Ultrafast Optical Physics II (SoSe 2019)
Lecture 2, April 12

(1) Susceptibility and Sellmeier equation

(2) Phase velocity and group velocity

(3) Linear pulse propagation and dispersion

Maxwell's Equations of isotropic and homogeneous media

Maxwell's Equations: Differential Form

Current due to free charges

Ampere's Law $$\vec{\nabla}\times\vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}, \qquad (2.1a)$$

Faraday's Law $$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}, \qquad (2.1b)$$

Gauss's Law $$\vec{\nabla}\cdot\vec{D} = \rho, \qquad (2.1c)$$

Free charge density

No magnetic charge $$\vec{\nabla}\cdot\vec{B} = 0. \qquad (2.1d)$$

Material Equations: Bring Life into Maxwell's Equations

$$\vec{D} = \epsilon_0\vec{E} + \vec{P}, \qquad (2.2a)$$

Polarization

$$\vec{B} = \mu_0\vec{H} + \vec{M}. \qquad (2.2b)$$

Magnetization

Derivation of wave equation

No free charges, No currents from free charges, Non magnetization

$$\left(\Delta - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = \mu_0\left(\cancel{\frac{\partial \vec{j}}{\partial t}} + \frac{\partial^2}{\partial t^2}\vec{P}\right) + \frac{\partial}{\partial t}\cancel{\nabla\times}\vec{M} + \nabla\left(\cancel{\nabla\cdot}\vec{E}\right). \qquad (2.4)$$

In the linear optics of isotropic media without free charges,

$$\nabla\cdot\vec{D} = 0 \longrightarrow \nabla\cdot\vec{E} = 0$$

Simplified wave equation:

$$\left(\Delta - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = \mu_0\frac{\partial^2}{\partial t^2}\vec{P}. \qquad (2.7)$$

Wave in vacuum **Source term**

Laplace operator: $\Delta = \vec{\nabla}^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$

Dielectric susceptibility and Helmholtz equation

$$\vec{P}(\vec{r},t) = \epsilon_0 \int dt' \, \chi(t-t') \, \vec{E}(\vec{r},t') \quad \Longrightarrow \quad \widetilde{\vec{P}}(\vec{r},\omega) = \epsilon_0 \tilde{\chi}(\omega) \widetilde{\vec{E}}(\vec{r},\omega)$$

$$\left(\Delta - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = \mu_0 \frac{\partial^2}{\partial t^2}\vec{P} \quad \Longrightarrow \quad \left(\Delta + \frac{\omega^2}{c_0^2}\right)\widetilde{\vec{E}}(\omega) = -\omega^2 \mu_0 \epsilon_0 \tilde{\chi}(\omega) \widetilde{\vec{E}}(\omega)$$

In a linear medium, dielectric susceptibility is independent of optical field

$$\left(\Delta + \frac{\omega^2}{c_0^2}(1+\tilde{\chi}(\omega))\right)\widetilde{\vec{E}}(\omega) = 0$$

$$1 + \chi(\omega) = n^2(\omega)$$

Can be complex

Refractive Index

Medium speed of light (dependent on frequency): $c(\omega) = c_0/\tilde{n}(\omega)$

Susceptibility calculated using Lorentz model

$$\underline{\tilde{\chi}}(\omega) = \tilde{\chi}_r(\omega) + \mathrm{j}\tilde{\chi}_i(\omega)$$

$$\omega_p = \left(\frac{Ne^2}{\varepsilon_0 m_0}\right)^{1/2}$$

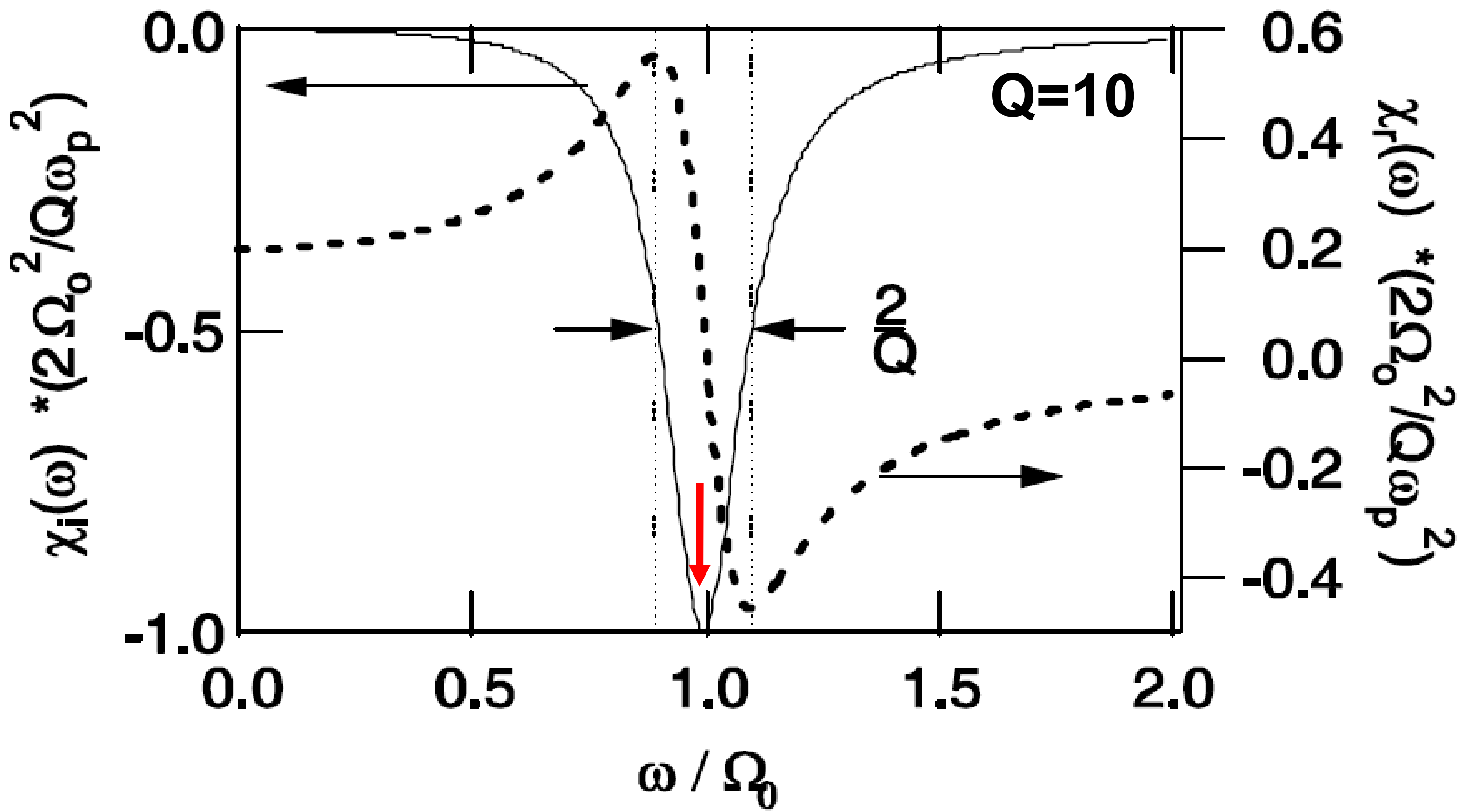
Plasma frequency

$$\tilde{\chi}_r(\omega) = \omega_p^2 \cdot \frac{\left(\Omega_0^2 - \omega^2\right)}{\left(\Omega_0^2 - \omega^2\right)^2 + \left(2\omega\frac{\Omega_0}{Q}\right)^2}$$

$$\tilde{\chi}_i(\omega) = -\omega_p^2 \cdot \frac{2\omega\frac{\Omega_0}{Q}}{\left(\Omega_0^2 - \omega^2\right)^2 + \left(2\omega\frac{\Omega_0}{Q}\right)^2}$$

Real and imaginary part of the susceptibility

$$\underline{\tilde{\chi}}(\omega) = \tilde{\chi}_r(\omega) + \mathrm{j}\tilde{\chi}_i(\omega)$$



Real part (dashed line) and imaginary part (solid line) of the susceptibility of the classical oscillator model for the dielectric polarizability

Real and imaginary part of the susceptibility

$$\underline{\tilde{\chi}}(\omega) = \tilde{\chi}_r(\omega) + \mathrm{j}\tilde{\chi}_i(\omega)$$

Example: EM-Wave polarized along x-axis and propagation along z-direction:

$$\underline{\vec{E}}(z,t) = \underline{E}_0 e^{j(\omega t - kz)}\,\vec{e}_x$$

$\sqrt{-1} = \mathrm{i}$	Physics notation
$\sqrt{-1} = \mathrm{j}$	Engineering notation

In general:

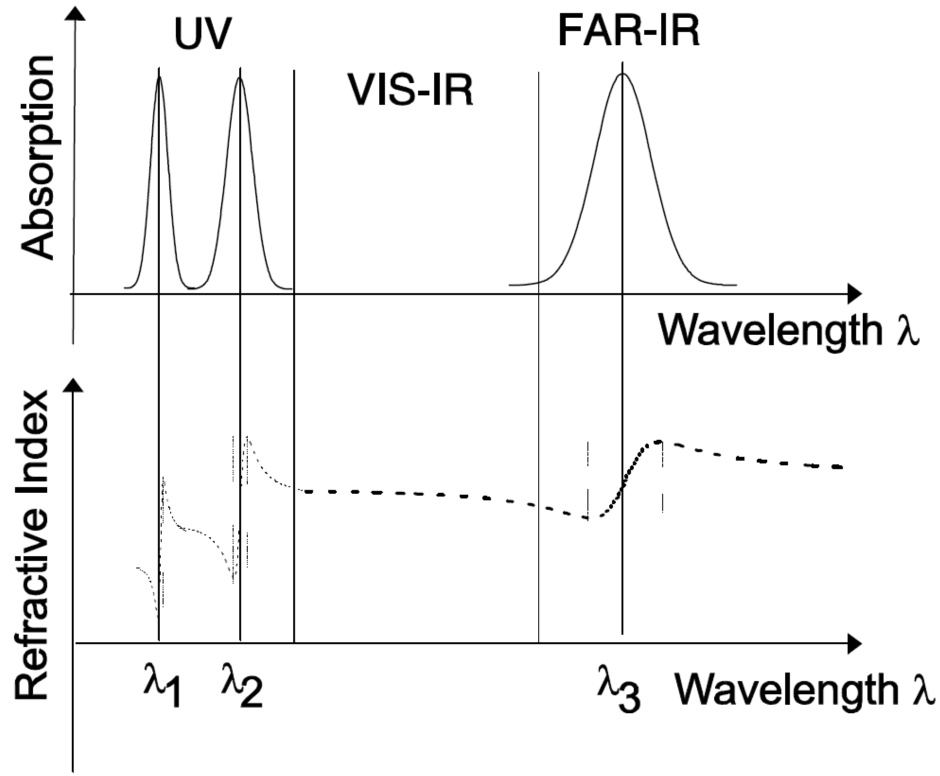
$$k(\omega) = \frac{\omega}{c_0}\tilde{n}(\omega) = \frac{\omega}{c_0}\sqrt{1+\tilde{\chi}(\omega)} = \frac{\omega}{c_0}\left(\tilde{n}_r(\omega) + j\tilde{n}_i(\omega)\right) = k_r(\omega) - j\alpha(\omega)$$

$$\boxed{\underline{\vec{E}}(z,t) = \underline{E}_0 e^{-\alpha\cdot z}\; e^{j(\omega t - k_r z)}\,\vec{e}_x}$$

Dispersion relation:

$$k_r(\omega) = \frac{\omega}{c_0} n_r(\omega)$$

Absorption and refractive index Vs. wavelength



$$\text{Classical Optics} \begin{cases} \frac{dn}{d\lambda} < 0 : \text{normal dispersion (blue refracts more than red)} \\ \frac{dn}{d\lambda} > 0 : \text{anomalous dispersion} \end{cases}$$

$$\text{Ultrafast Optics} \begin{cases} \frac{d^2n}{d\lambda^2} > 0 : \text{normal dispersion} \\ \text{short wavelengths slower than long wavelengths} \\ \frac{d^2n}{d\lambda^2} < 0 : \text{anomalous dispersion} \\ \text{short wavelengths faster than long wavelengths} \end{cases}$$

Sellmeier equations to model refractive index

If the frequency is far away from the absorption resonance $\left|\Omega_0^2 - \omega^2\right| >> 2\omega \frac{\Omega_0}{Q}$

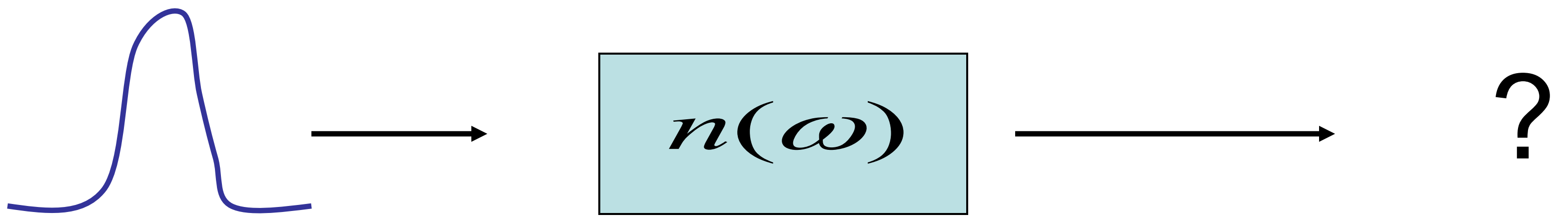
$$\tilde{\chi}_r(\omega) = \omega_p^2 \cdot \frac{\left(\Omega_0^2 - \omega^2\right)}{\left(\Omega_0^2 - \omega^2\right)^2 + \left(2\omega \frac{\Omega_0}{Q}\right)^2} \Rightarrow \tilde{\chi}_r(\omega) = \frac{\omega_p^2}{\Omega_0^2 - \omega^2}$$

