = 0$$

$$1 - \frac{N_f}{N_i} + \frac{1}{r}\ln\left(\frac{N_f}{N_i}\right) = 0$$

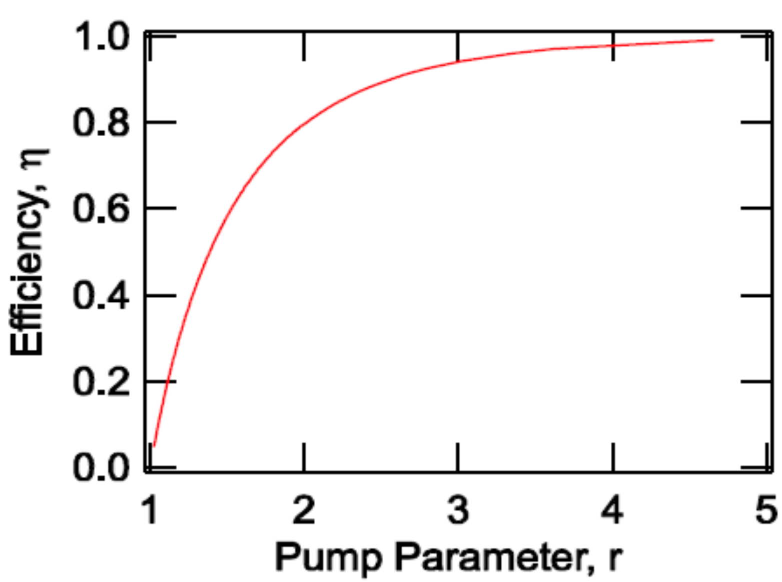


Assuming no internal losses, the pulse energy is: $E_P = (N_i - N_f)\, h f_L$

Energy extraction efficiency: $$\eta = \frac{N_i - N_f}{N_i}$$

$$\eta + \frac{1}{r}\ln(1-\eta) = 0$$

Energy extraction efficiency only depends on the pump parameter r.



Estimate of pulse width

We can estimate the pulse width of the emitted pulse by the ratio between pulse energy and peak power.

Emitted pulse energy can be written as $E_P = \eta(r) N_i h f_L$

Emitted pulse peak power: $P_{\max} = \frac{E_{sat}}{\tau_p}(r - 1 - \ln r)$

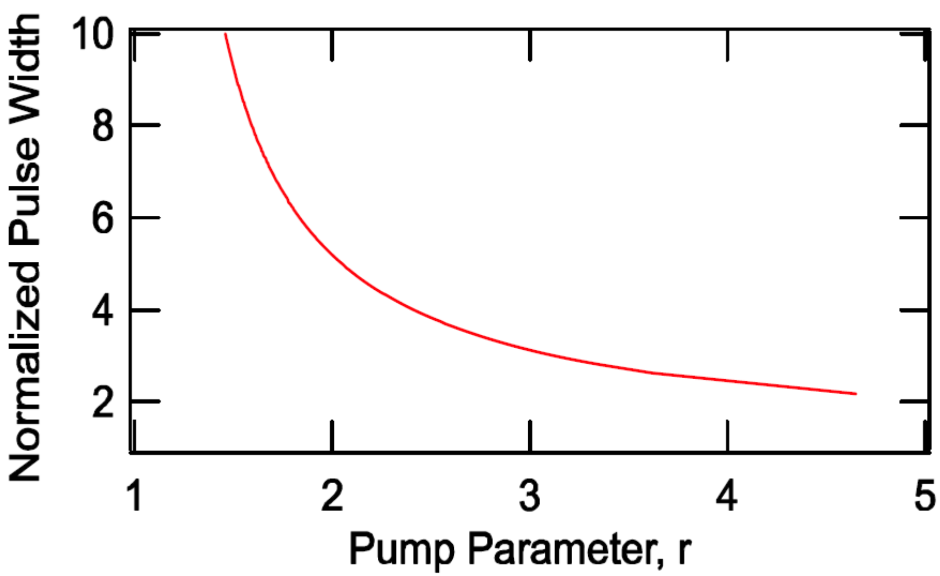
$$\tau_{Pulse} = \frac{E_P}{2l P_{peak}}$$

