Nonlinear Optics (WiSe 2017/18)

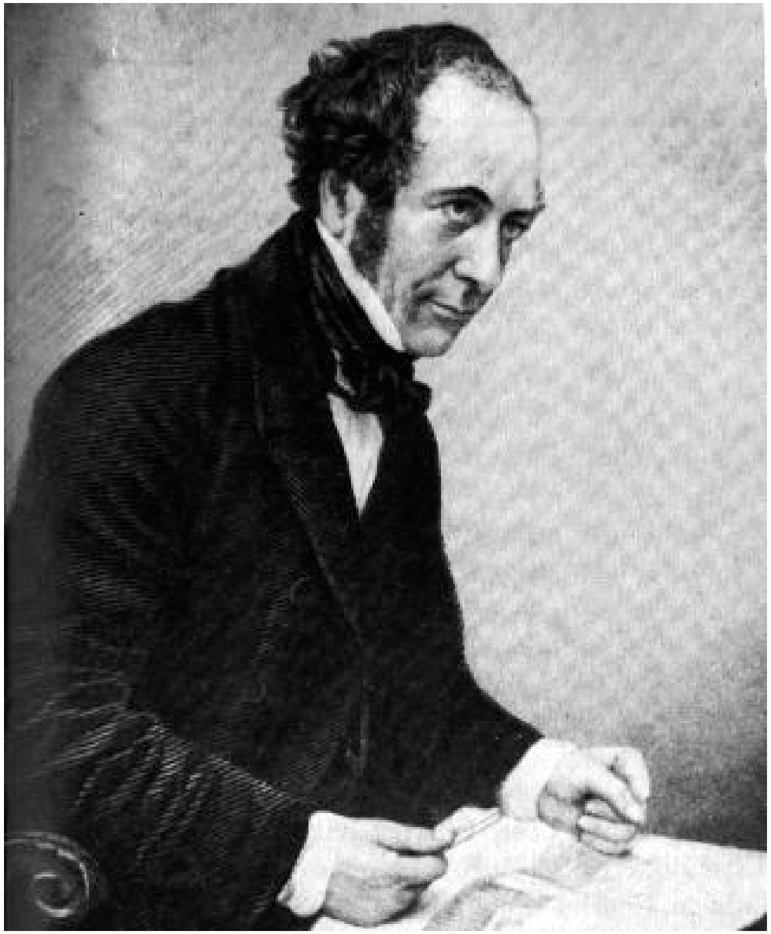
Lecture 13: November 30, 2017

8.3 Nonlinear Schrödinger Equation (NLSE)

$$-j\frac{\partial A(z,t)}{\partial z} = \frac{k''}{2}\frac{\partial^2 A}{\partial t^2} - \delta|A|^2 A. \qquad (8.15)$$

John Scott Russell

(1808-1882)



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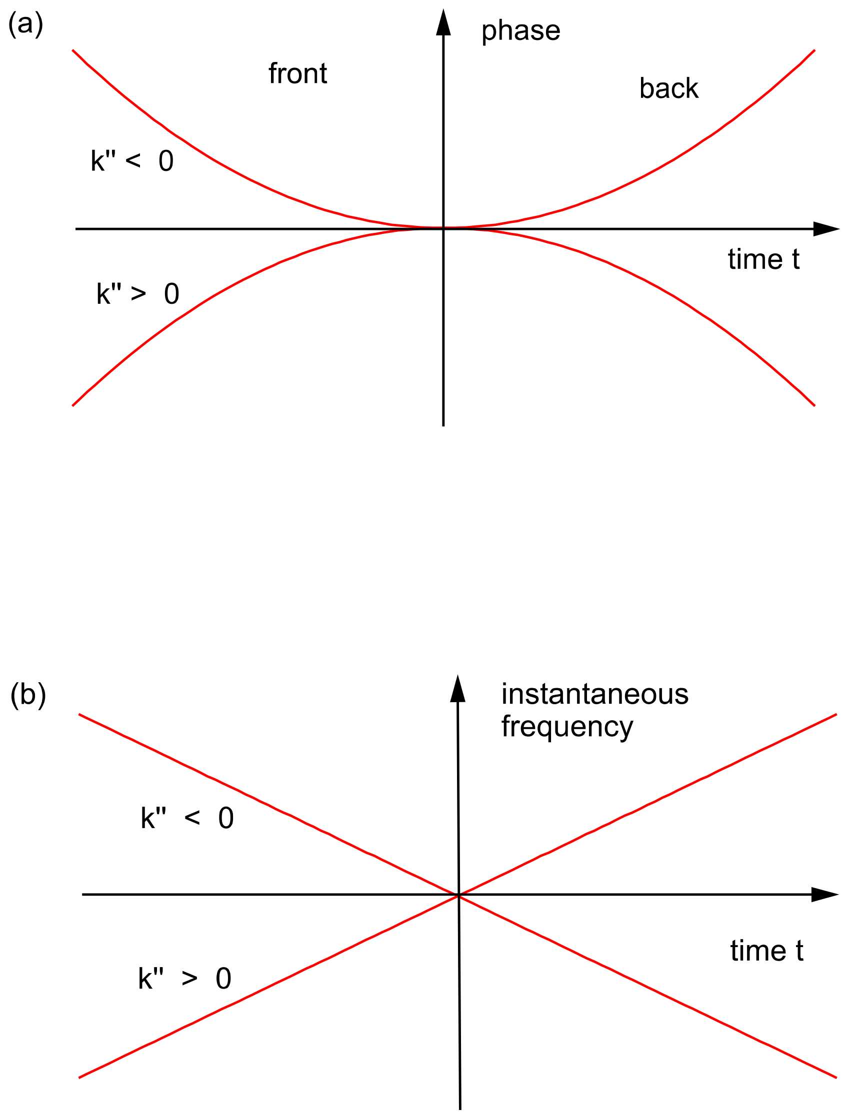


Figure 8.4: (a) Phase, (b) instantaneous frequency in a Gaussian pulse propagating in a positive dispersive medium.

John Scott Russel

In 1834, while conducting experiments to determine the most efficient design for canal boats, John Scott Russell made a remarkable scientific discovery. As he described it in his "Report on Waves":

Report of the fourteenth meeting of the British Association for the Advancement of Science, York, September 1844 (London 1845), pp 311-390, Plates XLVII-LVII).

Russell’s report

“I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed.”

Russell's report

"I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation."

Scott Russell Aqueduct

89.3m long, 4.13m wide,1.52m deep, On the union Canal, Near Heroit-Watt Univ.