Normally there are multiple resonant frequencies for the electronic oscillators. It means in general the refractive index will have the form

$$\boxed{n^2(\omega) = 1 + \sum_i A_i \frac{\omega_p^2}{\Omega_i^2 - \omega^2} = 1 + \sum_i a_i \frac{\lambda^2}{\lambda^2 - \lambda_i^2}}$$

	Fused Quartz	Sapphire
a_1	0.6961663	1.023798
a_2	0.4079426	1.058364
a_3	0.8974794	5.280792
λ_1^2	$4.679148 \cdot 10^{-3}$	$3.77588 \cdot 10^{-3}$
λ_2^2	$1.3512063 \cdot 10^{-2}$	$1.22544 \cdot 10^{-2}$
λ_3^2	$0.9793400 \cdot 10^{2}$	$3.213616 \cdot 10^{2}$

Linear propagation of a pulse



$$1+\chi(\omega)=n^2(\omega)$$

$$\left(\Delta+\frac{\omega^2}{c_0^2}(1+\tilde{\chi}(\omega))\right)\tilde{\vec{E}}(\omega)=0 \longrightarrow (\nabla^2+\frac{n^2(\omega)\omega^2}{c_0^2})E(\omega)=0$$

Neglecting diffraction (e.g. inside an optical waveguide)

$$\Delta=\frac{\partial^2}{\partial z^2}$$

Fourier transform
$$\left\{\begin{array}{l} E(z,t)=A(z,t)e^{j(\omega_0 t-k_0 z)} \\ E(z,\omega)=A(z,\omega-\omega_0)e^{-jk_0 z} \end{array}\right.$$

$$[\frac{d^2}{dz^2}+k^2(\omega)]E(z,\omega)=0$$

$$k(\omega)=\frac{n(\omega)\omega}{c_0}$$

$$\frac{d^2A}{dz^2}-2jk_0\frac{dA}{dz}+[k^2(\omega)-k_0^2]A=0$$

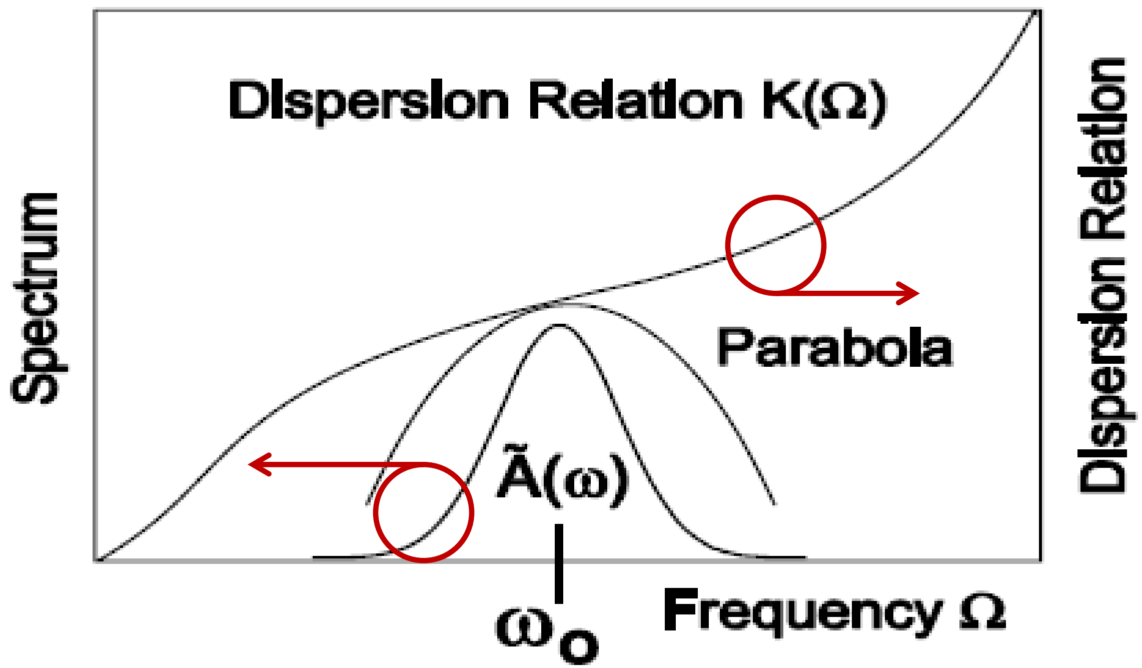
$$\frac{d^2A}{dz^2}-2jk_0\frac{dA}{dz}+\underbrace{[k(\omega)+k_0]}_{\approx 2k_0}[k(\omega)-k_0]A=0$$

Slowly varying amplitude approximation

$$\left|\frac{d^2A}{dz^2}\right|<<\left|2k_0\frac{dA}{dz}\right|$$

$$\boxed{\frac{dA}{dz}=-j[k(\omega)-k_0]A}$$

Linear pulse propagation



$$k(\omega)=\frac{n(\omega)\omega}{c_0}\approx k(\omega_0)+\left.\frac{dk}{d\omega}\right|_{\omega_0}(\omega-\omega_0)+\frac{1}{2}\left.\frac{d^2k}{d\omega^2}\right|_{\omega_0}(\omega-\omega_0)^2+\frac{1}{6}\left.\frac{d^3k}{d\omega^3}\right|_{\omega_0}(\omega-\omega_0)^3+\ldots$$

$$=k_0+k_1(\omega-\omega_0)+\frac{1}{2}k_2(\omega-\omega_0)^2+\frac{1}{2}k_3(\omega-\omega_0)^3+\ldots$$

$$\boxed{k_0=\frac{\omega_0}{c_0}n(\omega_0)\qquad k_1=\left.\frac{dk}{d\omega}\right|_{\omega_0}\qquad k_2=\left.\frac{d^2k}{d\omega^2}\right|_{\omega_0}\qquad k_3=\left.\frac{d^3k}{d\omega^3}\right|_{\omega_0}}$$

Group velocity Vs phase velocity

$$\frac{dA(z,\omega-\omega_0)}{dz} = -j[k(\omega)-k_0]A(z,\omega-\omega_0) \qquad k(\omega) = k_0 + \left.\frac{dk}{d\omega}\right|_{\omega=\omega_0}(\omega-\omega_0) + \frac{1}{2}\left.\frac{d^2k}{d\omega^2}\right|_{\omega=\omega_0}(\omega-\omega_0)^2 + \ldots$$

Let's first take a look at the effect of the first two terms $k(\omega) = k_0 + k^{(1)}(\omega-\omega_0)$ $\quad k^{(1)} = \left.\frac{dk}{d\omega}\right|_{\omega=\omega_0}$

$$\frac{dA(z,\omega-\omega_0)}{dz} = -jk^{(1)}(\omega-\omega_0)A(z,\omega-\omega_0) \longrightarrow A(z,\omega-\omega_0) = A(0,\omega-\omega_0)e^{-jk^{(1)}(\omega-\omega_0)z}$$

$$E(z,\omega) = A(z,\omega-\omega_0)e^{-jk_0z} = A(0,\omega-\omega_0)e^{-j[k_0+k^{(1)}(\omega-\omega_0)]z}$$

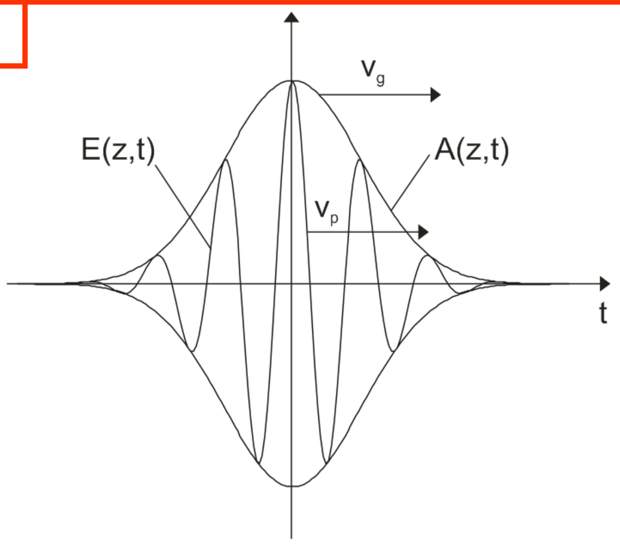
$$\boxed{E(z,t) = A(0,t-k^{(1)}z)e^{j(\omega_0 t-k_0 z)} = A\left(0,t-\frac{z}{V_g}\right)e^{j\omega_0\left(t-\frac{z}{V_p}\right)}}$$

Group velocity: travelling speed of the pulse envelope.

$$V_g = \frac{1}{k^{(1)}} = 1\Big/\left.\frac{dk}{d\omega}\right|_{\omega=\omega_0} = \frac{c_0}{n(\omega_0)+\omega_0\left.\frac{dn(\omega)}{d\omega}\right|_{\omega=\omega_0}}$$

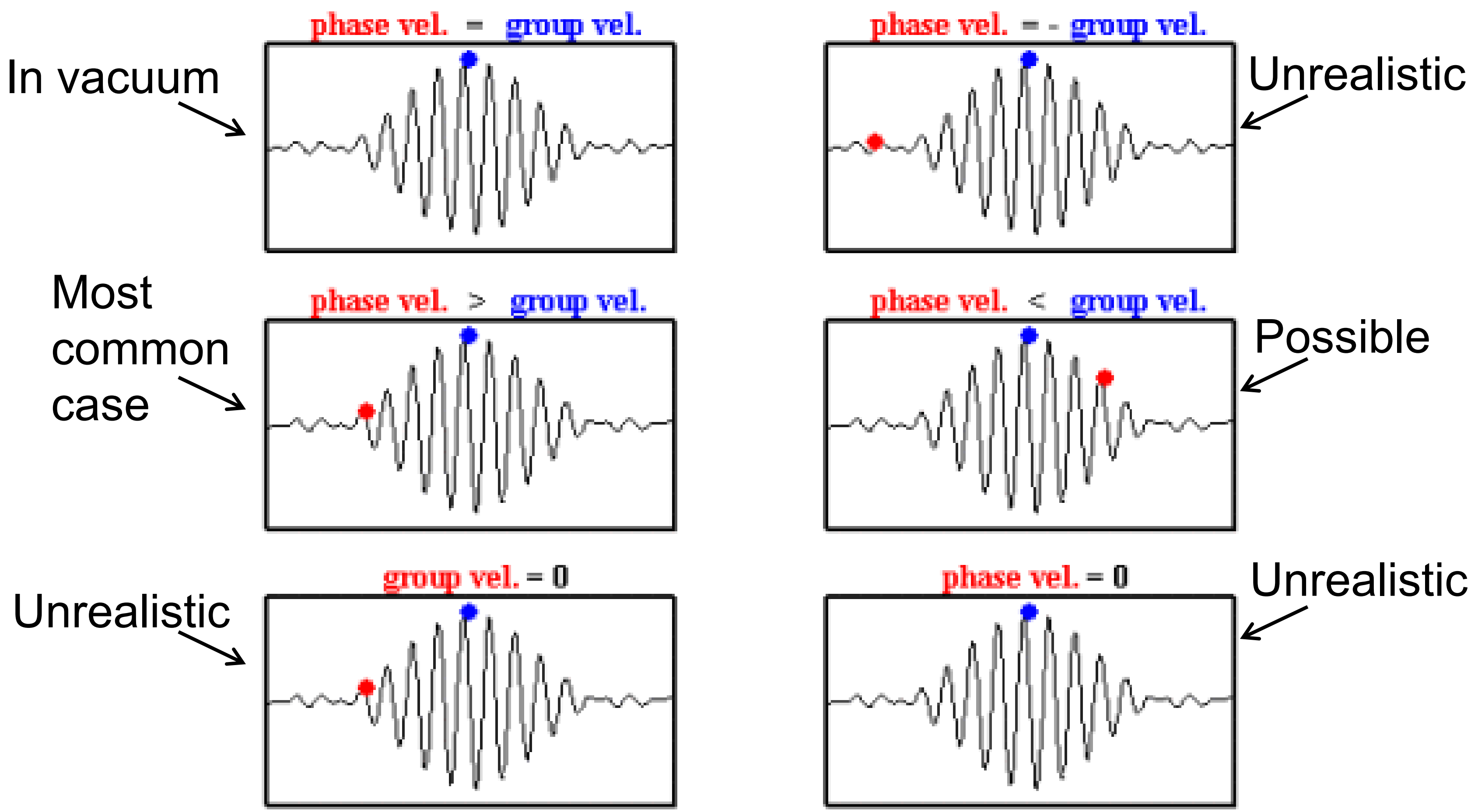
Phase velocity: travelling speed of the carrier wave under the pulse envelope.

$$V_p = \frac{\omega_0}{k_0} = \frac{c_0}{n(\omega_0)}$$



Electric field and pulse envelope in time domain

Group velocity Vs phase velocity



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Adapted from Rick Trebino's course slides

Calculating group velocity vs. wavelength

We more often think of the refractive index in terms of wavelength, so let's write the group velocity in terms of the vacuum wavelength λ_0.