$$= \tau_p \frac{\eta(r)}{(r - 1 - \ln r)} \frac{N_i h f_L}{2l E_{sat}}$$

$$\boxed{g_i = h f_L N_i / (2E_{sat})}$$

$$= \tau_p \frac{\eta(r)}{(r - 1 - \ln r)} \frac{g_i}{l}$$

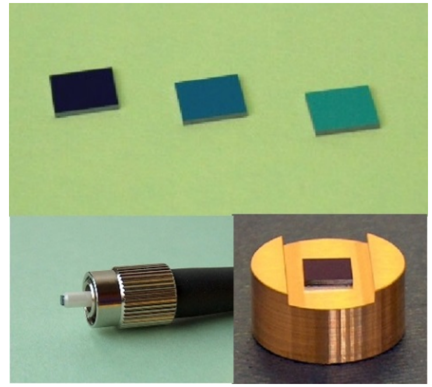
$$= \tau_p \frac{\eta(r) \cdot r}{(r - 1 - \ln r)}$$

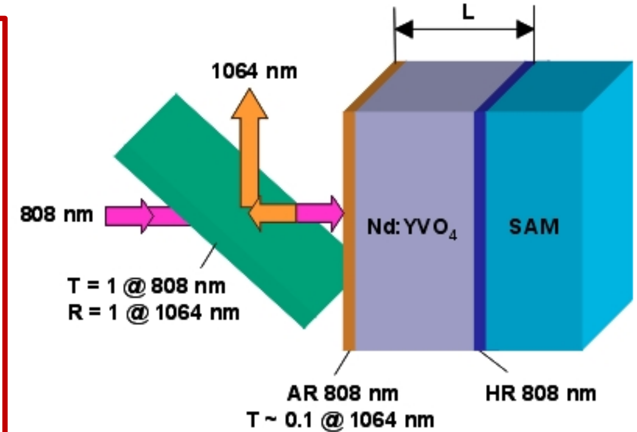


Passive Q-Switching: manage cavity loss using saturable absorber

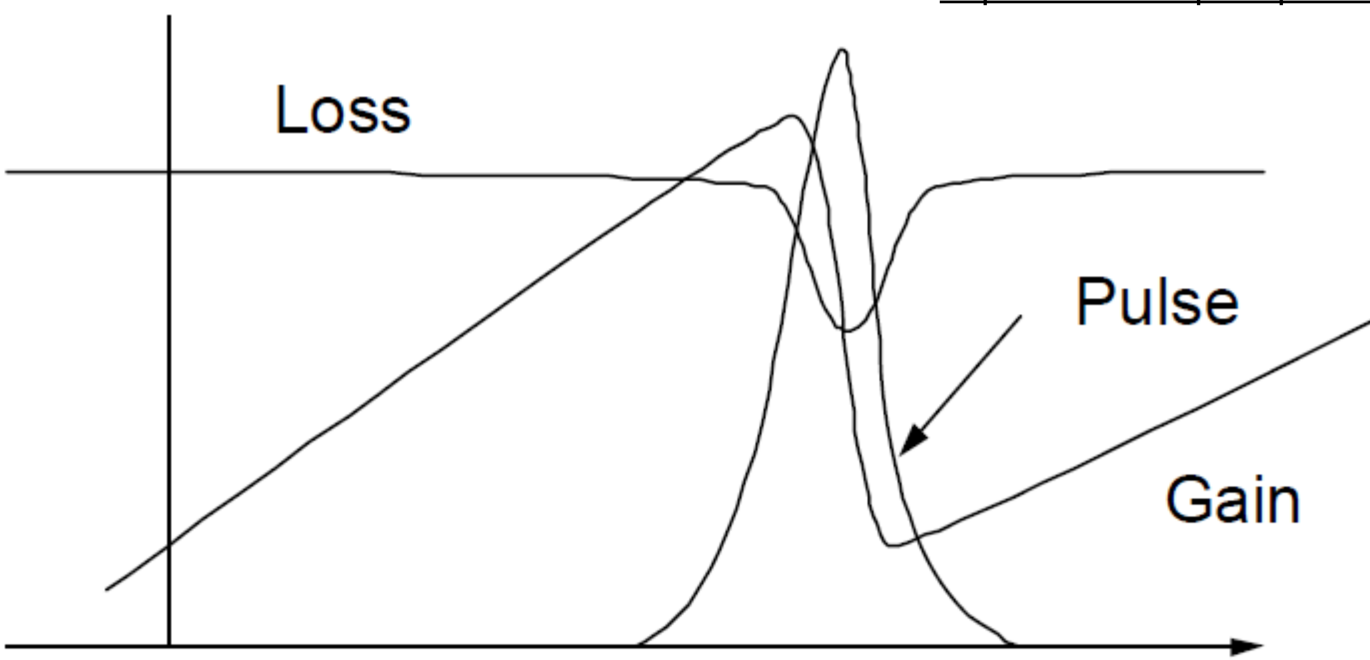
Saturable absorber: an optical passive device, which introduces large loss for low optical intensities and small loss at high optical intensities.