*www.spsu.edu/math/txu/research/presentations/**soliton/talk.ppt***

Scott Russell Aqueduct

*www.spsu.edu/math/txu/research/presentations/**soliton/talk.ppt***

A brief history (mainly for optical soliton)

- 1838 – observation of soliton in water
- 1895 – mathematical description of waves on shallow water surfaces, i.e. KdV equation
- 1972 – optical solitons arising from NLSE
- 1980 – experimental demonstration
- 1990's – soliton control techniques
- 2000's –understanding soliton in the context of supercontinuum generation

8.4 The Solitons of the NLSE

Without loss of generality, by normalization of the field amplitude $A = \frac{A'}{\tau}\sqrt{\frac{2D_2}{\delta}}$, the propagation distance $z = z' \cdot \tau^2/D_2$, and the time $t = t' \cdot \tau$, the NLSE (8.15) reads

$$j\frac{\partial A(z,t)}{\partial z} = \frac{\partial^2 A}{\partial t^2} + 2|A|^2 A. \tag{8.16}$$

8.4.1 The fundamental soliton

$$A_s(z,t) = A_0 \mathrm{sech}\left(\frac{t}{\tau}\right) e^{-j\theta}, \tag{8.17}$$

where θ is the nonlinear phase shift of the soliton

$$\theta = \frac{1}{2}\delta A_0^2 z. \tag{8.18}$$

$$\theta = \frac{|k''|}{2\tau^2} z. \tag{8.19}$$

Since the field amplitude $A(z,t)$ is normalized, such that the absolute square is the intensity, the soliton energy fluence is given by

$$w = \int_{-\infty}^{\infty} |A_s(z,t)|^2 dt = 2A_0^2 \tau. \tag{8.20}$$

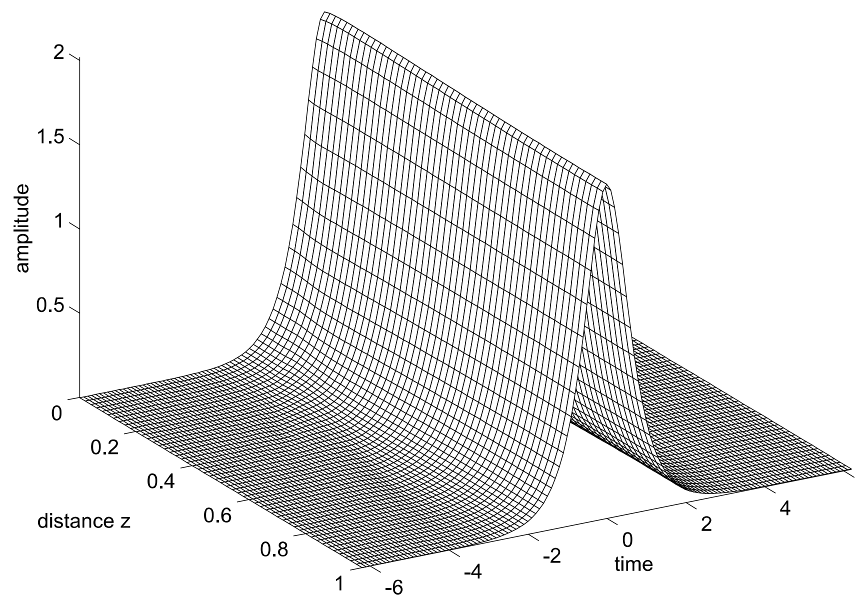
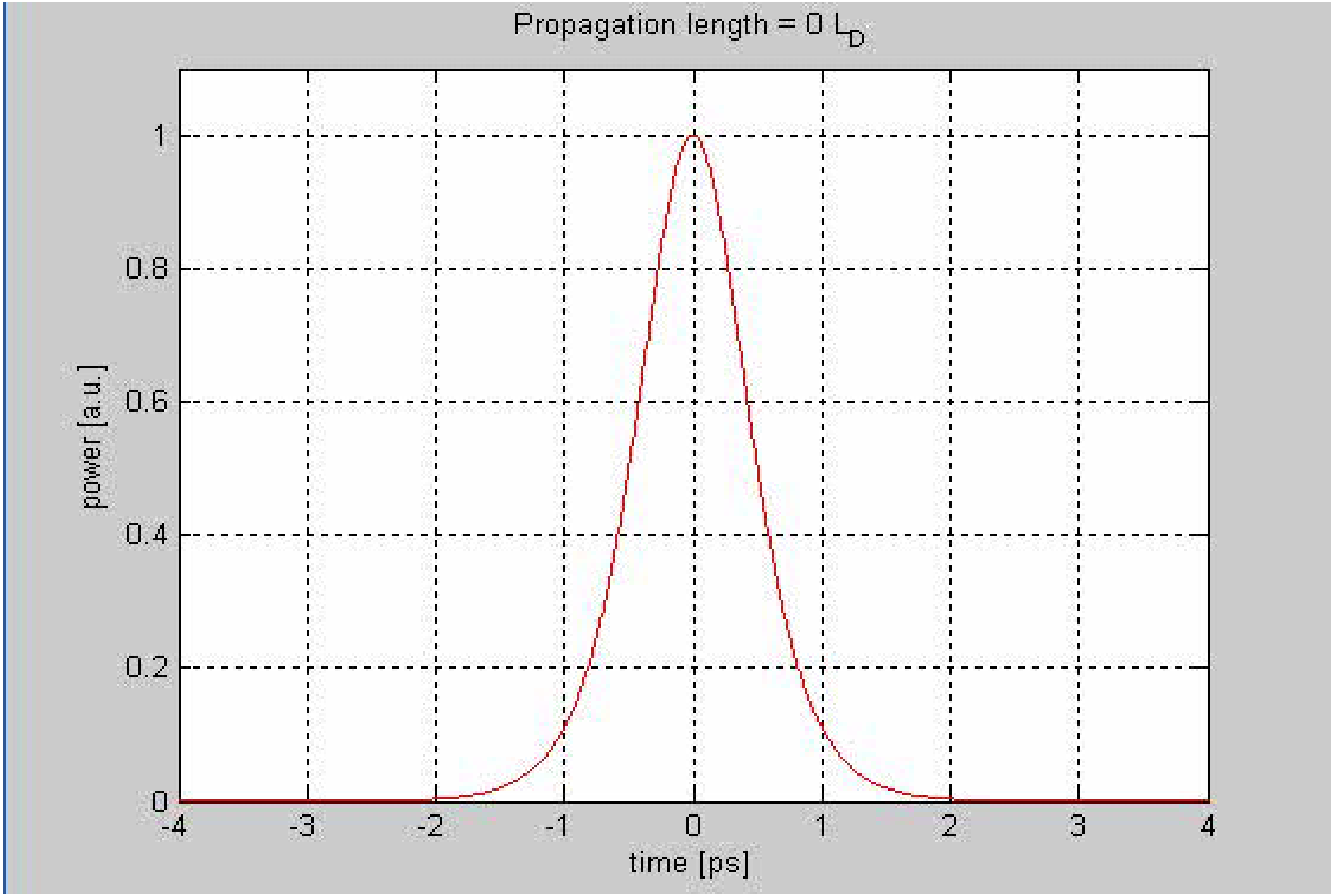


Figure 8.6: Fundamental soliton of the NLSE.

