University of Hamburg, Department of Physics

Nonlinear Optics

Kärtner/Mücke, WiSe 2017/2018

Problem Set 7

Issued: 7.12.'17 Due : 14.12.'17

Introduction to RK4 and split step Fourier method

The 4th order Runge Kutta (RK4) method $O(h5)$ is chosen in this scheme. For detail please see reference.[? ? ?]

In order to solve the nonlinear Schödinger equation, the split step Fourier method is necessary. Taking the self phase modulation (SPM) and group velocity dispersion (GVD) as an example.

$$\begin{aligned} \frac{\partial E(t,z)}{\partial z} &= \left(-i\frac{1}{2}\beta_2\frac{\partial^2}{\partial t^2} - i\frac{n_2\omega_0 I(t,z)}{c}\right)E(t,z) \\ &= (\mathcal{N}+\mathcal{L})E(t,z) \end{aligned} \tag{1}$$

$$\begin{aligned} \mathcal{N} &= -i\frac{n_2\omega_0 I(t,z)}{c} \\ \mathcal{L} &= -i\frac{1}{2}\beta_2\frac{\partial^2}{\partial t^2} \end{aligned}$$

where $\mathcal{N}$ is nonlinear operator in which n_2 is the nonlinear coefficient of SPM and $\mathcal{L}$ is linear operator in which β_2 is the coefficient of GVD. The numerical method used to solve equation (**??**) is the split step model.[? ?] The idea of the split step model is to calculate the linear operator in frequency domain and non linear operator in time domain separately.[?] In time domain if neglecting the linear operator and only the nonlinear operator is considered, one can get

$$\begin{aligned} \frac{\partial E(t,z)}{\partial z} &= \mathcal{N}E(t,z) \\ E(t,z+\frac{\triangle z}{2}) &= E(t,z)e^{\frac{\triangle z}{2}\mathcal{N}} \end{aligned} \tag{2}$$

In frequency domain if neglecting the nonlinear operator and only the linear operator is considered, one can get

$$\begin{aligned} \frac{\partial E(t,z)}{\partial z} &= \mathcal{L}E(t,z) \\ \frac{\partial \tilde{E}(\omega,Z)}{\partial Z} &= i\left(\frac{1}{2}\beta_2\omega^2\right)\tilde{E}(\omega,Z) \\ \tilde{E}(\omega,z+\frac{\triangle z}{2}) &= \tilde{E}(\omega,z)e^{\left(i\frac{1}{2}\beta_2\omega^2\right)\frac{\triangle z}{2}} \end{aligned}$$

After calculating the $\frac{\triangle z}{2}$ segment in both time and frequency separately, the total final numerical solution of equation (**??**) is[?]

$$E(t,z+\triangle z) = \mathcal{F}^{-1}\left[e^{\left(i\frac{1}{2}\beta_2\omega^2\right)\frac{\triangle Z}{2}}\mathcal{F}[e^{\frac{\triangle z}{2}\mathcal{N}}E(t,z)]\right] \tag{3}$$

where $\mathcal{F}$ represents Fourier transform. However generally equation (**??**) is not sufficient precise enough. The Runge Kutta method should be implemented rather than merely using exponential of the operators as solutions. In the following context, equation (**??**) is solved as an example using RK4 method.

$$\begin{aligned} \frac{\partial E(t,z)}{\partial z} &= -i\frac{n_2\omega_0 I(t,z)}{c}E(t,z) \\ f(E,t,z) &= -i\frac{n_2\omega_0 I(t,z)}{c}E(t,z) \end{aligned}$$

where the right hand side is written as $f(E, t, z)$ for convenience. The idea is to use the known data value at grid point z_i to calculate the next grid point z_{i+1}.[?]

$$
\begin{aligned}
k_1 &= f(E_i, t, z_i) \triangle z \\
k_2 &= f(E_i + \frac{k_1}{2}, t, z_i + \frac{\triangle z}{2}) \\
k_3 &= f(E_i + \frac{k_2}{2}, t, z_i + \frac{\triangle z}{2}) \\
k_4 &= f(E_i + k_3, t, z_i + \triangle z) \\
E(t, z_{i+1}) &= E(t, z_i) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\end{aligned}
$$

1 Homework 1, using RK4 solve SPM, Due Date 14./21.12.'17

With the techniques described above, write a code to simulate the evolution of an optical field under the influence of self-phase modulation. The nonlinear crystal will be BBO with an effective length of $L_{eff} = 2.5$mm and assume the initial pulse to have a waist size of $w = 1$mm, a fluence of $F = \frac{0.5}{\pi * w^2/2}$mJ/m^2 and a FWHM duration of 150fs.
We will have a look at a code example in class where the SPM code in frequency domain is given to you as a reference. Please write your own code of SPM using RK4 method in time domain.