$$T_R \frac{dq}{dt} = -\frac{q - q_0}{T_A} - \frac{q T_R P}{E_A}$$

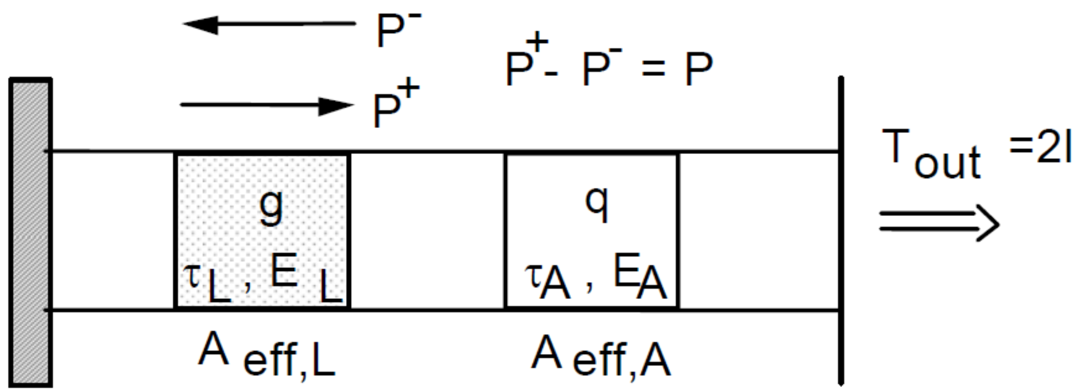




http://www.batop.de/products/products.html



Modeling of passively Q-switched laser



We assume small output coupling so that the laser power within one roundtrip can be considered position independent.

Rate equations for a passively Q-switched laser

Assume that the changes in the laser intensity, gain and saturable absorption are small on a time scale on the order of the round-trip time T_R in the cavity, (i.e. less than 20%).

$$
\begin{aligned}
T_R \frac{dP}{dt} &= 2(g - l - q)P \\
T_R \frac{dg}{dt} &= -\frac{g - g_0}{T_L} - \frac{g T_R P}{E_L} \\
T_R \frac{dq}{dt} &= -\frac{q - q_0}{T_A} - \frac{q T_R P}{E_A}
\end{aligned}
$$

Normalized upper-state lifetime of the gain medium and the absorber recovery time

$$
\begin{aligned}
T_L &= \tau_L / T_R \\
T_A &= \tau_A / T_R
\end{aligned}
$$

Saturation energies of the gain and the absorber

$$
\begin{aligned}
E_L &= h\nu A_{eff,L} / 2^* \sigma_L \\
E_A &= h\nu A_{eff,A} / 2^* \sigma_A
\end{aligned}
$$