Use the chain rule: $$\frac{dn}{d\omega} = \frac{dn}{d\lambda_0}\frac{d\lambda_0}{d\omega}$$

Now, $\lambda_0 = 2\pi c_0 / \omega$, so: $$\frac{d\lambda_0}{d\omega} = \frac{-2\pi c_0}{\omega^2} = \frac{-2\pi c_0}{(2\pi c_0/\lambda_0)^2} = \frac{-\lambda_0^2}{2\pi c_0}$$

Recalling that: $$\mathrm{v}_g = \left(\frac{c_0}{n}\right) / \left[1 + \frac{\omega}{n}\frac{dn}{d\omega}\right]$$

we have: $$\mathrm{v}_g = \left(\frac{c_0}{n}\right) / \left[1 + \frac{\cancel{2\pi c_0}}{n\lambda_0}\left\{\frac{dn}{d\lambda_0}\left(\frac{-\lambda_0^2}{\cancel{2\pi c_0}}\right)\right\}\right]$$

or:

$$\boxed{\mathrm{v}_g = \left(\frac{c_0}{n}\right) / \left(1 - \frac{\lambda_0}{n}\frac{dn}{d\lambda_0}\right)}$$

<u>*Adapted from Rick Trebino's course slides*</u>

Group-velocity dispersion (GVD)

What's effect of the 3rd term in the Taylor expansion of wave vector?

$$\frac{dA(z,\omega-\omega_0)}{dz} = -j[k(\omega)-k_0]A(z,\omega-\omega_0) \qquad k(\omega)=k_0+\left.\frac{dk}{d\omega}\right|_{\omega=\omega_0}(\omega-\omega_0)+\frac{1}{2}\left.\frac{d^2k}{d\omega^2}\right|_{\omega=\omega_0}(\omega-\omega_0)^2+\ldots$$

$$\boxed{k(\omega)=k_0+k^{(1)}(\omega-\omega_0)+\frac{1}{2}k^{(2)}(\omega-\omega_0)^2 \qquad k^{(1)}=\left.\frac{dk}{d\omega}\right|_{\omega=\omega_0} \qquad k^{(2)}=\left.\frac{d^2k}{d\omega^2}\right|_{\omega=\omega_0}}$$

$$\frac{dA(z,\omega-\omega_0)}{dz} = -j[\underline{\frac{1}{V_g}+\frac{1}{2}k^{(2)}(\omega-\omega_0)}](\omega-\omega_0)A(z,\omega-\omega_0) \qquad V_g=\frac{1}{k^{(1)}}$$

Group velocity becomes frequency dependent.

$$\boxed{\text{Group velocity dispersion (GVD)} \qquad k^{(2)}=\frac{d}{d\omega}\left(\frac{dk}{d\omega}\right)=\frac{d}{d\omega}\left(\frac{1}{V_g}\right)}$$

$$A(z,\omega-\omega_0)=A(0,\omega-\omega_0)e^{-j[\frac{1}{V_g}+\frac{1}{2}k^{(2)}(\omega-\omega_0)](\omega-\omega_0)z} \longrightarrow \left|A(z,\omega-\omega_0)\right|=\left|A(0,\omega-\omega_0)\right|$$

The pulse maintains its optical spectrum shape but acquires a quadratic spectral phase from GVD, which will change the pulse's temporal profile.

Group-velocity dispersion (GVD)

$$k^{(2)} = \frac{d}{d\omega}\left(\frac{1}{v_g}\right) = -\frac{1}{v_g{}^2(\omega_0)}\frac{dv_g}{d\omega} = \left(\frac{\lambda}{2\pi c}\right)\frac{\lambda^2}{c}\frac{d^2n}{d\lambda^2}$$

Positive GVD or normal dispersion

$$k^{(2)} > 0$$

$$\frac{dv_g}{d\omega} < 0$$

Low frequency travels faster

Negative GVD or anomalous dispersion

$$k^{(2)} < 0$$

$$\frac{dv_g}{d\omega} > 0$$

High frequency travels faster

Effect of GVD on pulse propagation

Gaussian Pulse:

$$\begin{aligned}\underline{E}(z &= 0,t) = \underline{A}(z=0,t)e^{\mathrm{j}\omega_0 t} \\ \underline{A}(z &= 0, t=t') = \underline{A}_0 \exp\left[-\frac{1}{2}\frac{t'^2}{\tau^2}\right]\end{aligned}$$

Pulse width

$$\frac{\partial \tilde{\underline{A}}(z,\omega)}{\partial z} = -\mathrm{j}\frac{k''\omega^2}{2}\tilde{\underline{A}}(z,\omega)$$

Substitute:

$$\tilde{\underline{A}}(z,\omega) = \tilde{\underline{A}}(z=0,\omega)\exp\left[-\mathrm{j}\frac{k''\omega^2}{2}z\right]$$

Gaussian Integral:

$$\boxed{\frac{1}{\sqrt{2\pi\sigma}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma}}e^{-jx\varsigma}dx = e^{-\frac{\sigma}{2}\varsigma^2} \text{ for } \mathrm{Re}\{\sigma\} \geq 0}$$

$$\tilde{\underline{A}}(z=0,\omega) = A_0\sqrt{2\pi}\tau\exp\left[-\frac{1}{2}\tau^2\omega^2\right]$$

Propagation of z distance:

$$\underline{\tilde{A}}(z,\omega) = A_0\sqrt{2\pi}\tau \exp\left[-\frac{1}{2}\left(\tau^2 + \mathrm{j}k''z\right)\omega^2\right]$$

$$\underline{A}(z,t') = A_0\left(\frac{\tau^2}{(\tau^2 + \mathrm{j}k''z)}\right)^{1/2} \exp\left[-\frac{1}{2}\frac{t'^2}{(\tau^2 + \mathrm{j}k''z)}\right]$$

Exponent Real and Imaginary Part:

$$\underline{A}(z,t') = A_0\underbrace{\left(\frac{\tau^2}{(\tau^2 + \mathrm{j}k''z)}\right)^{1/2}}_{\text{z-dependent phase shift, independent on time}} \exp\left[\underbrace{-\frac{1}{2}\frac{\tau^2 t'^2}{\left(\tau^4 + (k''z)^2\right)}}_{\text{determines pulse width}} + \underbrace{\mathrm{j}\frac{1}{2}k''z\frac{t'^2}{\left(\tau^4 + (k''z)^2\right)}}_{\text{temporal quadratic phase}}\right]$$

z-dependent phase shift, independent on time

determines pulse width

temporal quadratic phase

FWHM Pulse width:

$$\exp\left[-\frac{\tau^2(\tau'_{FWHM}/2)^2}{\left(\tau^4 + (k''z)^2\right)}\right] = 0.5$$

Initial pulse width:

$$\tau_{FWHM} = 2\sqrt{\ln 2}\;\tau$$

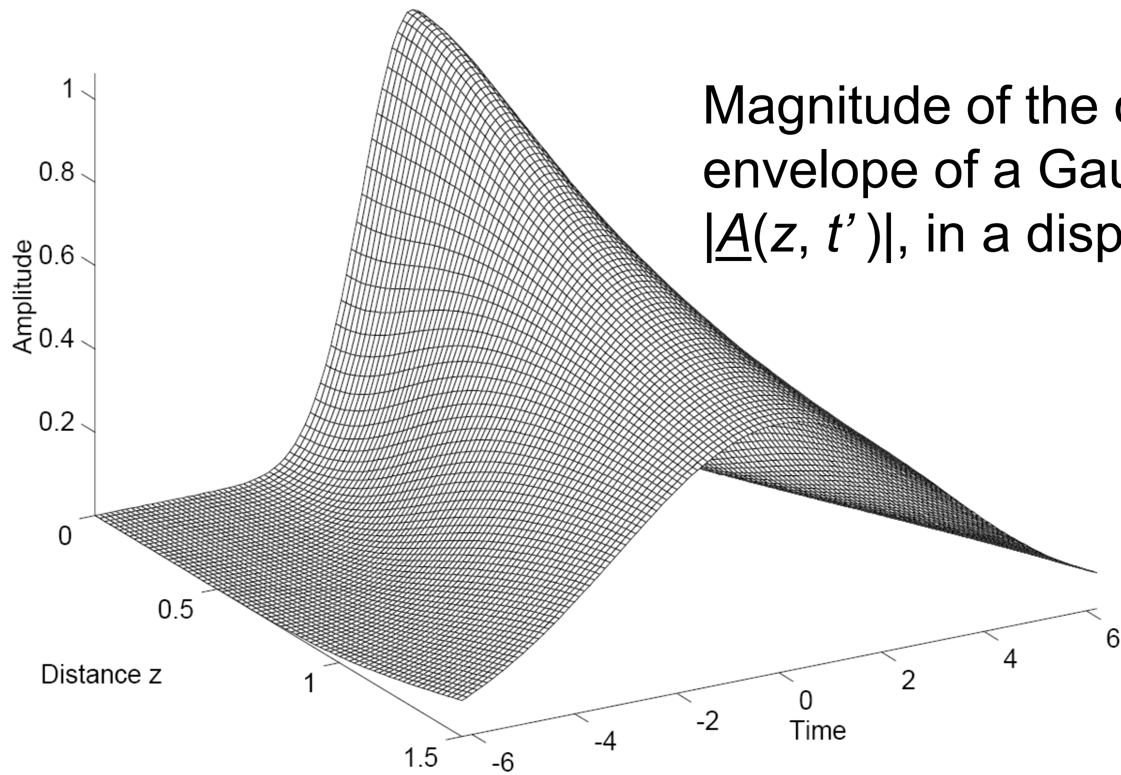
Initial pulse width:

$$\tau_{FWHM} = 2\sqrt{\ln 2}\ \tau$$

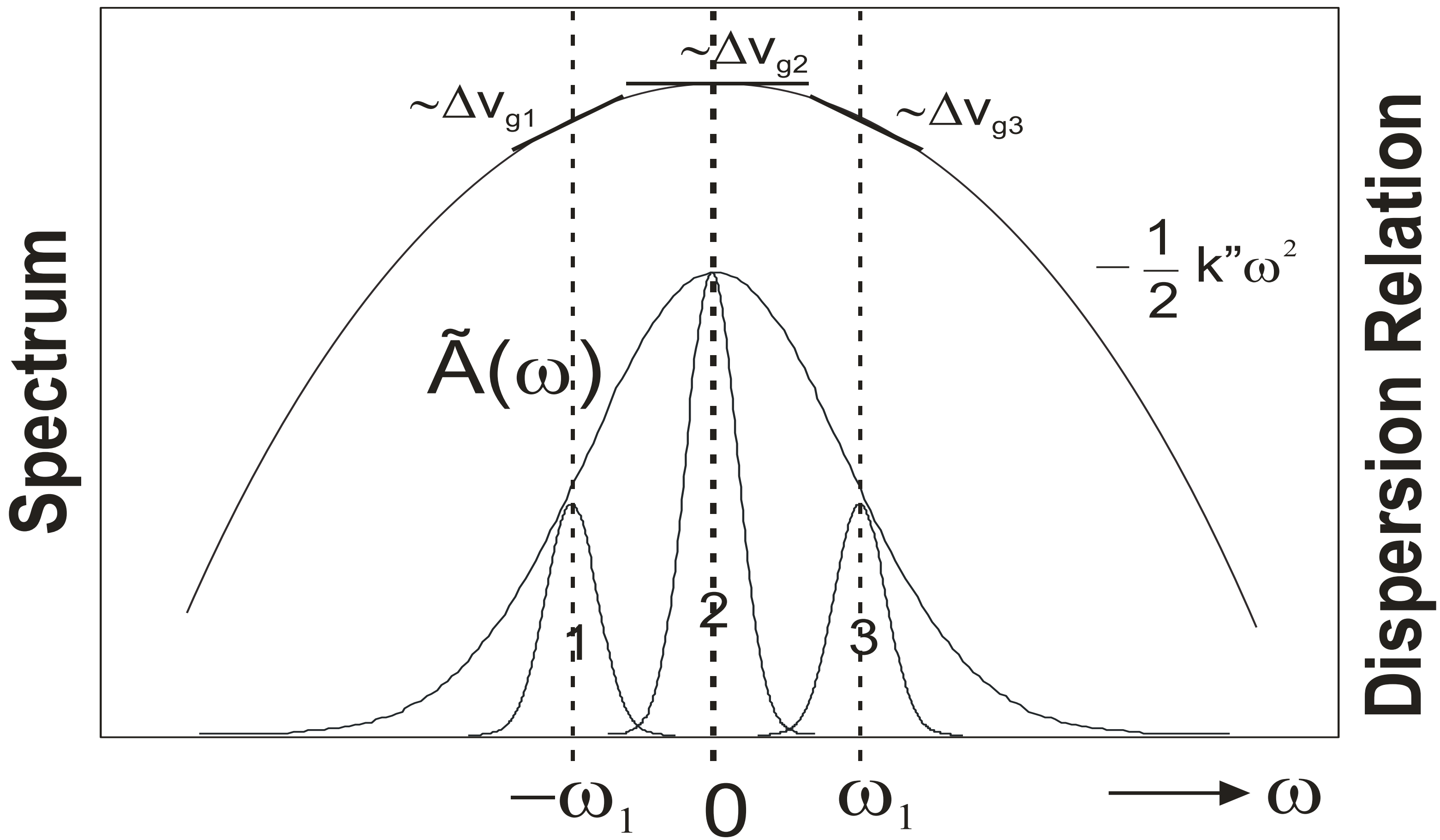
After propagation over a distance z=L:

$$\tau'_{FWHM} = 2\sqrt{\ln 2}\ \tau\sqrt{1+\left(\frac{k''L}{\tau^2}\right)^2} = \tau_{FWHM}\sqrt{1+\left(\frac{k''L}{\tau^2}\right)^2}$$