Propagation of fundamental soliton



Input: 1ps soliton centered at 1.55 um; medium: single-mode fiber

Important Relations

$$\boxed{\delta A_0^2 = \frac{2|D_2|}{\tau^2} \left(= \frac{|\beta_2|}{\tau^2}\right)} \Longrightarrow \boxed{A_s(z,t) = A_0 \operatorname{sech}\left(\frac{t}{\tau}\right) e^{-j\theta}}$$

(Balance between dispersion and nonlinearity)

Nonlinear phase shift soliton acquires during propagation:

$$\theta = \frac{1}{2}\delta A_0^2 z = \frac{|D_2|}{\tau^2} z$$

Area Theorem

$$\text{Pulse Area} = \int_{-\infty}^{\infty} |A_s(z,t)| dt = \pi A_0 \tau = \pi \sqrt{\frac{2|D_2|}{\delta}}.$$

Soliton Energy: $w = \int_{-\infty}^{\infty} |A_s(z,t)|^2 dt = 2A_0^2 \tau$ **Pulse width:** $\tau = \dfrac{4|D_2|}{\delta w}$

General fundamental soliton

$$A_s(z,t) = A_0 \mathrm{sech}(x(z,t)) e^{-j\theta(z,t)}, \tag{8.23}$$

with

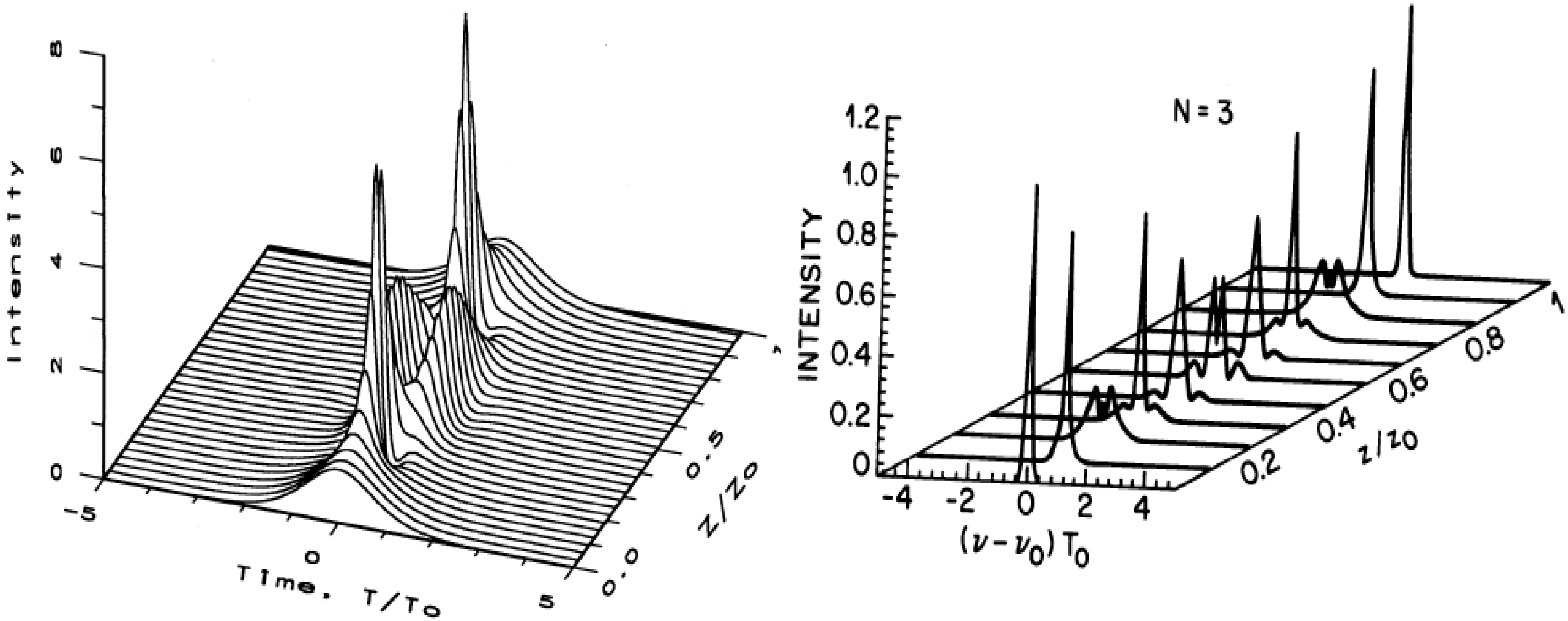
$$x = \frac{1}{\tau}(t - |k''| p_0 z - t_0), \tag{8.24}$$

and the generalized phase shift

$$\theta = p_0(t - t_0) + \frac{|k''|}{2}\left(\frac{1}{\tau^2} - p_0^2\right) z + \theta_0. \tag{8.25}$$

Higher-order Solitons: periodical evolution in both the time and the frequency domain

$$A_0\tau = N\sqrt{\frac{2|D_2|}{\delta}}, N = 1,2,3\ldots \qquad u(0,\tau) = N\mathrm{sech}(\tau)$$



G. P. Agrawal, *Nonlinear fiber optics* (2001)

Interaction between solitons (soliton collision)