Passively Q-switched laser: fast saturable absorber

Typical solid-state lasers:

$\tau_L = 100\ \mu s$ $\quad T_R = 10\ ns$ $\quad \tau_A = 1\text{-}100\ ps$

$T_L \approx 10^4 \quad T_A \approx 10^{-4}$ to 10^{-2}

Fast Saturable Absorber: $T_A << T_L$, the absorber will follow the instantaneous laser power:

$$T_R \frac{dq}{dt} = -\frac{q - q_0}{T_A} - \frac{q T_R P}{E_A} \xrightarrow{\text{Adiabatic solution}} q = \frac{q_0}{1 + P/P_A} \quad \text{with} \quad P_A = \frac{E_A}{\tau_A}$$

saturation power

New equations of motion:

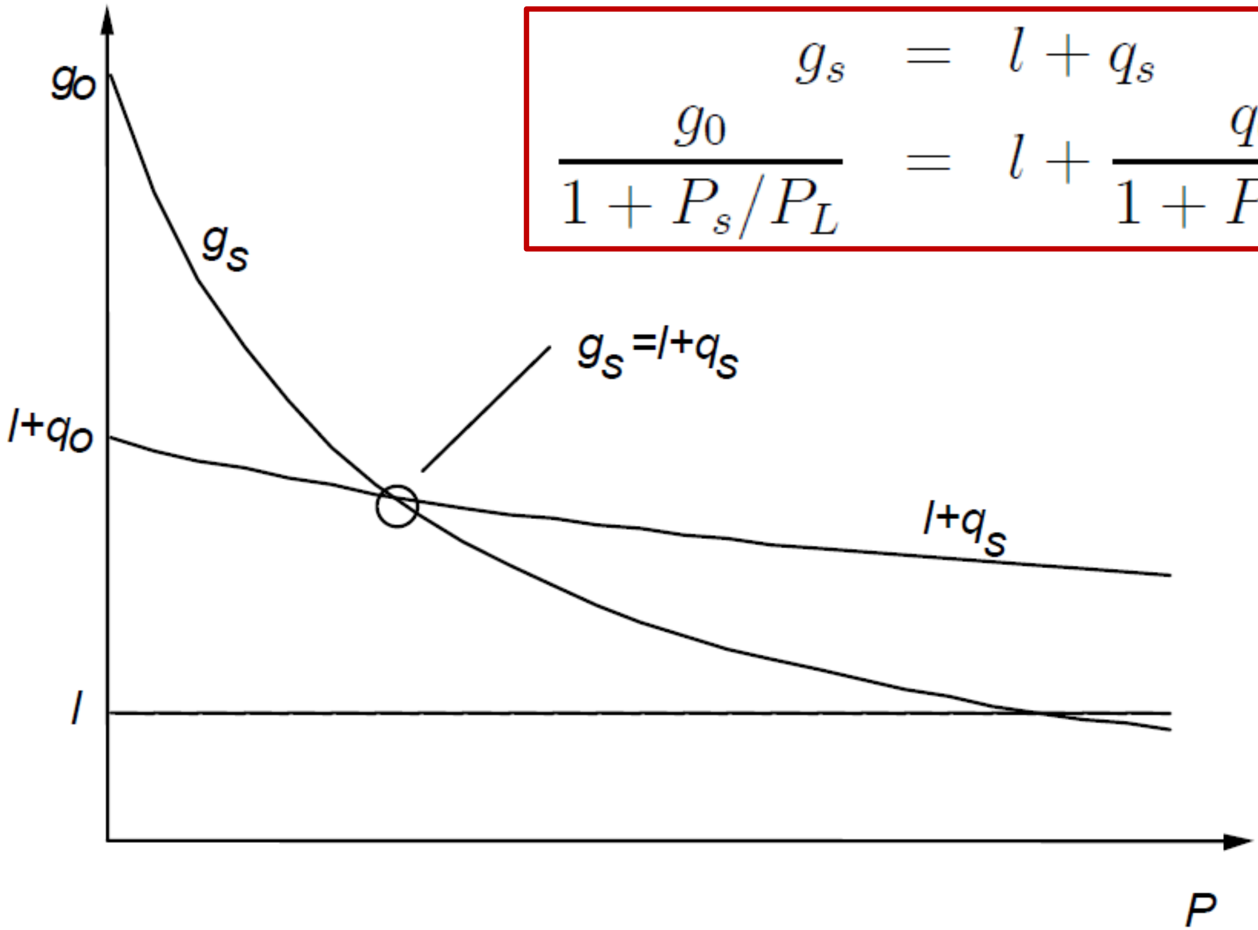
$$T_R \frac{dP}{dt} = 2(g - l - q(P))P$$