For large distances: $$\tau'_{FWHM} = 2\sqrt{\ln 2}\ \left|\frac{k''L}{\tau}\right| \quad \text{for} \quad \left|\frac{k''L}{\tau^2}\right| \gg 1$$



Magnitude of the complex envelope of a Gaussian pulse, $|\underline{A}(z, t')|$, in a dispersive medium



Decomposition of a pulse into wave packets with different center frequency. In a medium with dispersion the wave packets move at different relative group velocity

Instantaneous frequency and chirp

$$\underline{A}(z,t') = A_0 \underbrace{\left(\frac{\tau^2}{(\tau^2 + \mathrm{j}k''z)}\right)^{1/2}}_{\text{z-dependent phase shift, independent on time}} \exp\left[\underbrace{-\frac{1}{2}\frac{\tau^2 t'^2}{\left(\tau^4 + (k''z)^2\right)}}_{\text{determines pulse width}} \underbrace{+ \mathrm{j}\frac{1}{2}k''z\frac{t'^2}{\left(\tau^4 + (k''z)^2\right)}}_{\text{temporal quadratic phase}}\right]$$

After propagation of L distance: $\omega(z = L, t') = \dfrac{\partial}{\partial t'}\phi(L,t') = \dfrac{k''L}{\left(\tau^4 + (k''L)^2\right)}t'$

$$E(L,t') = \underline{A}(L,t')\exp(j\omega_0 t') \propto \exp[j\omega_0 t' + j\phi(L,t')]$$

$$\phi(z = L, t') = -\frac{1}{2}\arctan\left[\frac{k''L}{\tau^2}\right] + \frac{1}{2}k''L\frac{t'^2}{\left(\tau^4 + (k''L)^2\right)}$$

Instantaneous Frequency:

$$\omega_{inst}(t) \equiv \frac{\partial[\omega_0 t' + \phi(L,t')]}{\partial t'} = \omega_0 + \frac{\partial\phi(L,t')}{\partial t'}$$

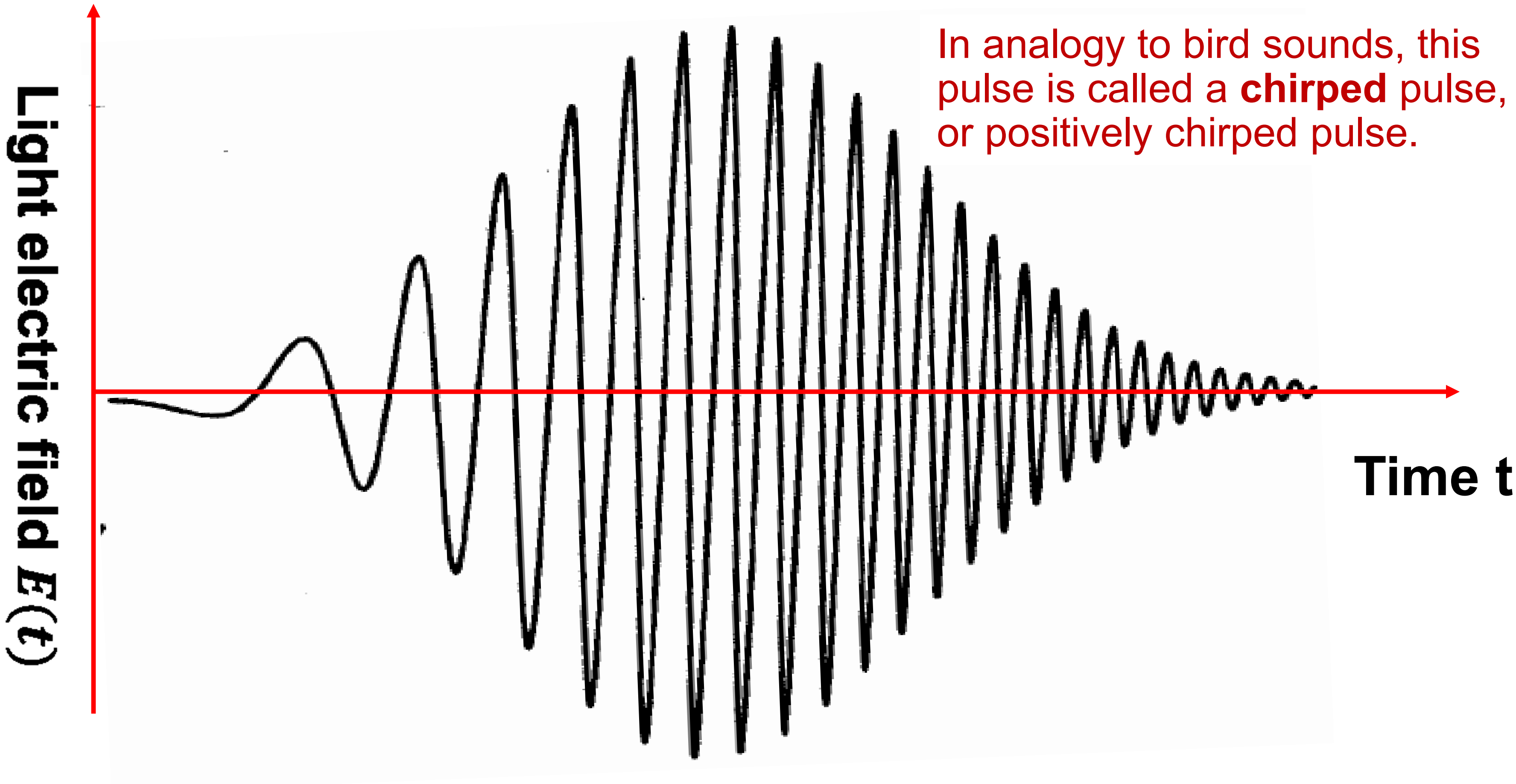
$$= \omega_0 + \frac{k''L}{\left(\tau^4 + (k''L)^2\right)}t'$$

Linearly chirped Gaussian pulse: positive chirp

$$\omega_{ins}(t') = \omega_0 + \frac{k''L}{(\tau^4 + (k''L)^2)} t'$$

For positive GVD, i.e., k”>0, lower frequency travels faster, and the instantaneous frequency linearly **INCREASES** with time.

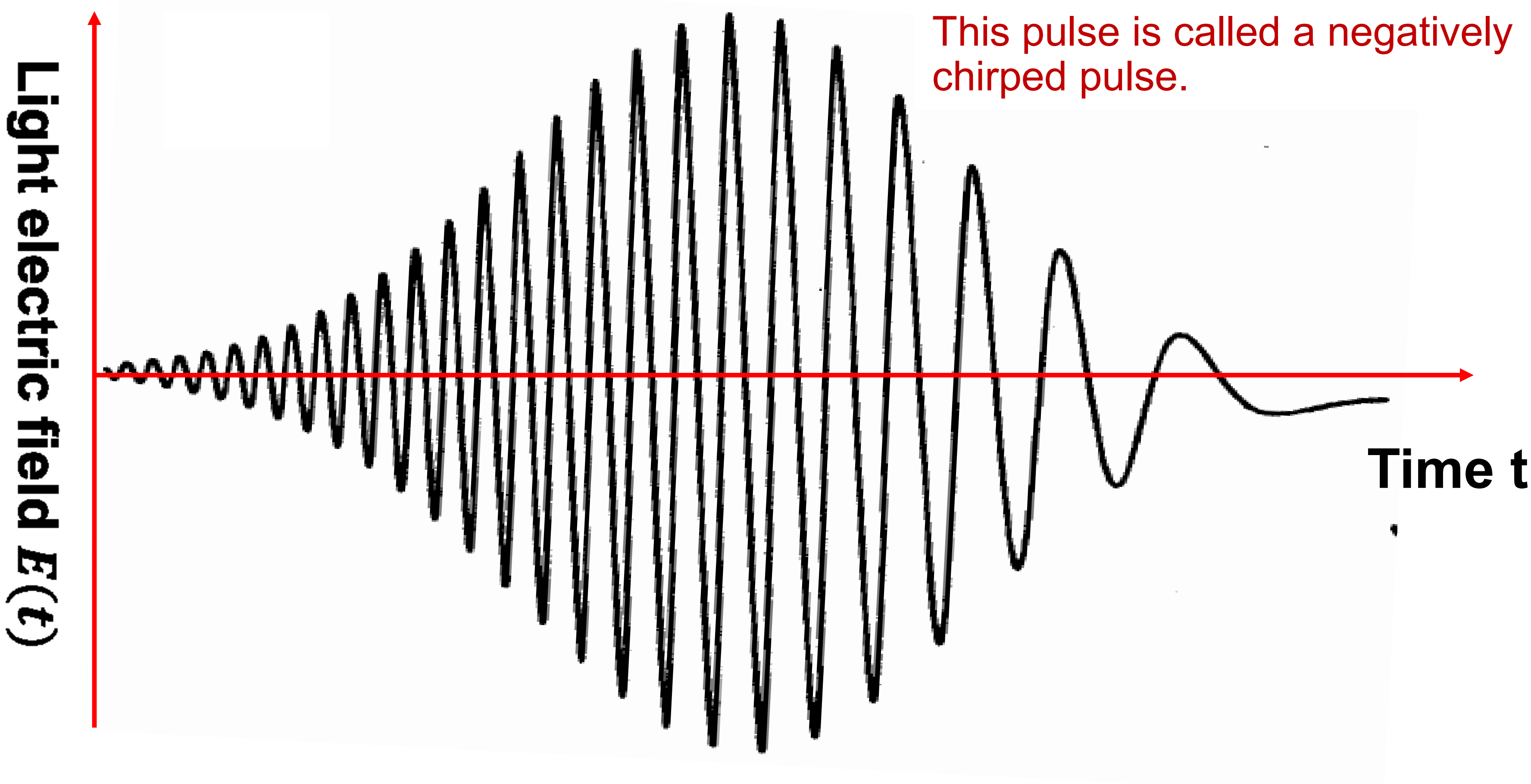
In analogy to bird sounds, this pulse is called a **chirped** pulse, or positively chirped pulse.



Linearly chirped Gaussian pulse: negative chirp

$$\omega_{ins}(t') = \omega_0 + \frac{k''L}{\left(\tau^4 + (k''L)^2\right)} t'$$

For negative GVD, i.e., k”<0, higher frequency travels faster.
The instantaneous frequency linearly **DECREASES** with time.

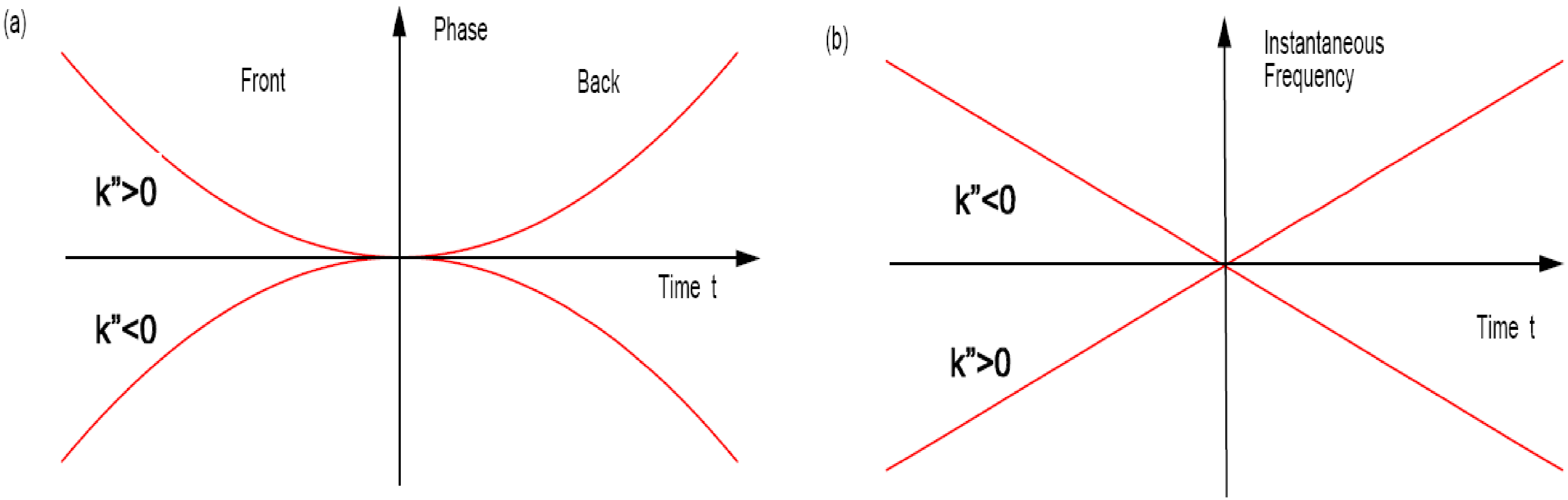


This pulse is called a negatively chirped pulse.

$$\phi(z=L,t') = -\frac{1}{2}\arctan\left[\frac{k''L}{\tau^2}\right] + \frac{1}{2}k''L\frac{t'^2}{\left(\tau^4+(k''L)^2\right)}$$

Instantaneous Frequency:

$$\omega_{ins}(t') = \omega_0 + \frac{k''L}{\left(\tau^4+(k''L)^2\right)}t'$$



(a) temporal Phase and (b) instantaneous frequency of a Gaussian pulse during propagation through a medium with positive or negative dispersion

Transform-limited pulse

$$\tilde{E}(\omega) = \int_{-\infty}^{\infty} E(t)\exp(-j\omega t)dt \qquad E(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{E}(\omega)\exp(j\omega t)d\omega$$

$\left|\tilde{E}(\omega)\right|^2$ has a spectrum bandwidth of $\Delta\nu$

$\left|E(t)\right|^2$ has a pulse duration of Δt

Both are measured at full-width at half-maximum (FWHM).