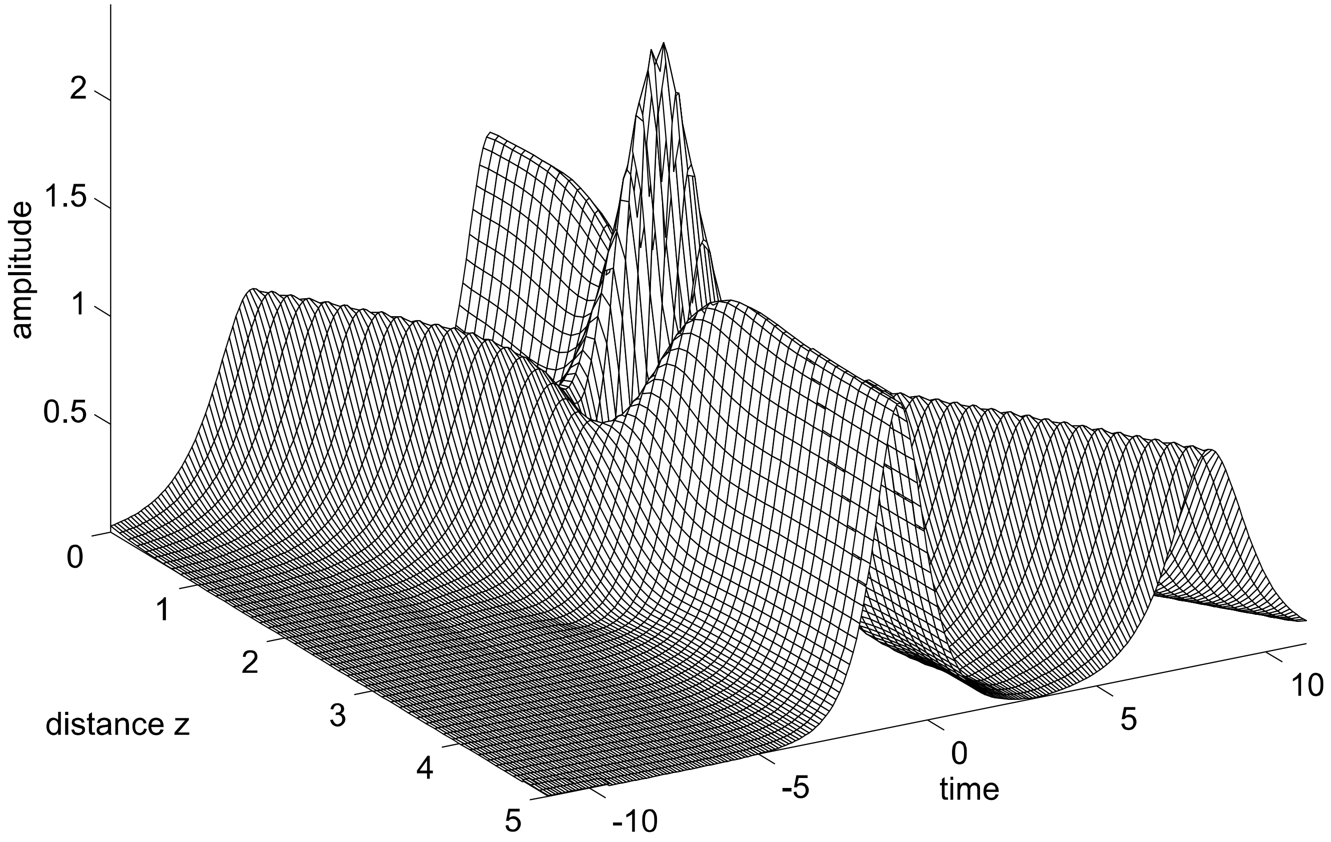
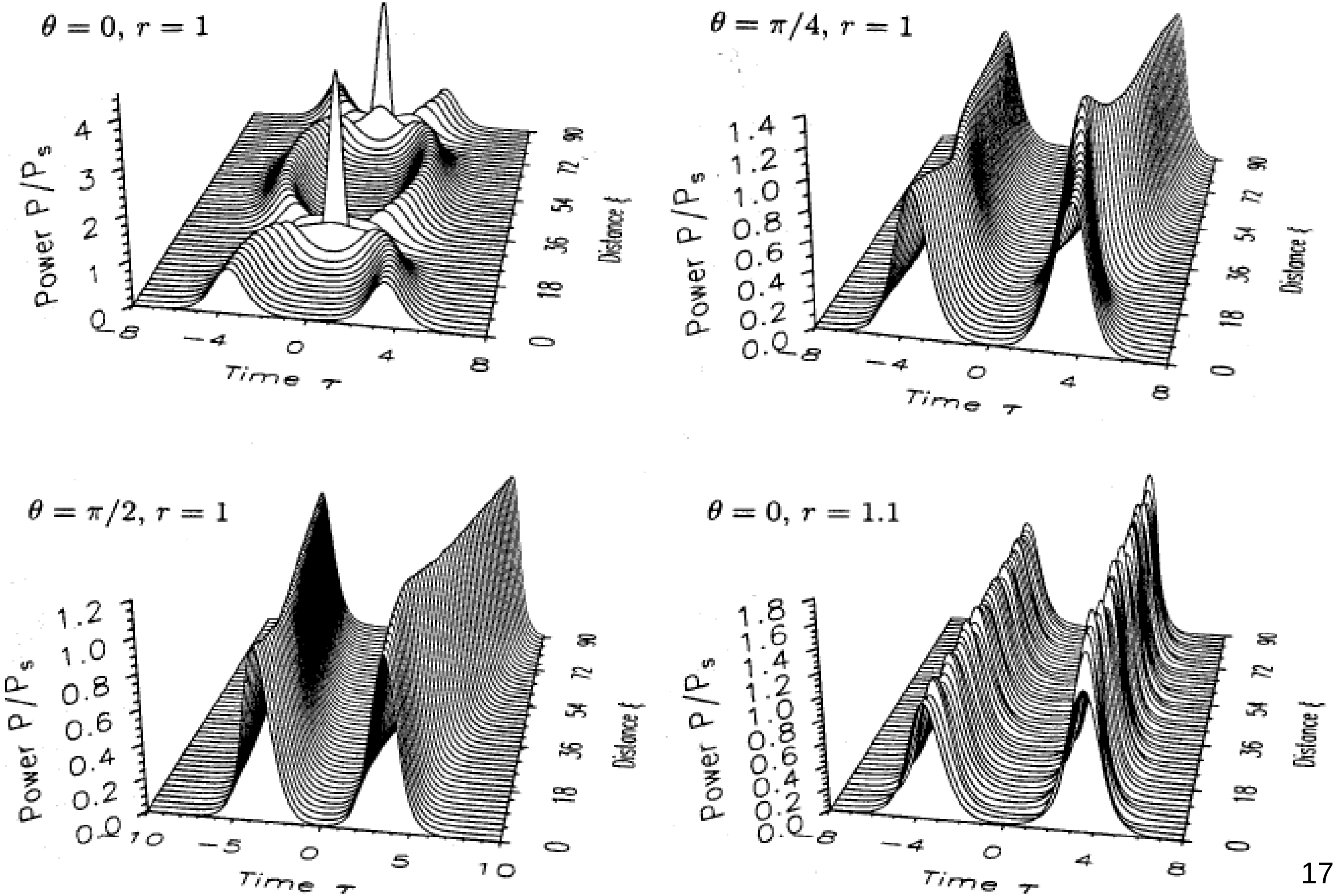


Figure 8.7: Soliton collision, both pulses recover completely.