$$T_R \frac{dg}{dt} = -\frac{g - g_0}{T_L} - \frac{g T_R P}{E_L}$$

Passively Q-switched laser: stationary solution

As in the case for the cw-running laser the stationary operation point of the laser is determined by the point of zero net gain:

$$\begin{aligned} g_s &= l + q_s \\ \frac{g_0}{1 + P_s/P_L} &= l + \frac{q_0}{1 + P_s/P_A} \end{aligned}$$



Graphical solution of the stationary operating point

Stability of stationary operating point: Passive Q-switching

To find the stability criterion, we linearize the system just as we have done for laser CW operation:

$$T_R \frac{d}{dt}\begin{pmatrix} \Delta P_0 \\ \Delta g_0 \end{pmatrix} = A \begin{pmatrix} \Delta P_0 \\ \Delta g_0 \end{pmatrix}, \text{ with } A = \begin{pmatrix} -2\left.\frac{dq}{dP}\right|_{cw} P_s & 2P_s \\ -\frac{g_s T_R}{E_L} & -\frac{T_R}{\tau_{stim}} \end{pmatrix}$$

We look for the eigen solution:

$$\frac{d}{dt}\begin{pmatrix} \Delta P_0(t) \\ \Delta g_0(t) \end{pmatrix} = s \begin{pmatrix} \Delta P_0(t) \\ \Delta g_0(t) \end{pmatrix}$$

$$\boxed{s = \frac{1}{2}\left(\gamma_Q - \frac{1}{\tau_{stim}}\right) \pm j\sqrt{\gamma_Q \frac{r}{\tau_L} + \frac{r-1}{\tau_p \tau_L} - \left(\frac{\gamma_Q - \frac{1}{\tau_{stim}}}{2}\right)^2}}$$

Growth rate introduced by the saturable absorber that destabilizes the laser relaxation oscillation:

$$\gamma_Q = -\frac{2}{T_R}\left.\frac{dq}{dP}\right|_{cw} P_s$$

Q-switching happens when $\gamma_Q > \frac{1}{\tau_{stim}}$

Passive Q-switching: a numerical example

τ_L=250μs, T_R=4ns, 2l=0.1, $2q_o$=0.005, $2g_o$=2, P_L/P_A=100.