Uncertainty principle:

$$\underbrace{\Delta\nu\Delta t}_{\text{Time Bandwidth Product (TBP)}} \geq \underbrace{K}_{\text{A number depending only on pulse shape}}$$

For a given optical spectrum, there exist a lower limit for the pulse duration. If the equality is reached, we say the pulse is a transform-limited pulse.

To get a shorter transform-limited pulse, one needs a broader optical spectrum.

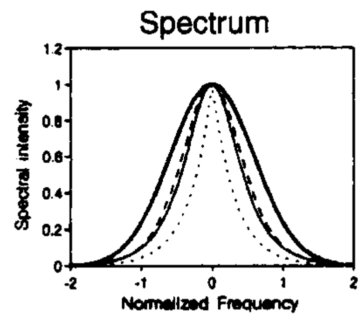
Temporal and spectral shapes and TBPs of typical ultrashort pulses

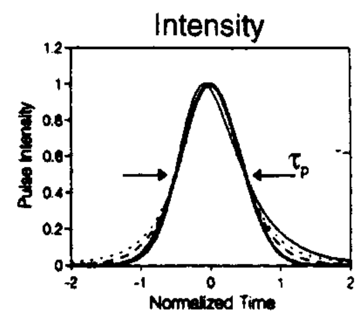
Diels and Rudolph, Femtosecond Phenomena

Common pulse envelopes (with τ_p = Intensity FWHM):

Gaussian pulse	$\mathcal{E}(t)$	$\propto$	$\exp[-1.385(t/\tau_p)^2]$
sech - pulse	$\mathcal{E}(t)$	$\propto$	$\mathrm{sech}[1.763(t/\tau_p)]$
Lorentzian pulse	$\mathcal{E}(t)$	$\propto$	$[1+1.656(t/\tau_p)^2]^{-1}$
asymm. sech pulse	$\mathcal{E}(t)$	$\propto$	$[\exp(t/\tau_p)+\exp(-3t/\tau_p)]^{-1}$

Spectra for pulses with the same pulse width





Field envelope	Intensity profile	τ_p (FWHM)	Spectral profile	$\Delta\omega_p$ (FWHM)	TBP
Gauss	$e^{-2(t/\tau_G)^2}$	$1.177\tau_G$	$e^{-(\omega\tau_G)^2/2}$	$2.355/\tau_G$	0.441
sech	$\mathrm{sech}^2(t/\tau_s)$	$1.763\tau_s$	$\mathrm{sech}^2(\pi\omega\tau_s/2)$	$1.122/\tau_s$	0.315
Lorentz	$[1+(t/\tau_L)^2]^{-2}$	$1.287\tau_L$	$e^{-2\|\omega\|\tau_L}$	$0.693/\tau_L$	0.142
asymm. sech	$\left[e^{t/\tau_a}+e^{-3t/\tau_a}\right]^{-2}$	$1.043\tau_a$	$\mathrm{sech}(\pi\omega\tau_a/2)$	$1.677/\tau_a$	0.278
rectang.	1 for $\|t/\tau_r\| \le \frac{1}{2}$, 0 else	τ_r	$\mathrm{sinc}^2(\omega\tau_r)$	$2.78/\tau_r$	0.443

Some definitions

$$k(\omega) = n(\omega)\frac{\omega}{c} = k_0 + k_1(\omega - \omega_0) + \frac{1}{2}k_2(\omega - \omega_0)^2 + \ldots$$

$$k_1 = \frac{1}{v_g} = \frac{1}{c}(n + \omega\frac{dn}{d\omega}) = \frac{1}{c}(n - \lambda\frac{dn}{d\lambda})$$ Unit: s/m

$$k_m = \left(\frac{d^m k}{d\omega^m}\right)_{\omega=\omega_0}$$

v_g Group velocity

$$k_2 = \frac{d}{d\omega}(\frac{1}{v_g}) = -\frac{1}{v_g{}^2(\omega_0)}\frac{dv_g}{d\omega} = \frac{1}{c}(2\frac{dn}{d\omega} + \omega\frac{d^2n}{d\omega^2}) = (\frac{\lambda}{2\pi c})\frac{\lambda^2}{c}\frac{d^2n}{d\lambda^2}$$

k_2 Group velocity dispersion (GVD) Unit: s^2/m

Note: more often, $\beta(\omega)$ is used to replace $k(\omega)$ and β_2 is GVD.

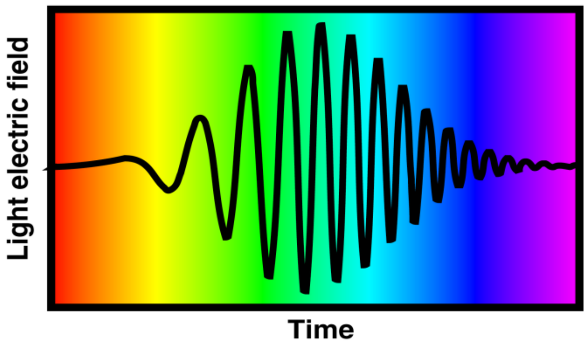
GVD changes the pulse duration and introduces chirp

$$k_2 = \frac{d}{d\omega}\left(\frac{1}{v_g}\right) = -\frac{1}{v_g^{\,2}(\omega_0)}\frac{dv_g}{d\omega} = \left(\frac{\lambda}{2\pi c}\right)\frac{\lambda^2}{c}\frac{d^2 n}{d\lambda^2}$$

Positive GVD or normal dispersion

$$k_2 > 0 \qquad \frac{dv_g}{d\omega} < 0$$

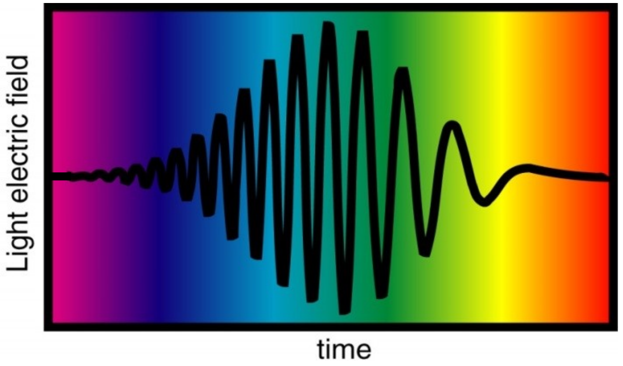
Red faster, positive chirp



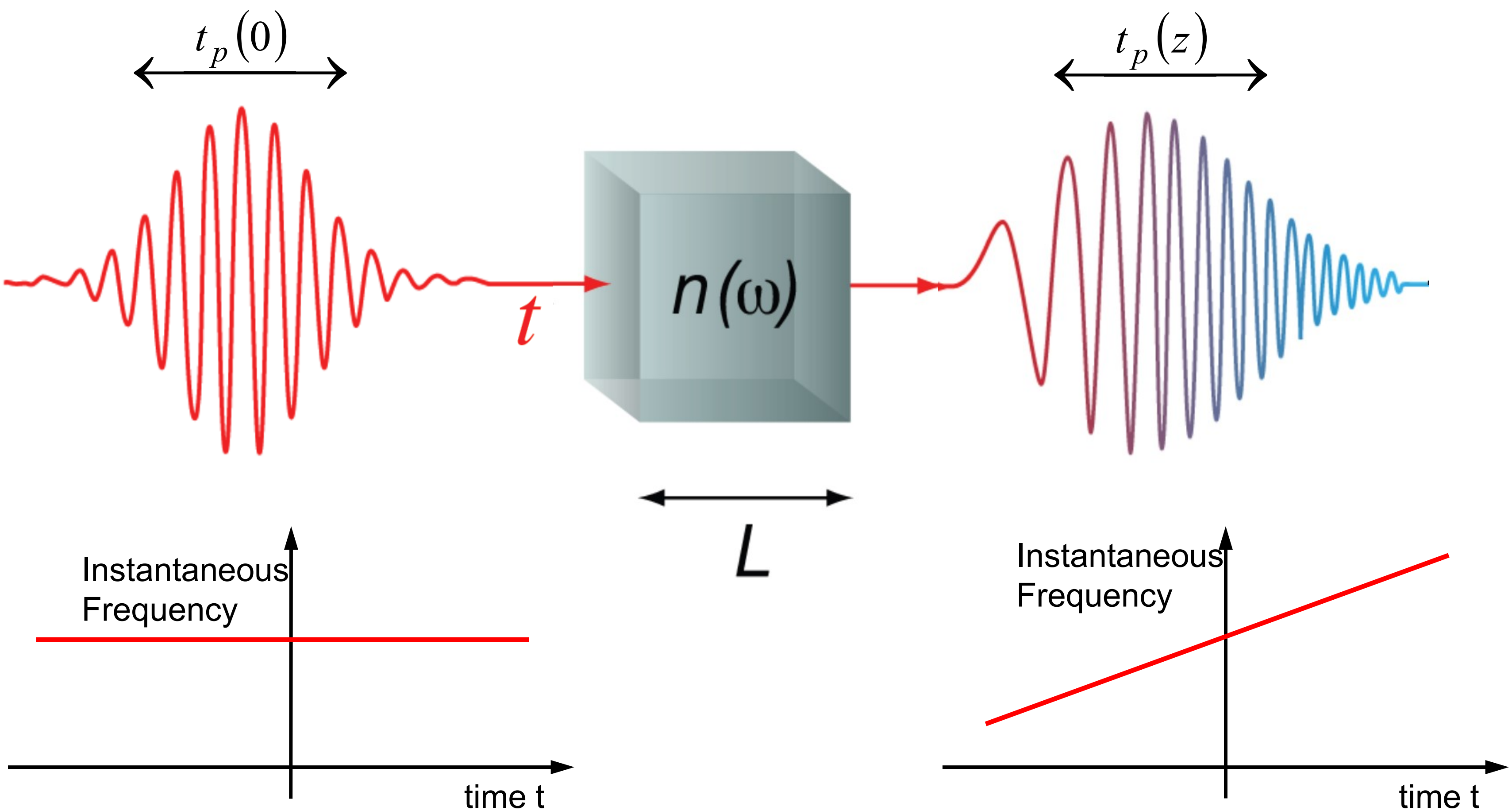
Negative GVD or anomalous dispersion

$$k_2 < 0 \qquad \frac{dv_g}{d\omega} > 0$$

Blue faster, negative chirp



Pulse travels through a dispersive bulk medium



Transform-limited pulse

Positive chirp

Group Delay & Group Delay Dispersion

$$\varphi(\omega) = k(\omega)z = \varphi_0 + \varphi_1(\omega-\omega_0) + \frac{1}{2}\varphi_2(\omega-\omega_0)^2 + \frac{1}{6}\varphi_3(\omega-\omega_0)^3 + \ldots$$