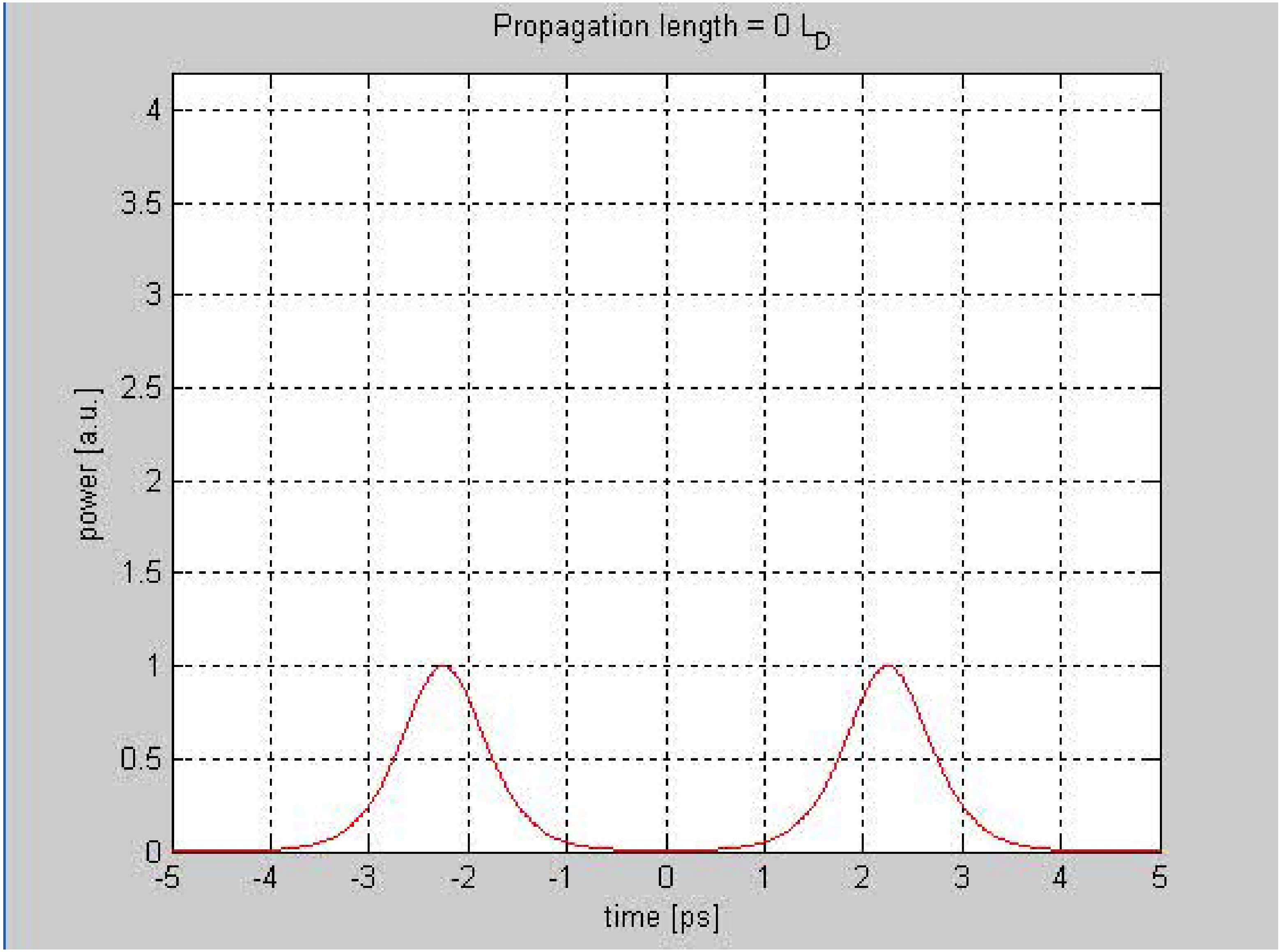
Interaction of two solitons at the same center frequency

Input to NLSE: $u(0,\tau) = \mathrm{sech}(\tau - q_0) + r\,\mathrm{sech}[r(\tau + q_0)]e^{i\theta}$