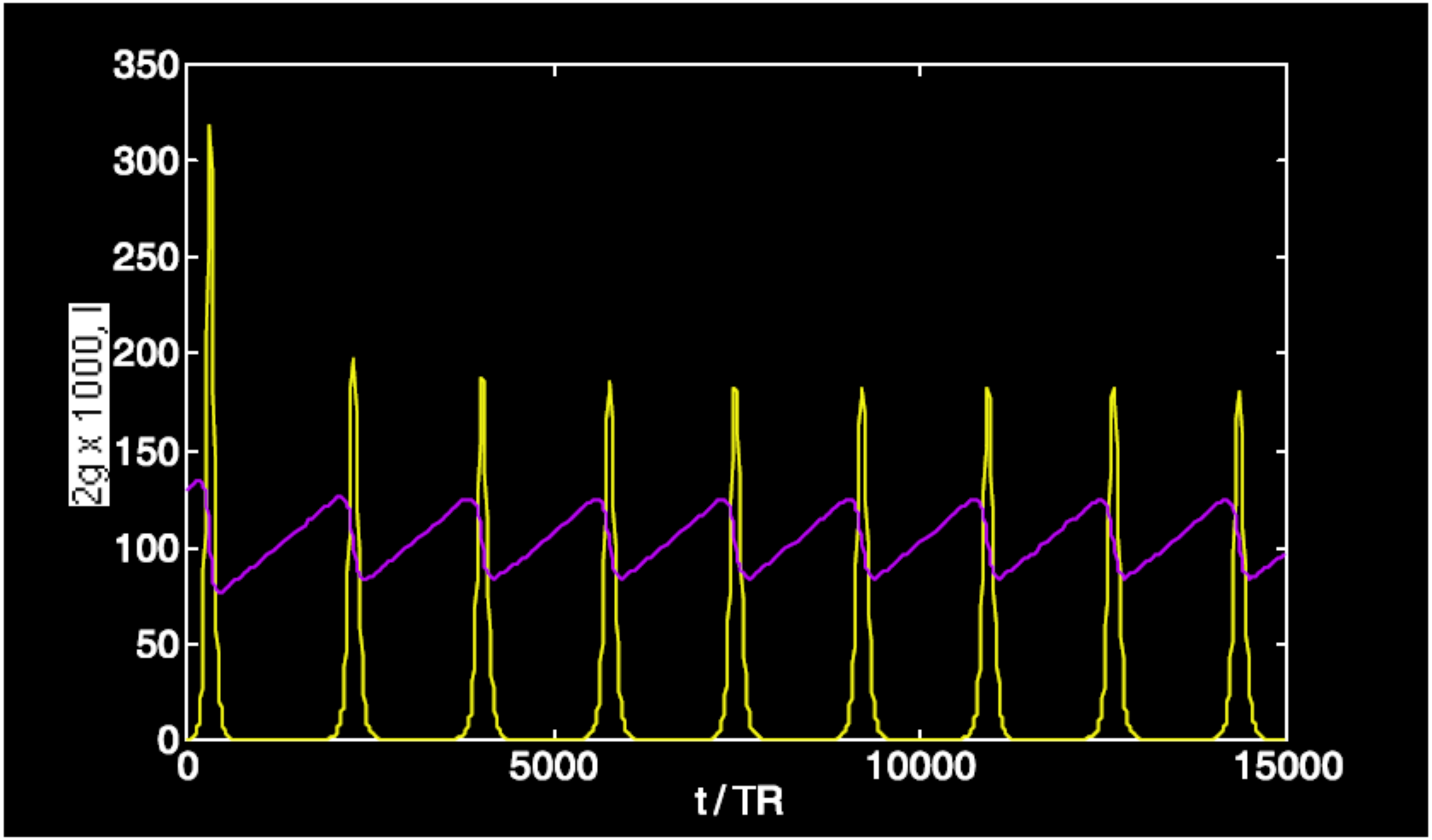