2 Homework 2, numerical method of OPA, Due Date after Christmas

For simplicity, let's consider the situation when the nonlinear Schödinger equation only contains two term for the pump—SPM and second order polarisation of difference frequency generation (DFG); one term for signal and idler (DFG). In the following context, everything is converted to frequency domain. It is the same concept as the split step Fourier transform, but the difference is that here we don't Fourier transform the field, we transform the terms on the right hand side. Please write a code to solve equation (**??**,**??**,**??**) using the same method as homework 1. Note: you are suggested to solve everything in frequency domain, but of course you could also do that in time domain as you wish.

$$
\frac{\partial E_p(\omega, z)}{\partial z} = P_p^{(NL)}(\omega) - \mathcal{F}[i\frac{n_2\omega_0 I_p(t, z)}{c}E_p(t, z)] \tag{4}
$$

$$
\frac{\partial E_s(\omega, z)}{\partial z} = P_s^{(NL)}(\omega) \tag{5}
$$

$$
\frac{\partial E_i(\omega, z)}{\partial z} = P_i^{(NL)}(\omega) \tag{6}
$$

It is very very VERY important to know that the polarization terms in all the equations about are the results of all the possible frequencies combinations. For example, if we want to calculation the polarization term of signal

$$
\begin{aligned}
P_s^{(2)}(f_s) \quad &= \chi^{(2)} \int_{-\infty}^{\infty} E^*(f_p - f_s)E(f_p)df_p \qquad (7)\\
&= \chi^{(2)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (E(t_1)e^{i2\pi t_1(f_p - f_s)})^* dt_1 E(t_2)e^{i2\pi t_2 f_p}dt_2 df_p \\
&= \chi^{(2)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(t_1)^* E(t_2)e^{i2\pi(t_2 - t_1)f_p}e^{i2\pi t_1 f_s}dt_1 dt_2 df_p \\
&= \chi^{(2)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(t_1)^* E(t_2)\delta(t_2 - t_1)e^{i2\pi t_1 f_s}dt_1 dt_2 \\
&= \chi^{(2)} \int_{-\infty}^{\infty} |E(t_1)|^2 e^{i2\pi t_1 f_s}dt_1
\end{aligned}
$$

A code example is given here.

```
function [f_ir] =P_NL(chi_2,u_p,u_s,u_i,k_p,k_s,k_i,f_s,n_s,z0)
%note that u_s u_i u_p are the envelop of the field , e_i,e_s,e_p are the field
    c=3e8;
    e_i=u_i.*exp(-1i*k_i*z0);
    e_s=u_s.*exp(-1i*k_s*z0);
    e_p=u_p.*exp(-1i*k_p*z0);


    df=f_s(2)-f_s(1);
    N=length(f_s);
%inverse Fourier transform field of pump E_p and idler E_i* to time domain
    buff1 = fftshift(ifft(ifftshift(e_p))).*N*df;
    buff2 = conj(fftshift(ifft(ifftshift(e_i))).*N*df);
%  Fourier transform the result back to frequency domain. the exp(1i*k_s*z0) ...
    comes from the phase matching
    p_s =chi_2*fftshift(fft(ifftshift(buff2.*buff1))).*exp(1i*k_s*z0)./(N*df);

    f_ir =-0.5i*(2*pi)^2*f_s.^2.*p_s./(c^2.*k_s);

end
```

References

[] Thorsten Hohage and Frank Schmidt. *On the numerical solution of nonlinear Schrödinger type equations in fiber optics*. ZIB, 2002.

[] Mathematics. Help with using the runge-kutta 4th order method on a system of 2 first order ode's., 2014. URL http://math.stackexchange.com/questions/721076/help-with-using-the-runge-kutta-4th-order-method-on-a-system-of-2-first-order-od. [Online; accessed 6-Dec-2016].

[] Pablo Suarez. An introduction to the split step fourier method using matlab, 2013. [https://www.researchgate.net/publication/281441538_An_introduction_to_the_Split_Step_Fourier_Method_using_MATLAB; accessed 23-April-2016].

[] Hanquan Wang. Numerical studies on the split-step finite difference method for nonlinear schrödinger equations. *Applied mathematics and computation*, 170(1):17–35, 2005.

[] Wikipedia. Runge–kutta methods — wikipedia, the free encyclopedia, 2016. URL https://en.wikipedia.org/w/index.php?title=Runge%E2%80%93Kutta_methods&oldid=752035966. [Online; accessed 6-Dec-2016].

[] Michael Zeltkevic. Runge-kutta methods, 1998-04-15. URL http://web.mit.edu/10.001/Web/Course_Notes/Differential_Equations_Notes/node5.html. [Online; accessed 6-Dec-2016].