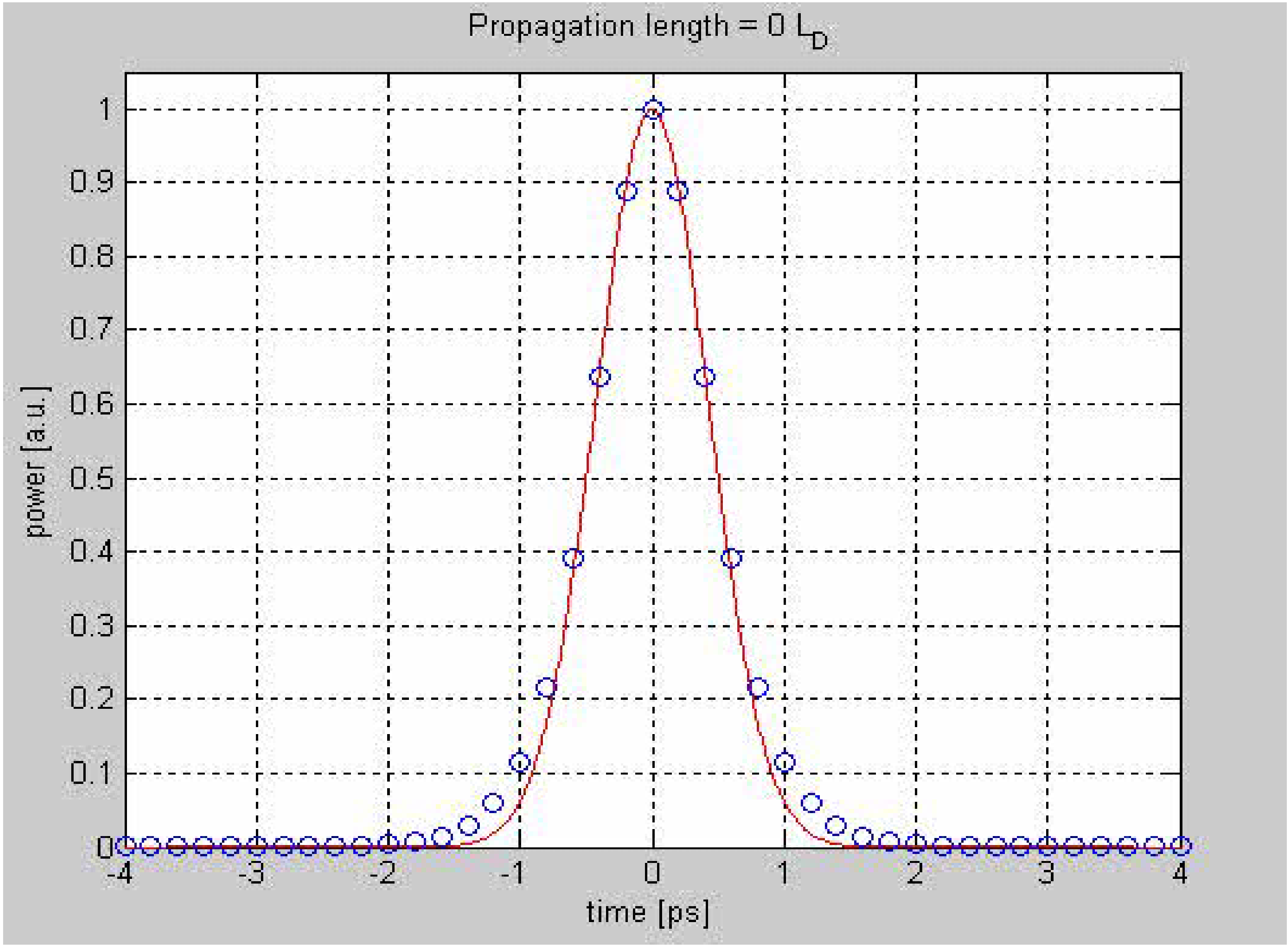


G. P. Agrawal, *Nonlinear fiber optics* (2001)

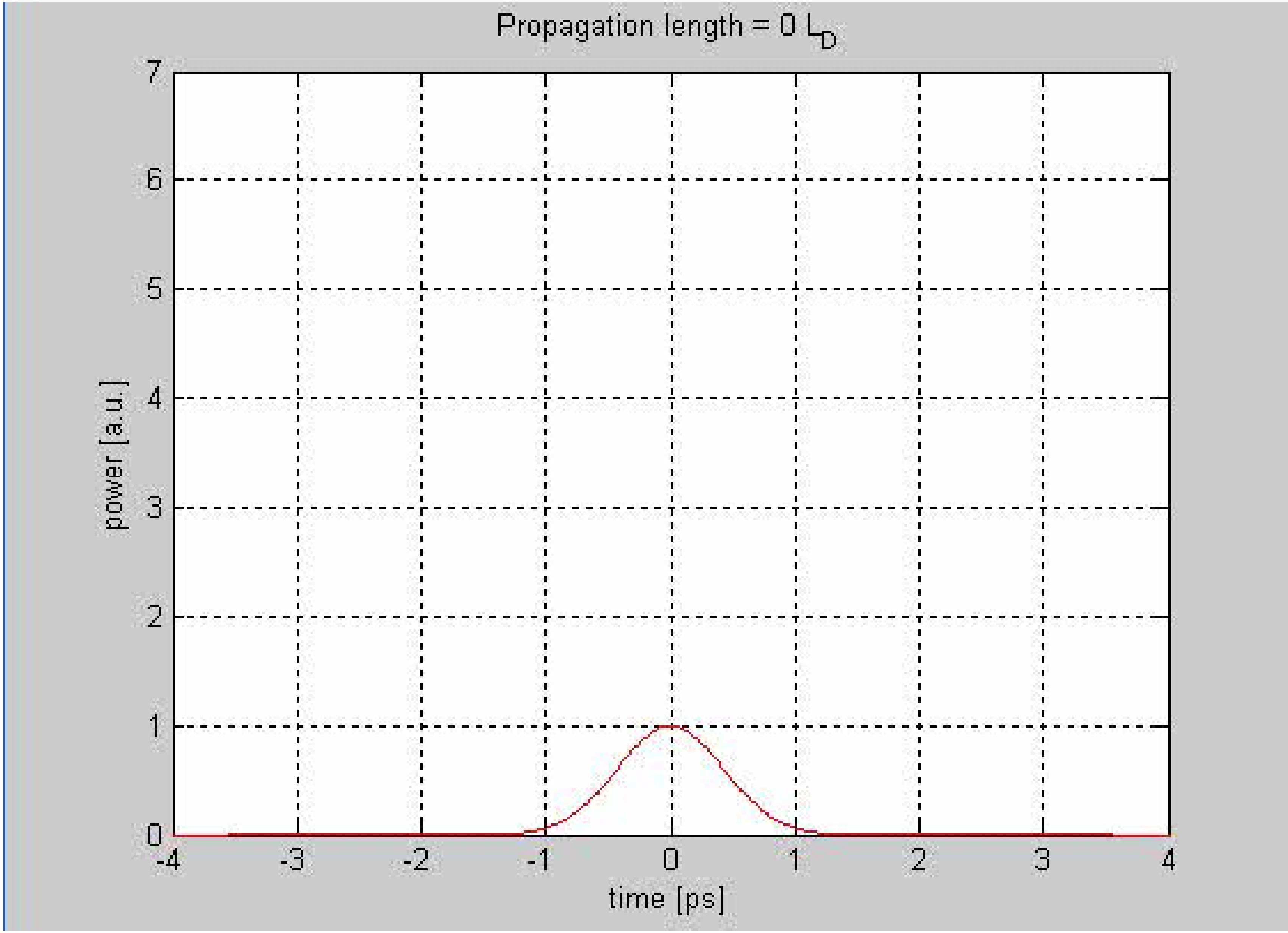
Interactions of two solitons



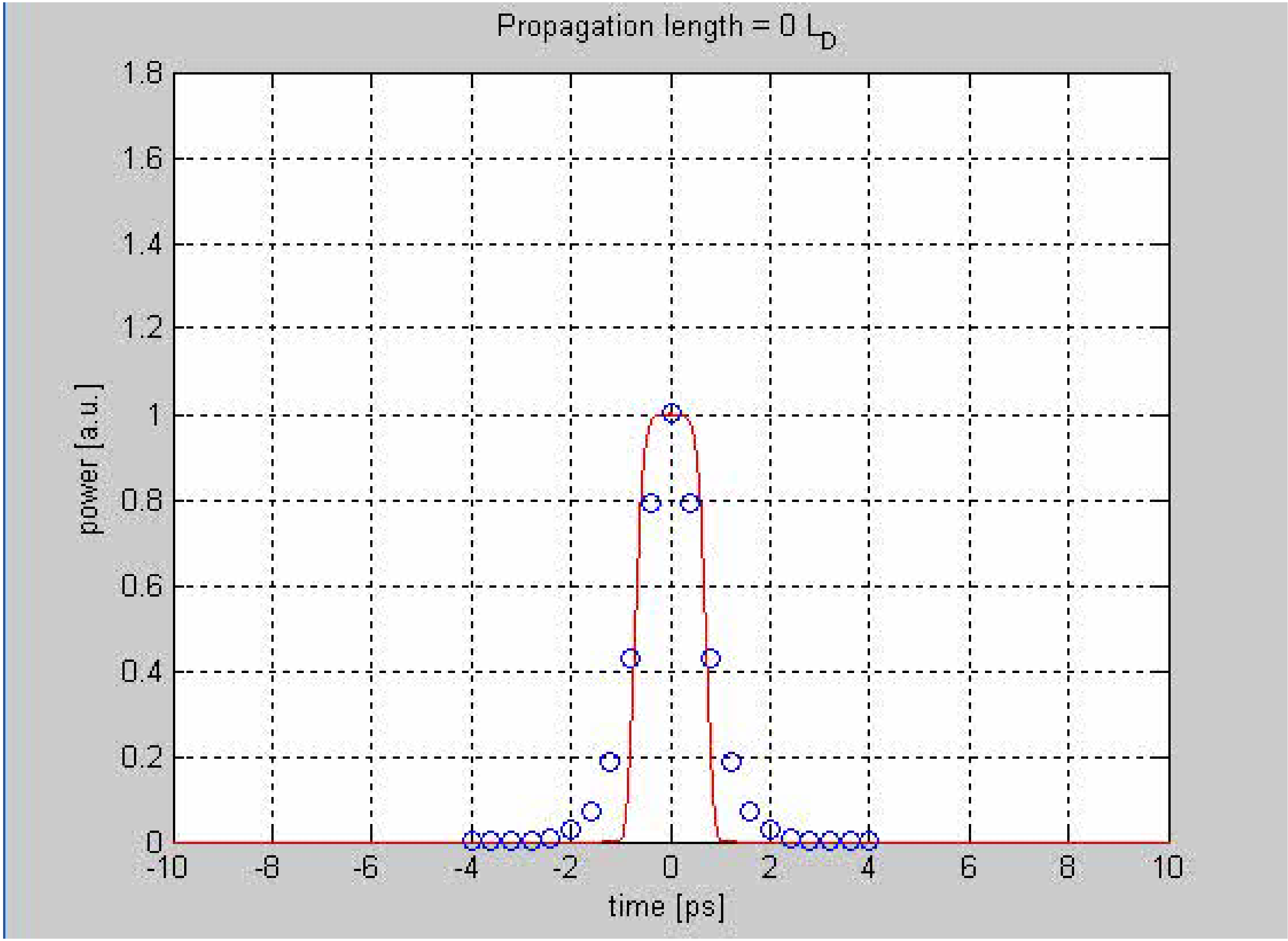
From Gaussian pulse to soliton



Gaussian pulse to 3-order soliton



Evolution of a super-Gaussian pulse to soliton