$$\varphi_1 = \frac{z}{v_g} = \tau_g$$ Group delay, in fs

$$\varphi_m = \left(\frac{d^m\varphi}{d\omega^m}\right)_{\omega=\omega_0}$$

$$\varphi_2 = \frac{d\tau_g}{d\omega}$$

Group delay dispersion (GDD), in fs^2

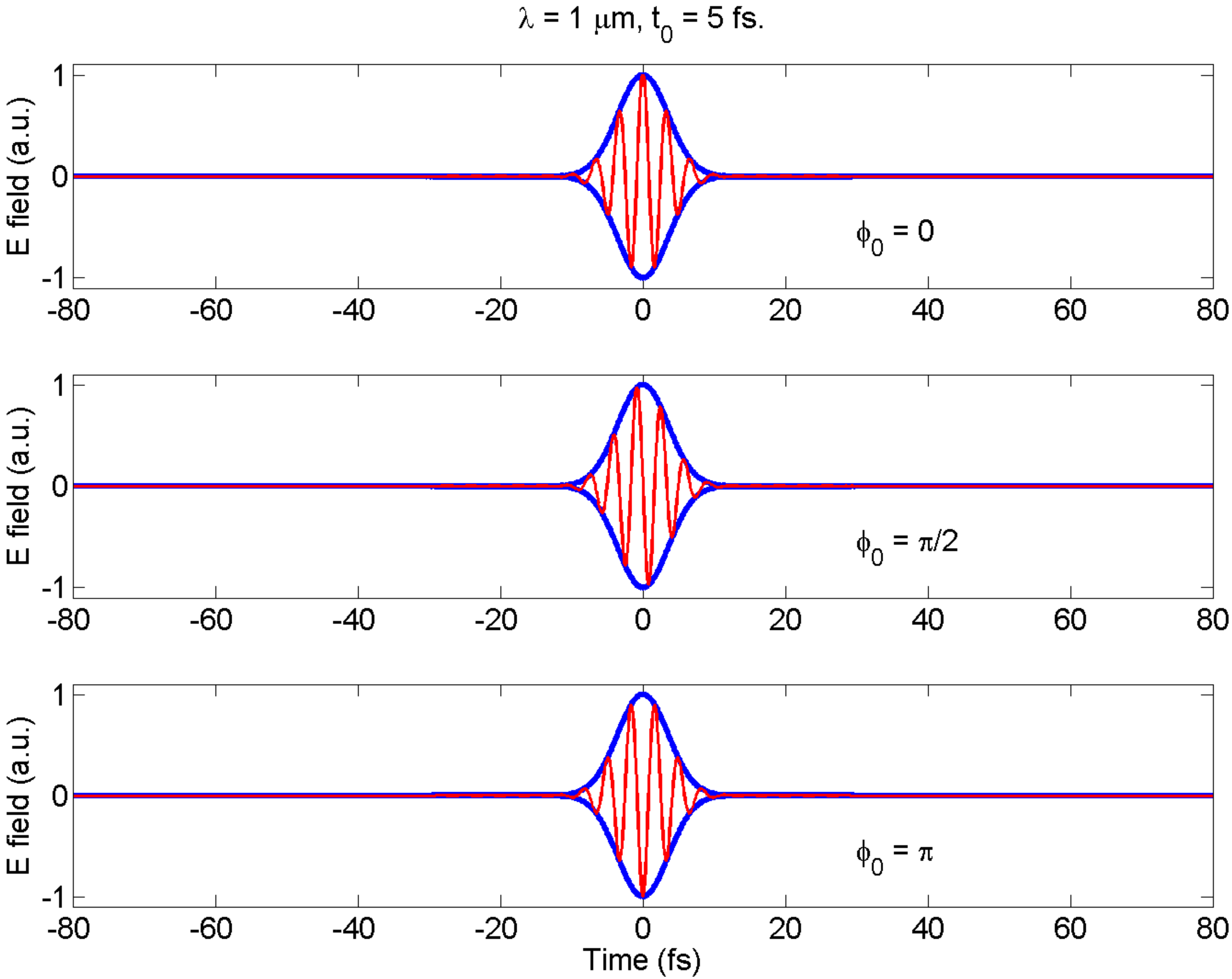
GDD > 0, positive dispersion
GDD < 0, negative dispersion

φ_3 Third order dispersion (TOD), in fs^3

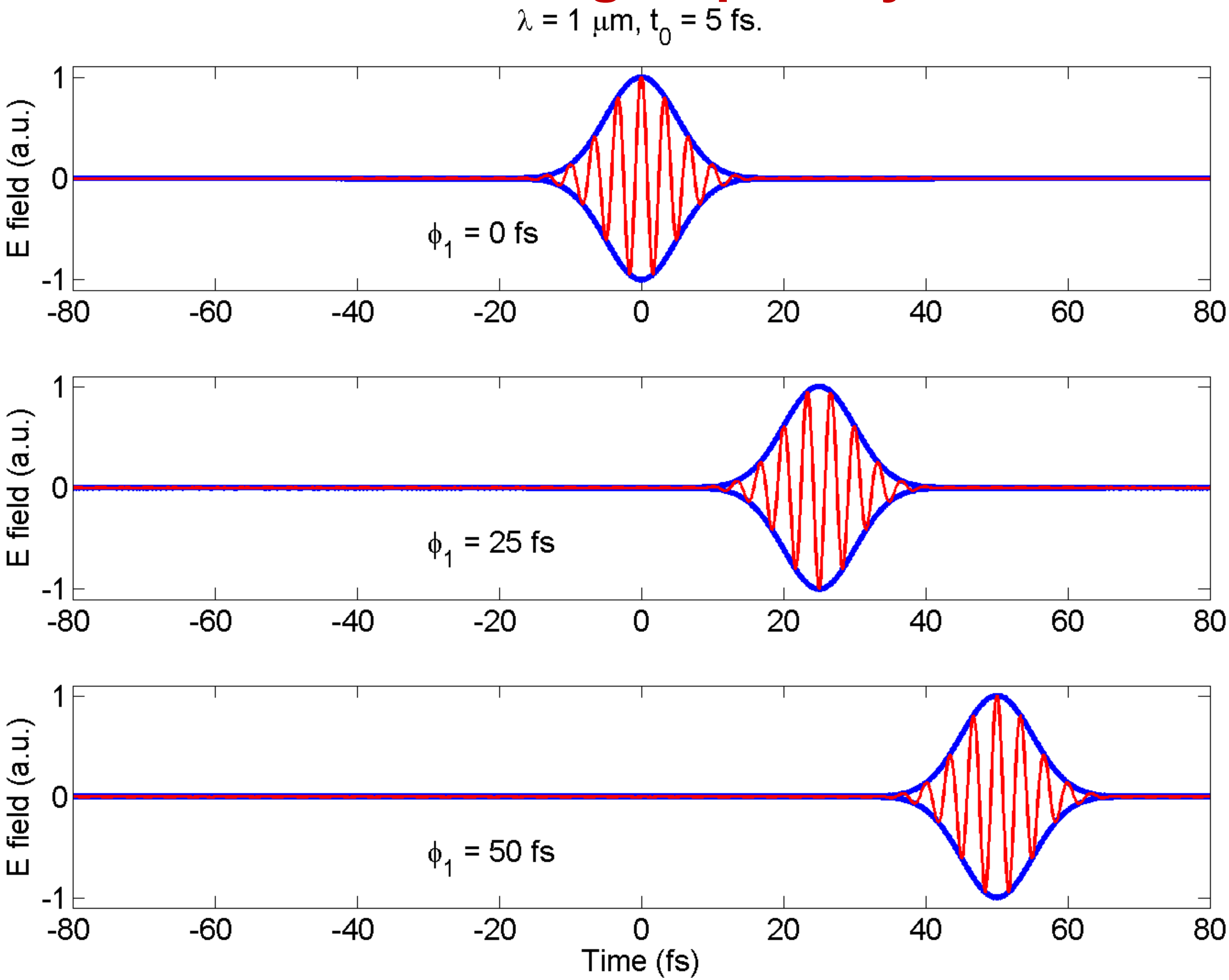
φ_4 Fourth order dispersion, in fs^4

<u>Group delay shift the time origin of the pulse envelope while GDD changes its shape.</u>