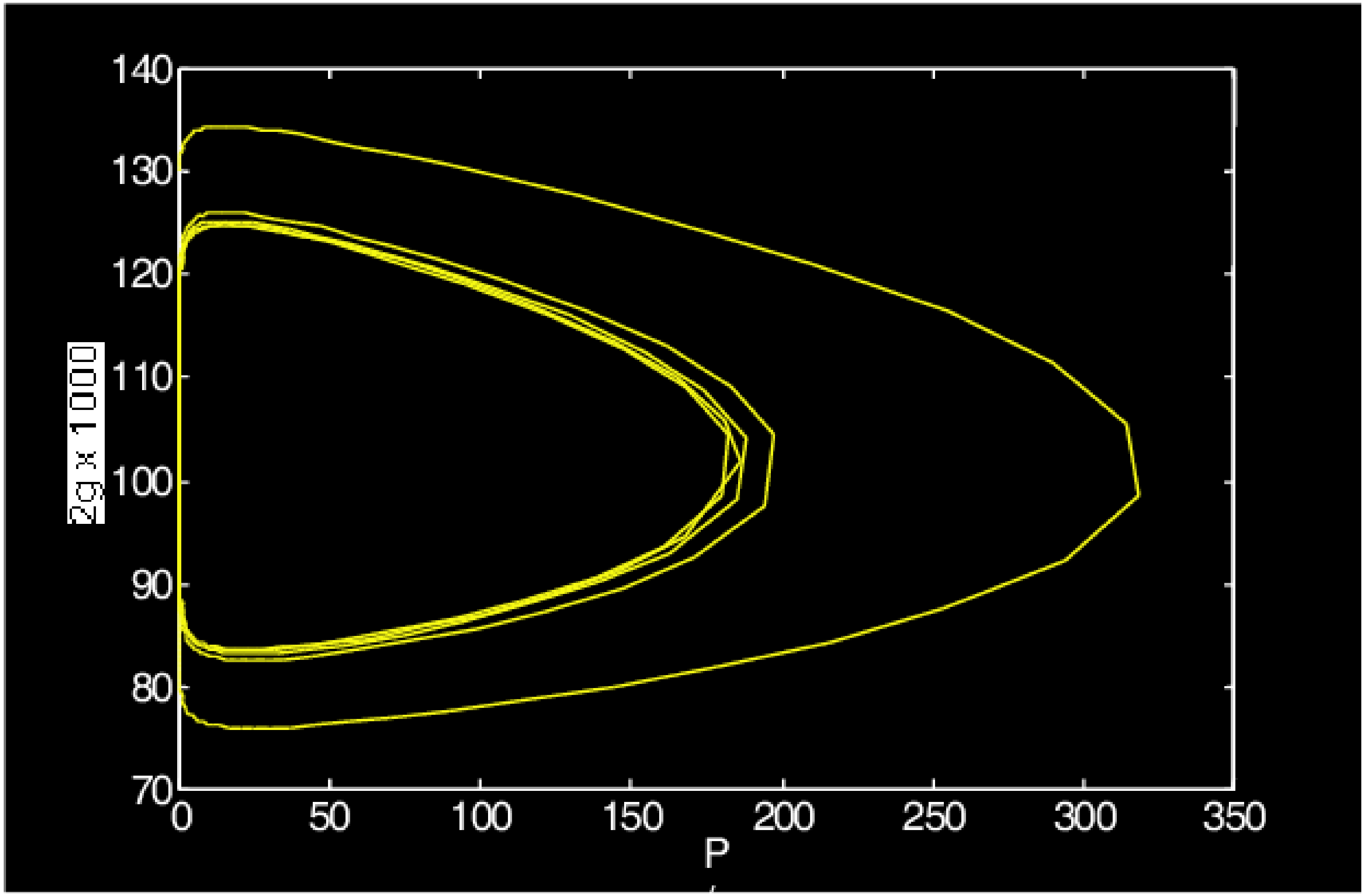