Rogue wave

Find more information from New York times:
http://www.nytimes.com/2006/07/11/science/11wave.html

One more Rogue wave

Standard Solution of PDEs

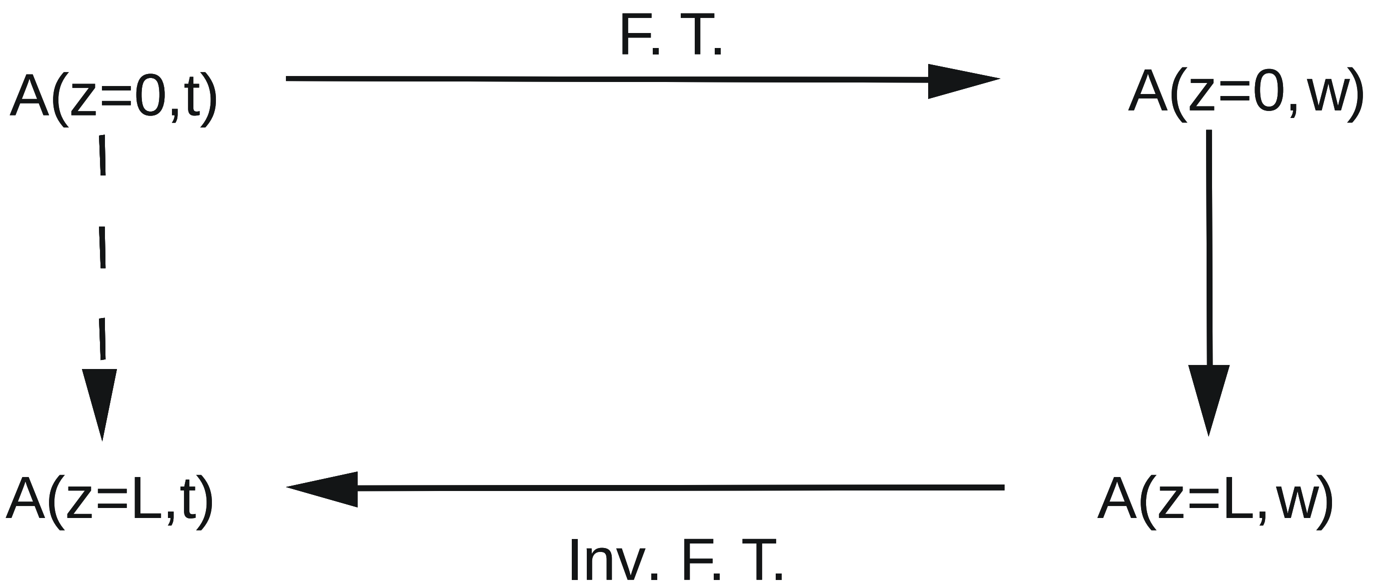


Figure 8.9: Fourier Transform method for the solution of linear time invariant PDEs.