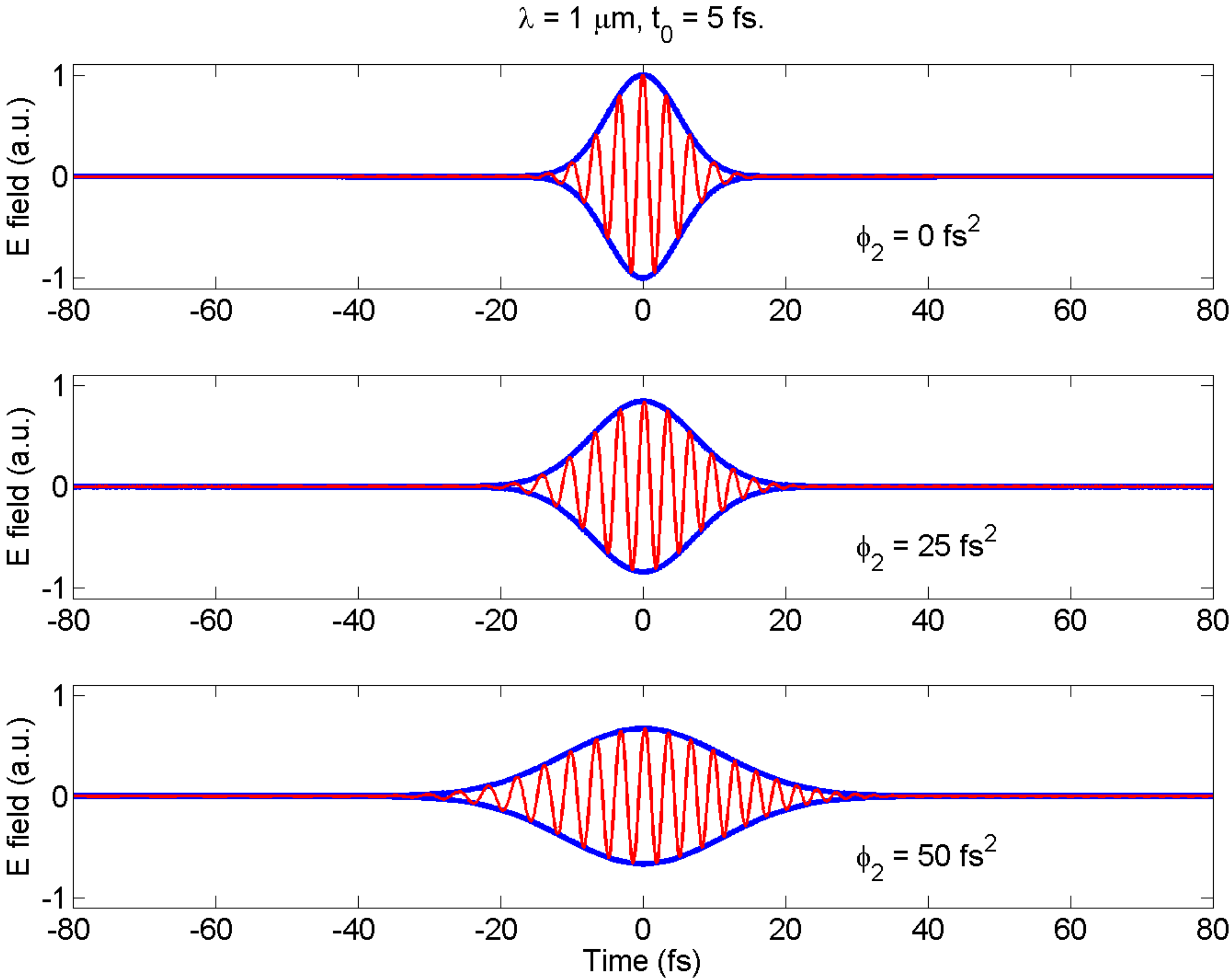
Effect of absolute phase



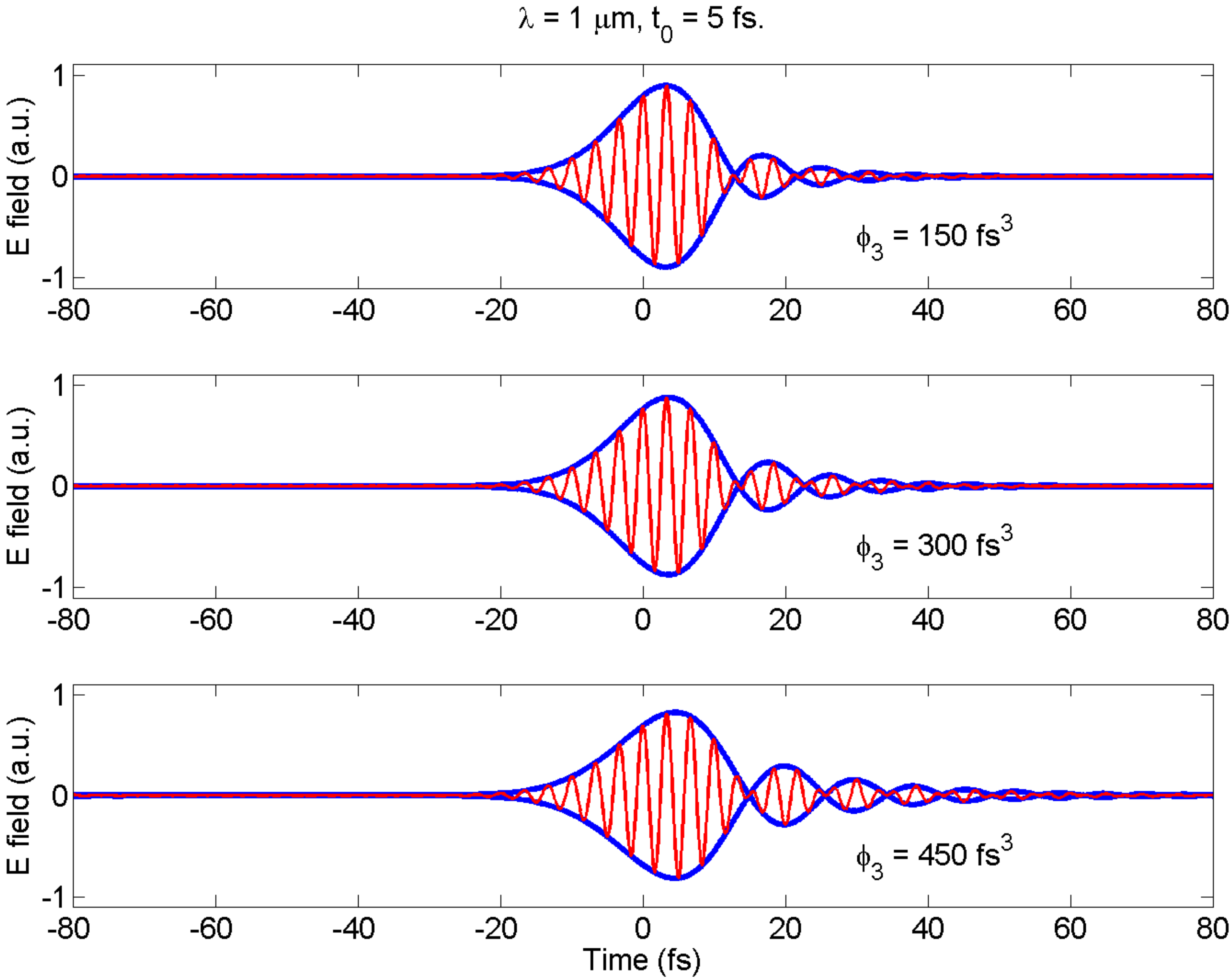
Effect of group delay



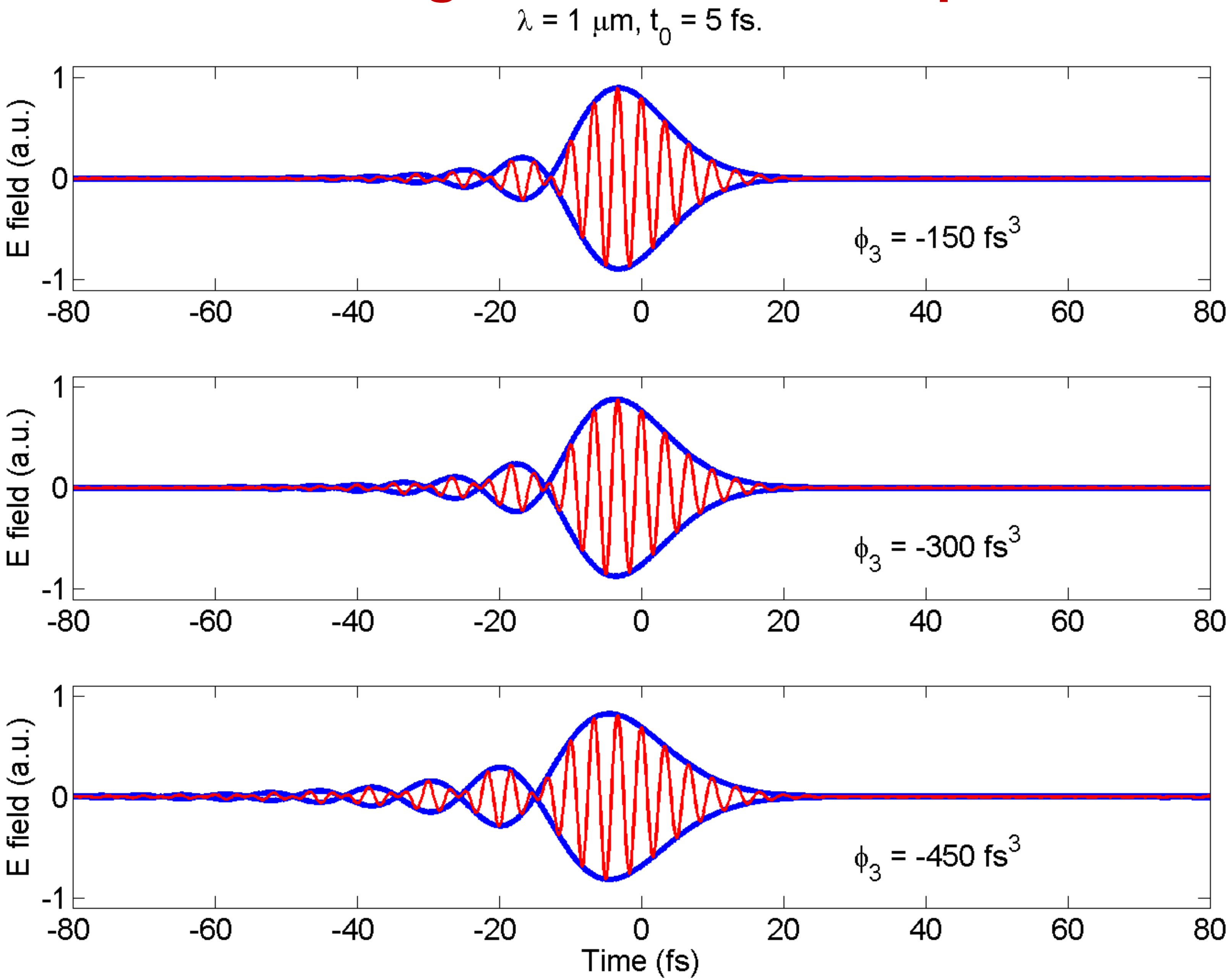
Effect of positive 2nd order dispersion



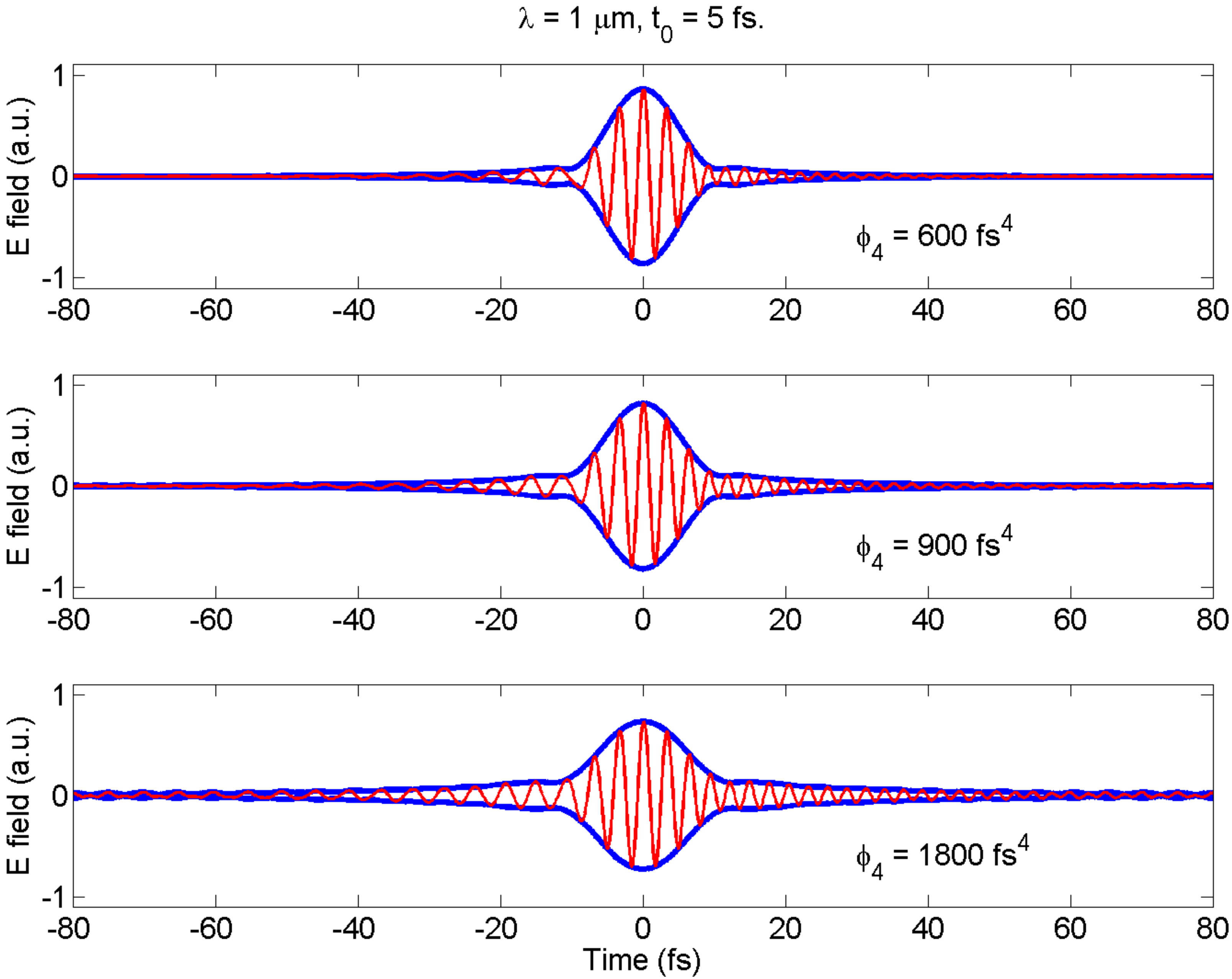
Effect of positive 3rd order dispersion



Effect of negative 3rd order dispersion



Effect of positive 4th order dispersion



Dispersion parameters for various materials

material	λ [nm]	$n(\lambda)$	$\frac{dn}{d\lambda}\cdot 10^{-2}\left[\frac{1}{\mu m}\right]$	$\frac{d^2n}{d\lambda^2}\cdot 10^{-1}\left[\frac{1}{\mu m^2}\right]$	$\frac{dn^3}{d\lambda^3}\left[\frac{1}{\mu m^3}\right]$	$T_g\left[\frac{fs}{mm}\right]$	$GDD\left[\frac{fs^2}{mm}\right]$	$TOD\left[\frac{fs^3}{mm}\right]$
BK7	400	1,5308	-13,17	10,66	-12,21	5282	120,79	40,57
	500	1,5214	-6,58	3,92	-3,46	5185	86,87	32,34
	600	1,5163	-3,91	1,77	-1,29	5136	67,52	29,70
	800	1,5108	-1,97	0,48	-0,29	5092	43,96	31,90
	1000	1,5075	-1,40	0,15	-0,09	5075	26,93	42,88
	1200	1,5049	-1,23	0,03	-0,04	5069	10,43	66,12
SF10	400	1,7783	-52,02	59,44	-101,56	6626	673,68	548,50
	500	1,7432	-20,89	15,55	-16,81	6163	344,19	219,81
	600	1,7267	-11,00	6,12	-4,98	5980	233,91	140,82
	800	1,7112	-4,55	1,58	-0,91	5830	143,38	97,26
	1000	1,7038	-2,62	0,56	-0,27	5771	99,42	92,79
	1200	1,6992	-1,88	0,22	-0,10	5743	68,59	107,51
Sapphire	400	1,7866	-17,20	13,55	-15,05	6189	153,62	47,03
	500	1,7743	-8,72	5,10	-4,42	6064	112,98	39,98
	600	1,7676	-5,23	2,32	-1,68	6001	88,65	37,97
	800	1,7602	-2,68	0,64	-0,38	5943	58,00	42,19
	1000	1,7557	-1,92	0,20	-0,12	5921	35,33	57,22
	1200	1,7522	-1,70	0,04	-0,05	5913	13,40	87,30
Quartz	300	1,4878	-30,04	34,31	-54,66	5263	164,06	46,49
	400	1,4701	-11,70	9,20	-10,17	5060	104,31	31,49
	500	1,4623	-5,93	3,48	-3,00	4977	77,04	26,88
	600	1,4580	-3,55	1,59	-1,14	4934	60,66	25,59
	800	1,4533	-1,80	0,44	-0,26	4896	40,00	28,43
	1000	1,4504	-1,27	0,14	-0,08	4880	24,71	38,73
	1200	1,4481	-1,12	0,03	-0,03	4875	9,76	60,05

Linear propagation equation for pulse envelope

$$\widetilde{E}(z,\omega)=\widetilde{E}(0,\omega)\exp[-jk(\omega)z]$$

$$\boxed{\frac{\partial}{\partial z}\widetilde{E}(z,\omega)=-jk\widetilde{E}(z,\omega)}$$

$$E(z,t)=A(z,t)\exp[j(\omega_0 t-k_0 z)] \Rightarrow \widetilde{E}(z,\omega)=\widetilde{A}(z,\omega-\omega_0)\exp(-jk_0 z)$$

$$\frac{\partial}{\partial z}\widetilde{A}(z,\omega-\omega_0)-jk_0\widetilde{A}(z,\omega-\omega_0)=-jk\widetilde{A}(z,\omega-\omega_0)$$

$$\frac{\partial}{\partial z}\widetilde{A}(z,\omega-\omega_0)=-j[\sum_{n=1}^{\infty}\frac{k^{(n)}}{n!}(\omega-\omega_0)^n]\widetilde{A}(z,\omega-\omega_0)$$

$$\omega-\omega_0 \longrightarrow \omega$$

$$\frac{\partial}{\partial z}\widetilde{A}(z,\omega)=-j[\sum_{n=1}^{\infty}\frac{k^{(n)}}{n!}\omega^n]\widetilde{A}(z,\omega)$$

In the time domain

$$\boxed{\frac{\partial}{\partial z}A(z,t)=-j[\sum_{n=1}^{\infty}\frac{k^{(n)}}{n!}(-j\frac{\partial}{\partial t})^n]A(z,t)}$$

Linear propagation equation for pulse envelope

$$\frac{\partial}{\partial z}A(z,t) = -k_1\frac{\partial}{\partial t}A(z,t) - j\left[\sum_{n=2}^{\infty}\frac{k^{(n)}}{n!}\left(-j\frac{\partial}{\partial t}\right)^n\right]A(z,t)$$

$$\Downarrow$$

$$\frac{\partial}{\partial z}A(z,t) + \frac{1}{V_g}\frac{\partial}{\partial t}A(z,t) = -j\left[\sum_{n=2}^{\infty}\frac{k^{(n)}}{n!}\left(-j\frac{\partial}{\partial t}\right)^n\right]A(z,t)$$