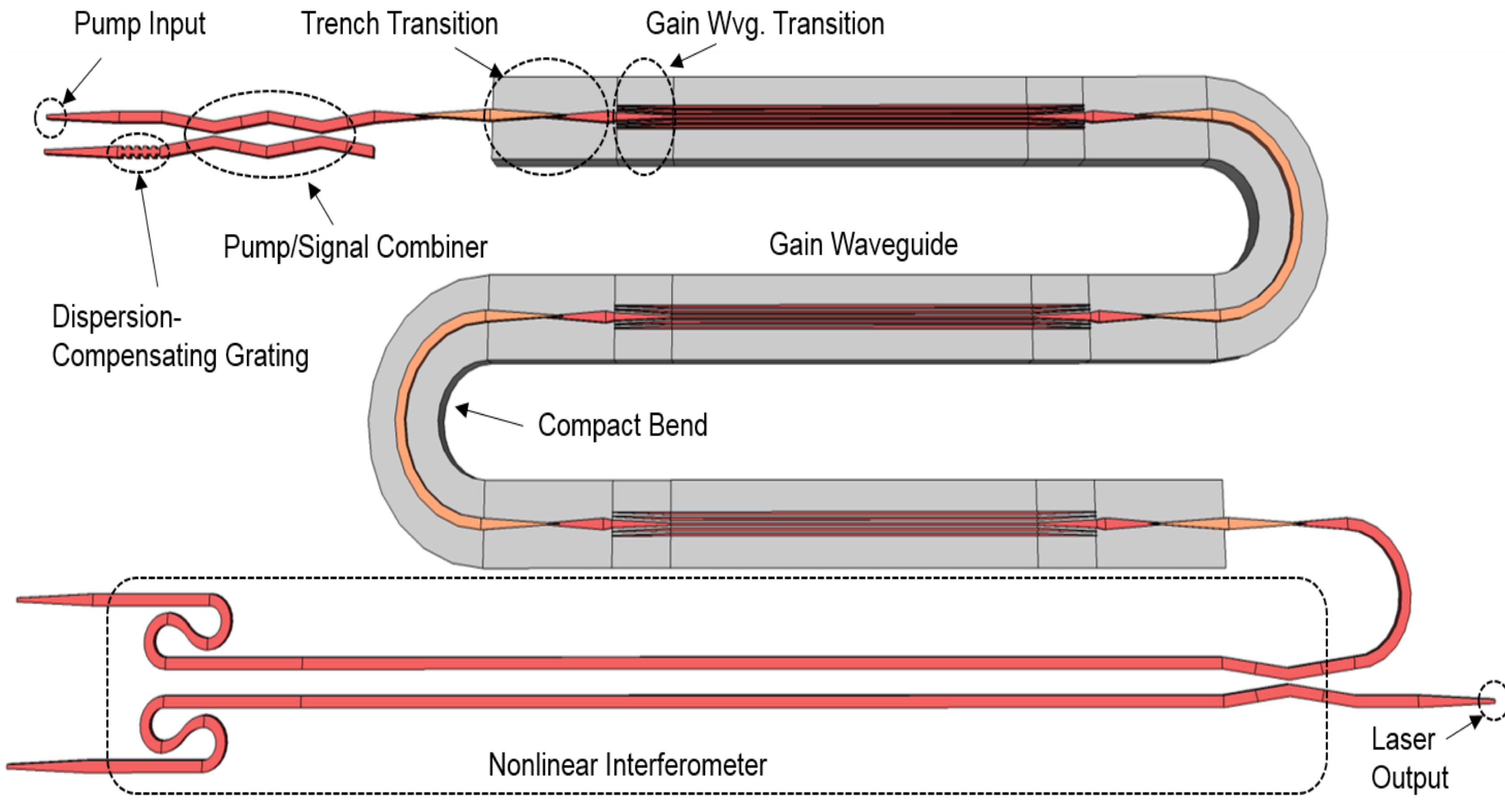
Fig. 4.28: Gain and output power as a function of time.



τ_L=250μs, T_R=4ns, 2l=0.1, $2q_o$=0.005, $2g_o$=2, P_L/P_A=100.

Fig. 4.27: Phase space solution for rate equations.

Unintentional Q-Switching of Integrated Rare-Earth Doped Mode-Locked Lasers on a CMOS Platform



K. Shtyrkova, et al, "Integerated CMOS-compatible Q-switched mode-locked lasers at 1900 nm with an on-chop artificial saturable absorber," Optics Express **27:**(3) 3542-3556 (2019).

Unintentional Q-Switching of Integrated Rare-Earth Doped Mode-Locked Lasers on a CMOS Platform