3.3.4 Inverse Scattering Theory

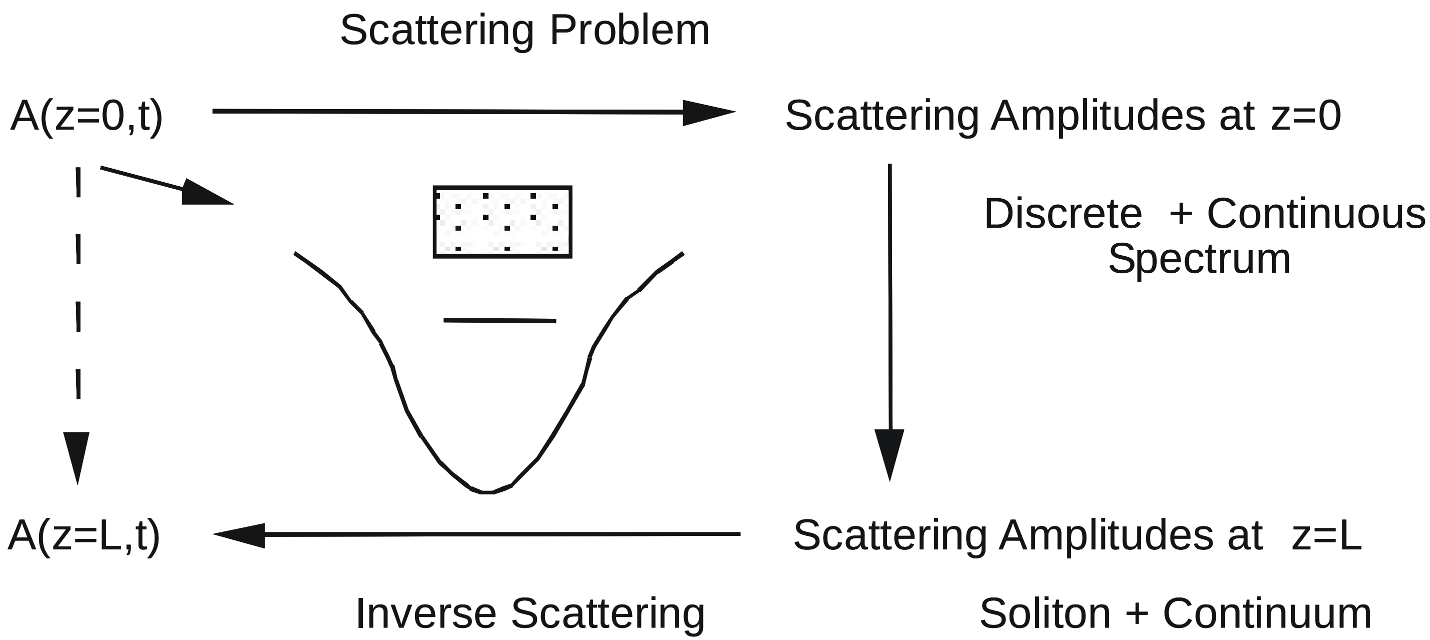


Figure 8.10: Schematic representation for the inverse scattering theory for the solution of integrable nonlinear partial differential equations

Rectangular Shaped Initial Pulse and Continuum Generation

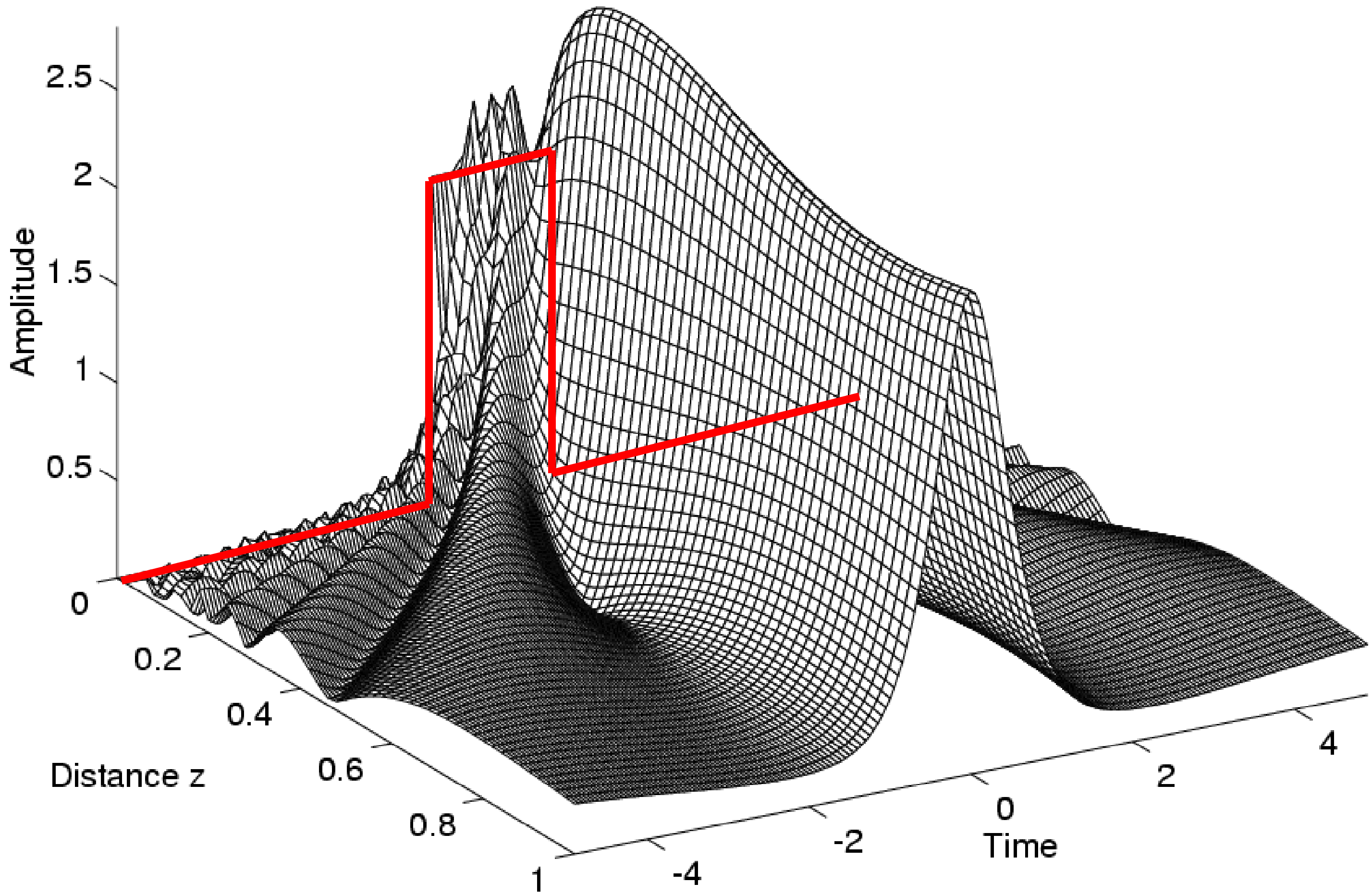


Figure 8.11: Solution of the NSE for a rectangular shaped initial pulse