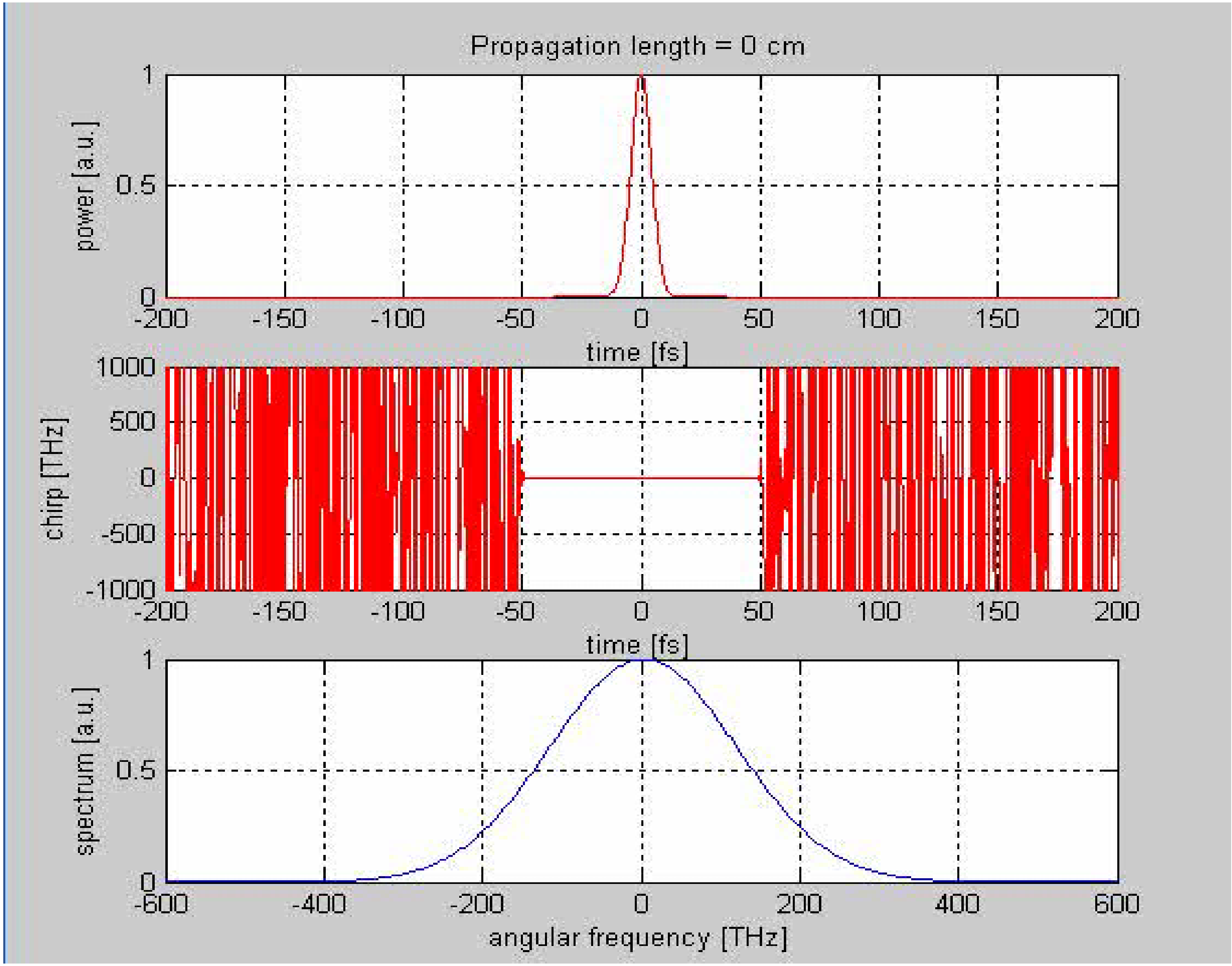
$$t - \frac{z}{V_g} \longrightarrow t' \quad \Rightarrow \quad \frac{\partial}{\partial z}A(z,t') = -j\left[\sum_{n=2}^{\infty}\frac{k^{(n)}}{n!}\left(-j\frac{\partial}{\partial t'}\right)^n\right]A(z,t')$$

$$\Downarrow$$

In a frame of reference moving with the pulse at the group velocity:

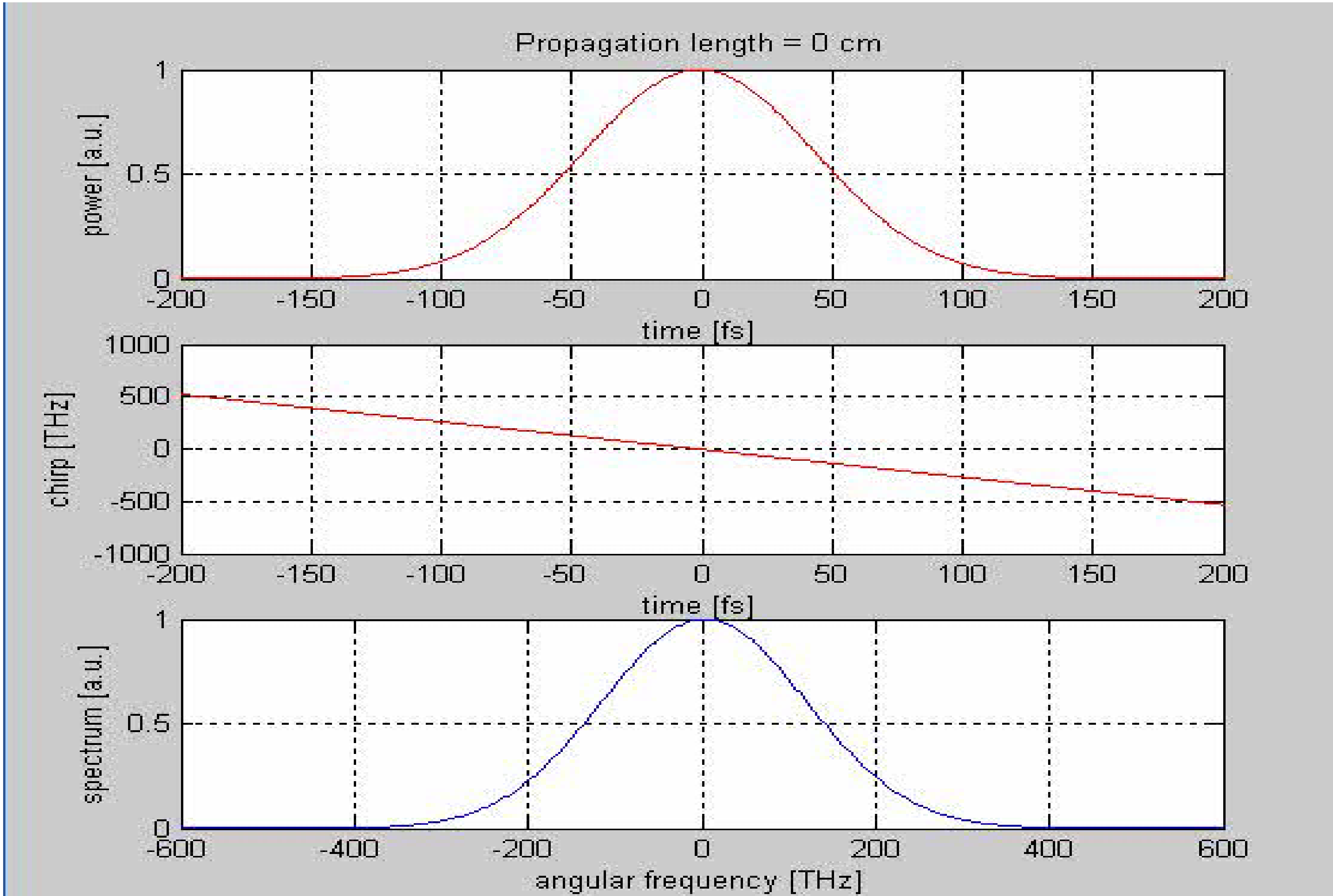
$$\boxed{\frac{\partial}{\partial z}\tilde{A}(z,\omega) = -j\left[\sum_{n=2}^{\infty}\frac{k^{(n)}}{n!}\omega^n\right]\tilde{A}(z,\omega)}$$

Effect of negative GVD



GVD $\beta_2 = -25\, ps^2 / km$ **Input pulse duration:10fs**

Effect of positive GVD



GVD $\beta_2 = 25\, ps^2 / km$ The output of last slide is taken as the input here.

Real and imaginary part of the susceptibility

$$\underline{\tilde{\chi}}(\omega) = \tilde{\chi}_r(\omega) + j\tilde{\chi}_i(\omega)$$

Example: EM-Wave polarized along x-axis and propagation along z-direction:

$$\underline{\vec{E}}(z,t) = \underline{E}_0 e^{j(\omega t - kz)} \, \vec{e}_x$$

In general:

$$k(\omega) = \frac{\omega}{c_0}\tilde{n}(\omega) = \frac{\omega}{c_0}\sqrt{1+\tilde{\chi}(\omega)} = \frac{\omega}{c_0}\left(\tilde{n}_r(\omega) + j\tilde{n}_i(\omega)\right) = k_r(\omega) - j\alpha(\omega)$$

$$\boxed{\underline{\vec{E}}(z,t) = \underline{E}_0 e^{-\alpha\cdot z} \; e^{j(\omega t - k_r z)} \, \vec{e}_x}$$

Dispersion relation: $$k_r(\omega) = \frac{\omega}{c_0} n_r(\omega)$$

Besides dispersion, a medium may introduce loss or gain

$$\underline{\tilde{n}}(\Omega) = n_r(\Omega) + jn_i(\Omega)$$

Refractive index + gain and/or loss

$$\underline{\tilde{n}}(\Omega) = \sqrt{1 + \underline{\tilde{\chi}}(\Omega)}$$

for: $\left|\underline{\tilde{\chi}}(\Omega)\right| << 1$

$$\underline{\tilde{n}}(\Omega) \approx 1 + \frac{\underline{\tilde{\chi}}(\Omega)}{2}$$

Complex Lorentzian close to resonance : $\Omega \approx \Omega_0$

$$\underline{\chi}(\omega) = \frac{\omega_p^2}{(\Omega_0^2 - \omega^2) + 2j\omega\frac{\Omega_0}{Q}} \longrightarrow \underline{\tilde{\chi}}(\Omega) = \frac{-j\chi_0}{1 + jQ\frac{\Omega - \Omega_0}{\Omega_0}}$$

Maximum absorption: $\chi_0 = Q\frac{\omega_p^2}{2\Omega_0^2}$

Half Width Half Maximum linewidth (HWHM): $\Delta\Omega = \frac{\Omega_0}{Q}$

Real and imaginary parts:

$$\begin{aligned}\widetilde{\chi}_r(\Omega) &= \frac{-\chi_0 \frac{(\Omega-\Omega_0)}{\Delta\Omega}}{1+\left(\frac{\Omega-\Omega_0}{\Delta\Omega}\right)^2},\\ \widetilde{\chi}_i(\Omega) &= \frac{-\chi_0}{1+\left(\frac{\Omega-\Omega_0}{\Delta\Omega}\right)^2},\end{aligned}$$

Complex wave number in lossy medium:

$$\underline{\tilde{K}}(\Omega) = \frac{\Omega}{c_0}\left(1+\frac{1}{2}\left(\widetilde{\chi}_r(\Omega)+\mathrm{j}\widetilde{\chi}_i(\Omega)\right)\right)$$

Redefine group velocity: e.g. at line center:

$$v_g^{-1} = \left.\frac{\partial K_r(\Omega)}{\partial\Omega}\right|_{\Omega_0} = \frac{1}{c_0}\left(1-\frac{\chi_0}{2}\frac{\Omega_0}{\Delta\Omega}\right)$$

Change in group velocity can be positive or negative

Absorption:

$$K = \frac{\Omega}{c_0} \qquad \alpha(\Omega) = -\frac{K}{2}\tilde{\chi}_i(\Omega)$$

For a wavepacket (optical pulse) with carrier frequency $\omega_0 = \Omega_0 \quad K_0 = \frac{\Omega_0}{c_0}$

$$\left.\frac{\partial \underline{\tilde{A}}(z,\omega)}{\partial z}\right|_{(loss)} = -\alpha(\Omega_0+\omega)\tilde{A}(z,\omega) = \frac{-\chi_0 K_0/2}{1+\left(\frac{\omega}{\Delta\Omega}\right)^2}\underline{\tilde{A}}(z,\omega)$$

Parabolic loss or gain approximation:

$$\left.\frac{\partial \underline{A}(z,t')}{\partial z}\right|_{(loss)} = -\frac{\chi_0 K_0}{2}\left(1+\frac{1}{\Delta\Omega^2}\frac{\partial^2}{\partial t^2}\right)\underline{A}(z,t')$$

Gain: $g = -\frac{\chi_0 K_0}{2}$

$$\left.\frac{\partial \underline{A}(z,t')}{\partial z}\right|_{(gain)} = g\left(1+\frac{1}{\Omega_g^2}\frac{\partial^2}{\partial t^2}\right)\underline{A}(z,t')$$

HWHM – gain bandwidth